QUADRATURE FORMULAS FOR SINGULAR INTEGRALS WITH HILBERT KERNEL*1)

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Abstract

In this paper, we first establish the quadrature formulae of proper integrals with weight by trigonometric interpolation. Then we use the method of separation of singularity to derive the quadrature formulae of corresponding singular integrals with Hilbert kernel. The trigonometric precision, the estimate of the remainder and the convergence of each quadrature formula derived here are also established.

§ 1. Introduction

We shall consider the numerical evaluation of singular integrals with Hilbert kernel of the form

$$I(f, x)=\int_0^{2\pi} w(t)f(t)\operatorname{ctg}\frac{1}{2}(t-x)dt, \quad x\in[0, 2\pi), \tag{1.1}$$

where $w(t)$ is a given non-negative function with period 2π which is known as the weight function, and $f(t)$ is a function with period 2π. $w(t)$ and $f(t)$ are assumed further to be Hölder-continuous for the existence of (1.1)[1].

The investigations on numerical evaluation of singular integrals with Cauchy kernel are rather complete[2-6]. But the results of investigations on numerical evaluation for singular integrals with Hilbert kernel are not many up to now, except for some special cases.

In 1974, M. M. Chawla and T. R. Ramakrishnan discussed the numerical evaluation of (1.1) for the case $w(t)=1$. They assumed that $f(z)$ is a 2π-periodic function analytic on the rectangular domain $D_r=\{z,\ 0\leqslant \operatorname{Re}(z)\leqslant 2\pi,\ -r\leqslant \operatorname{Im}(z)\leqslant r,\ r>0\}$ with the boundary B_r, which is written as $f\in AP(B_r)$. For such functions, evaluating directly the contour integral

$$(4\pi i)^{-1}\int_{B_r}\left[f(z)\operatorname{ctg}\frac{1}{2}(z-x)/\sin(nz)\right]\operatorname{ctg}\frac{1}{2}(z-t)dz$$

by the analyticity of $f(z)$, they obtained the following quadrature formula[7]:

$$\int_0^{2\pi} f(t)\operatorname{ctg}\frac{1}{2}(t-x)dt=Q_n(f, x)+R_n(f, x), \tag{1.2}$$

where

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$$Q_n(f, x)=\begin{cases}\frac{\pi}{n}\sum_{k=0}^{2n-1} f(t_k)\operatorname{ctg}\frac{1}{2}(t_k-x)+2\pi f(x)\operatorname{ctg}(nx), & x\neq t_k,\ k=0, 1, \cdots, 2n-1,\\ \frac{\pi}{n}\sum_{k=0}^{2n-1}{}' f(t_k)\operatorname{ctg}\frac{1}{2}(t_k-x)+2\pi f'(t_s)/n, & x=t_s,\ 0\leqslant s\leqslant 2n-1,\end{cases}$$

$$R_n(f, x)=(4\pi i)^{-1}\int_{B_r}\left[f(z)\operatorname{ctg}\frac{1}{2}(z-x)S_n(z)/\sin(nz)\right]dz,$$

where $t_k=k\pi/n$,

$$S_n(z)=\int_0^{2\pi}\sin(nt)\operatorname{ctg}\frac{1}{2}(z-t)\,dt,$$

and $\sum'$ denotes the summation for k except $k=s$.

In 1983, N. I. Ioakimidis rediscussed the quadrature formula (1.2) for the purpose of numerical solution of the singular integral equation with Hilbert kernel

$$ay(t)+\frac{b}{2\pi}\int_0^{2\pi}y(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt+\int_0^{2\pi}k(t, x)y(t)dt=f(x),\quad 0\leqslant x<2\pi \tag{1.3}$$

with constant coefficients a and b. He extended the result of Chawla and Ramakrishnan to the case that the number of nodes may also be odd. Only assuming $f\in C'_{2\pi}$, he obtained[8]

$$\int_0^{2\pi}f(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt\approx\frac{2\pi}{n}\sum_{k=0}^{n-1}f(t_k)\operatorname{ctg}\frac{1}{2}(t_k-x)+2\pi\operatorname{ctg}\frac{1}{2}nx f(x),$$
$$x\neq t_k,\ k=0, 1, \cdots, n-1, \tag{1.4}$$

where $t_k=2k\pi/n$.

Ioakimidis pointed out that the quadrature formula (1.4) is exact when $f(t)=\sin(jt)$ $(j=-n+1, -n+2, \cdots, n-1)$, $f(t)=\cos(jt)$ $(j=-n, -n+1, \cdots, n)$. But he gave neither the estimate of the remainder of (1.4) nor the convergence of (1.4). Ioakimidis' method applied to the derivation of (1.4) is the method of separation of singularity. By this method, (1.4) is converted directly to the classical quadrature formula for periodic functions.

The investigations of the numerical evaluation of (1.1) in the case that $w(t)$ is a general weight function, to the author's knowledge, have not appeared in literature until now. It is very natural to consider the weight function in the investigation of the numerical evaluation of singular integrals with Cauchy kernel. In general, the weight function possesses the weak singularity at the end-points of the interval of integration, and hence separates this singularity from the integrand. Particularly, this separation is one of the foundations of the numerical method of singular integral equations with Cauchy kernel[9-11]. In the singular integral with Hilbert kernel (1.1) we require the weight $w(t)$ be Hölder continuous. Hence its separation from the function $f(t)$ is often thought unnecessary, but recently the author found that the theory of numerical evaluation of the singular integral (1.1) with weight $w(t)$ is very significant in practice. In concrete terms, the quadrature formula (1.4) can only apply to the numerical solution of the singular integral equation (1.3) with the constant coefficients, which is discussed by Ioakimidis. For the numerical method of general singular integral equations with Hilbert kernel, it is a material matter to separate the weight $w(t)$ from the function $f(t)$. Thus, the quadrature formula of (1.1) is essential (this problem will be discussed in another paper by the author). Therefore it is significant in practice to investigate the numerical evaluation of (1.1).

In this paper, we first use the trigonometric interpolation to establish the quadrature formula of the proper integral. Then we use the separation of singularity to derive the quadrature formula of (1.1). The trigonometric precision, the estimate of the remainder and the convergence of each quadrature formula derived here are established.

§ 2. Quadrature Formulas of Proper Integrals

First of all, we discuss the quadrature formulas of the proper integral

$$I(f)=\int_0^{2\pi} w(t)f(t)dt, \tag{2.1}$$

which will be used directly for the numerical solution of singular integral equations with Hilbert kernel.

Notations:

The family of all the trigonometric polynomials of the form

$$a_0+\sum_{j=1}^{n}[a_j\cos(jt)+b_j\sin(jt)] \tag{2.2}$$

is written as H_n^T.

The class of all the trigonometric polynomials of the form

$$a\sin(nt+\theta)+T_{n-1}(t),\quad T_{n-1}\in H_{n-1}^T,\quad n\geqslant 1,\quad 0\leqslant\theta<\pi \tag{2.3}$$

is written as $H_n^T(\theta)$.

It is obvious that any trigonometric polynomial of order n can not belong to two different classes $H_n^T(\theta_1)$ and $H_n^T(\theta_2)$ $(\theta_1\neq\theta_2)$.

Now we consider the approximate evaluation of (2.1). It is well-known that the general form of the mechanical quadrature formula of (2.1) is

$$\int_0^{2\pi} w(t)f(t)\,dt\approx\sum_{k=1}^{n}H_kf(t_k). \tag{2.4}$$

The right-hand side is known as the quadrature sum.

We may arbitrarily choose the nodes t_k and the coefficients H_k, in as much as the structure of the quadrature formula is arbitrary. Certainly, we hope that (2.4) will possess the best accuracy for the class of trigonometric polynomials through the choice of t_k and H_k.

If (2.4) is exact for trigonometric polynomials in H_m^T and not so for a certain trigonometric polynomial of order $m+1$, we then say (2.4) possesses the trigonometric precision of order m. We mention that (2.4) does not possess the trigonometric precision of order n, however we choose t_k and H_k. This may be easily seen by considering the trigonometric polynomial $f(t)=\prod_{k=1}^{n}\sin^2\frac{1}{2}(t-t_k)$.

Thus, (2.4) possesses the trigonometric precision of order $n-1$ at most. It will be seen that there actually exists such a quadrature formula.

We assume that (2.4) is of trigonometric precision of order $n-1$, the existence of which is assumed for the time being. Under this assumption, we discuss some principles for choosing the nodes and the coefficients.

We call

$$\Pi_n(t)=\prod_{k=1}^{n}\sin\left(t-\frac{1}{2}t_k\right) \tag{2.5}$$

the nodal trigonometric polynomial of (2.4) of order n.

Let

$$\Delta_n(t)=\Pi_n(t/2)=\prod_{k=1}^{n}\sin\frac{1}{2}(t-t_k). \tag{2.6}$$

Lemma 1. *If (2.4) possesses the trigonometric precision of order $n-1$, then $\Pi_n(t)$ and any trigonometric polynomial in H_{n-1}^T are orthogonal with respect to the weight $w(2t)$ on $[0, 2\pi]$.*

For simplicity, we call such a trigonometric polynomial the orthogonal trigonometric polynomial with respect to the weight $w(2t)$ thereafter.

Proof.
$$\int_0^{2\pi} w(2t)\Pi_n(t)\cos(jt)\,dt$$
$$=\int_0^{\pi} w(2t)\Pi_n(t)\cos(jt)\,dt+(-1)^{n+j}\int_0^{\pi} w(2t)\Pi_n(t)\cos(jt)\,dt$$
$$=[1+(-1)^{n+j}]\int_0^{\pi} w(2t)\Pi_n(t)\cos(jt)\,dt.$$

When $n+j$ is odd,

$$\int_0^{2\pi} w(2t)\Pi_n(t)\cos(jt)\,dt=0.$$

When $n+j$ is even,

$$\int_0^{2\pi} w(2t)\Pi_n(t)\cos(jt)\,dt=2\int_0^{\pi} w(2t)\Pi_n(t)\cos(jt)\,dt$$
$$=\int_0^{2\pi} w(t)\Delta_n(t)\cos\frac{1}{2}jt\,dt.$$

Thus, we have

$$\Delta_n(t)\cos\frac{1}{2}jt\in H_{n-1}^T,\quad 0\leq j<n-1;$$

thereby

$$\int_0^{2\pi} w(t)\Delta_n(t)\cos\frac{1}{2}jt\,dt=\sum_{k=1}^{n}H_k\Delta_n(t_k)\cos\frac{1}{2}jt_k=0.$$

Similarly,

$$\int_0^{2\pi} w(2t)\Pi_n(t)\sin(jt)\,dt=0,\quad j=1,2,\cdots,n-1.$$

This lemma is proved.

Lemma 2. *If (2.4) possesses the trigonometric precision of order $n-1$, then*

$$H_k=\prod_{\substack{j=1\\j\neq k}}^{n}\left[\sin\frac{1}{2}(t_k-t_j)\right]^{-1}\int_0^{2\pi} w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\,dt,$$
$$k=1,2,\cdots,n. \tag{2.7}$$

Proof. Since

$$f(t)=\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\in H_{n-1}^T,$$

the quadrature formula (2.4) is exact for $f(t)$, i.e.,

$$\int_0^{2\pi}w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\,dt=\sum_{j=1}^{n}H_jf(t_j)=H_kf(t_k).$$

It means

$$H_k=\prod_{\substack{j=1\\j\neq k}}^{n}\left[\sin\frac{1}{2}(t_k-t_j)\right]^{-1}\int_0^{2\pi}w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\,dt.$$

We introduce the notation

$$\Delta_n^*(z)=\int_0^{2\pi}w(t)\Delta_n(t)\cos^{n-2}\frac{1}{2}(t-z)\operatorname{ctg}\frac{1}{2}(t-z)\,dt, \tag{2.8}$$

which is known as the associated function of $\Delta_n(t)$ with respect to the weight $w(t)$. When $z\in[0, 2\pi)$, the $\Delta_n^*(z)$ is understood in the principal value sense. If $\Pi_n(t)$ is the orthogonal trigonometric polynomial of order n with respect to the weight $w(2t)$, we call it directly as the associated function of Δ_n. This function plays an important role in the following discussions. We now point out a fundamental property of this function.

For simplicity, let $(\Delta_n^*)'(z)$ denote the derivative of $\Delta_n^*(z)$ in the set $[0, 2\pi)$ throughout this paper. We have

$$(\Delta_n^*)'(t_k)=\frac{1}{2}\int_0^{2\pi}w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\operatorname{ctg}\frac{1}{2}(t-t_k)\,dt$$

$$+\frac{1}{2}(n-1)\int_0^{2\pi}w(t)\Delta_n(t)\cos^{n-2}\frac{1}{2}(t-t_k)\,dt. \tag{2.9}$$

In fact, it follows from the continuity of the principal value integral[1].

Note. In general, $(\Delta_n^*)'(z)$ does not exist since we only assume $w(t)$ is Hölder continuous.

Now we rewrite the coefficients given by (2.7) as

$$H_k=\Delta_n^*(t_k)/2\Delta_n'(t_k). \tag{2.10}$$

Theorem 1. *If the trigonometric polynomial* (2.5) *determined by the nodes of* (2.4) *is the orthogonal trigonometric polynomial of order n with respect to the weight* $w(2t)$ *and the coefficients are given in* (2.7), *then* (2.4) *possesses the trigonometric precision of order* $n-1$.

Proof. We choose again $n-1$ distinct points $t_{n+1}, t_{n+2}, \cdots, t_{2n-1}$ besides $t_1, t_2, \cdots, t_n$. When $f\in H_{n-1}^T$, we know

$$f(t)=\sum_{k=1}^{2n-1}f(t_k)\prod_{\substack{j=1\\j\neq k}}^{2n-1}\frac{\sin\frac{1}{2}(t-t_j)}{\sin\frac{1}{2}(t_k-t_j)}.$$

Hence

$$H_k=\frac{1}{\prod\limits_{\substack{j=1\\j\neq k}}^{2n-1}\sin\frac{1}{2}(t_k-t_j)}\int_0^{2\pi}w(t)\prod_{\substack{j=1\\j\neq k}}^{2n-1}\sin\frac{1}{2}(t-t_j)\,dt.$$

By the orthogonality of $\Pi_n(t)$, we have, when $k>n$,

$$H_k=0.$$

In fact,

$$\int_0^{2\pi} w(t)\prod_{\substack{j=1\\j\neq k}}^{2n-1}\sin\frac{1}{2}(t-t_j)dt=\int_0^{2\pi} w(t)\,\Delta_n(t)\prod_{\substack{j=n+1\\j\neq k}}^{2n-1}\sin\frac{1}{2}(t-t_j)dt$$

$$=\int_0^{2\pi} w(2t)\,\Pi_n(t)\prod_{\substack{j=n+1\\j\neq k}}^{2n-1}\sin\left(t-\frac{1}{2}t_j\right)dt=0;$$

when $1\leqslant k\leqslant n$,

$$\int_0^{2\pi} w(t)\prod_{\substack{j=1\\j\neq k}}^{2n-1}\sin\frac{1}{2}(t-t_j)dt$$

$$=\int_0^{2\pi} w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\left[\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t-t_j)-\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t_k-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\right]dt$$

$$+\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t_k-t_j)\int_0^{2\pi} w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)dt.$$

In order to prove that coefficients H_k's in the present case are given by (2.7), we should only prove that the above integral vanishes:

$$\int_0^{2\pi} w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\left[\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t-t_j)-\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t_k-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\right]dt$$

$$=\int_0^{2\pi} w(t)\Delta_n(t)\left\{\left[\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t-t_j)-\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t_k-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\right]\Big/\right.$$

$$\left.\sin\frac{1}{2}(t-t_k)\right\}dt$$

$$=\int_0^{2\pi} w(2t)\Pi_n(t)\left[\prod_{j=n+1}^{2n-1}\sin\left(t-\frac{1}{2}t_j\right)-\prod_{j=n+1}^{2n-1}\sin\frac{1}{2}(t_k-t_j)\cos^{n-1}\left(t-\frac{1}{2}t_k\right)\right]$$

$$\times\operatorname{cosec}\left(t-\frac{1}{2}t_k\right)dt$$

$$=0,$$

in which we have used the orthogonality of $\Pi_n(t)$.

Thus, when $f\in H_{n-1}^T$, we get

$$\int_0^{2\pi} w(t)f(t)\,dt=\sum_{k=1}^{n}H_kf(t_k),$$

where H_k's are just those given by (2.7).

Now, we prove the existence of the orthogonal trigonometric polynomial of the form (2.5).

In the following, we introduce the space $L_w^2[0,\ 2\pi]$ equipped with the scalar product

$$(f,\ g)=\int_0^{2\pi} w(2t)f(t)g(t)\,dt. \tag{2.11}$$

Using the Gram-Schmidt orthogonalizing process, we obtain the following

Lemma 3. *There is a unique orthogonal trigonometric polynomial of order n (up to a constant factor) with respect to the weight $w(2t)$ in every class $H_n^T(\theta)$*

$(0\leqslant\theta<\pi)$.

This follows from the fact that

$$1,\ \cos t,\ \sin t,\ \cdots,\ \cos(n-1)t,\ \sin(n-1)t,\ \sin(nt+\theta)$$

is a linearly independent set in $L^2_w[0, 2\pi]$.

We show that the orthogonal trigonometric polynomial of order n with respect to the weight $w(2t)$ is of the form (2.5). To do so, we first discuss some properties of $T_n(t)$.

Lemma 4. *If $T_n\in H^T_n$, $\neq 0$ and is the orthogonal trigonometric polynomials with respect to the weight $w(2t)$, then it is of order n.*

Lemma 5. *If both $T^1_n\in H^T_n$ and $T^2_n\in H^T_n$ are orthogonal trigonometric polynomials with respect to the weight $w(2t)$, so is their linear combination.*

Lemma 6. *If $T_n(t)$ is the orthogonal trigonometric polynomial with respect to the weight $w(2t)$, so is $T_n(\pi+t)$.*

The proofs of these lemmas are simple.

Lemma 7. *If $T_n(t)$ is the orthogonal trigonometric polynomial with respect to the weight $w(2t)$, then*

$$T_n(t)=(-1)^n T_n(\pi+t).$$

Proof. Write

$$T_n(t)=a_n\cos(nt)+b_n\sin(nt)+\cdots+a_0;$$

then

$$T_n(\pi+t)=(-1)^n a_n\cos(nt)+(-1)^n b_n\sin(nt)+\cdots+a_0.$$

By Lemma 6 and 5, we know $T_n(t)-(-1)^nT_n(\pi+t)$ is the orthogonal trigonometric polynomial with respect to the weight $w(2t)$. Noting $T_n(t)-(-1)^nT_n(\pi+t)\in H^T_{n-1}$, the conclusion follows by Lemma 4.

Lemma 8. *If $T_n(t)$ is the orthogonal trigonometric polynomial of order n with respect to the weight $w(2t)$, then $T_n(t)$ possesses $2n$ simple zeros in $[0, 2\pi)$, n of which, $t_1, t_2, \cdots, t_n$ are located in $[0, \pi)$ and the others are just $\pi+t_k$ $(k=1, 2, \cdots, n)$.*

Proof. Suppose that the zeros of $T_n(t)$ in $[0, \pi)$ are $t_1, t_2, \cdots, t_m$ with order $k_1, k_2, \cdots, k_m$ respectively. By Lemma 7 we know that it has and only has the zeros $\pi+t_1, \pi+t_2, \cdots, \pi+t_m$ with order $k_1, k_2, \cdots, k_m$ in $[\pi, 2\pi)$.

Without loss of generality, we assume $k_1, k_2, \cdots, k_p$ are odd and others, if any, even; then the sign of $T_n(t)\prod_{j=1}^{p}\sin(t-t_j)$ does not change. Thereby

$$\int_0^{2\pi} w(2t)T_n(t)\prod_{j=1}^{p}\sin(t-t_j)dt\neq 0.$$

Again from the orthogonality of T_n and noting $m\leqslant n$, we get

$$p=m=n.$$

Now, it is easy to obtain

Theorem 2. *There exists a unique orthogonal trigonometric polynomial of order n with respect to the weight $w(2t)$ in each class $H^T_n(\theta)$ $(0\leqslant\theta<\pi)$*

$$\Pi_n(t)=a\prod_{j=1}^{n}\sin\left(t-\frac{1}{2}t_j\right),\quad a\neq 0, \tag{2.12}$$

where $\frac{1}{2}t_j$ *are the n distinct zeros of* $\Pi_n(t)$ *in* $[0, \pi)$.

In (2.12), by appropriately choosing the coefficient a, we may get $\Pi_n(t)$ of the form $\Pi^\theta_n(t)=\sin(nt+\theta)+T_{n-1}(t)$ $(T_{n-1}\in H^T_{n-1})$.

Lemma 9. $\Pi^\theta_n(t)=\cos\theta\Pi^0_n(t)+\sin\theta\Pi^{\frac{\pi}{2}}_n(t)$.

Proof. By Lemma 5, we know that $\cos\theta\Pi^0_n(t)+\sin\theta\Pi^{\frac{\pi}{2}}_n(t)$ is the orthogonal trigonometric polynomial of order n with respect to the weight $w(2t)$. Then by Lemma 3 we know that it is just $\Pi^\theta_n(t)$.

Now, we may construct (2.4) based on $\Pi_n(t)$.

Let $\Delta_n(t)=\Pi_n\left(\frac{1}{2}t\right)$. Taking the zeros $t_1, t_2, \cdots, t_n$ of $\Delta_n(t)$ in $[0, 2\pi)$ as the nodes and evaluating the coefficients by (2.7), we obtain the quadrature formula of (2.1) which is stated in the following theorem.

Theorem 3.

$$I(f)=Q_n(f)+R_n(f), \tag{2.13}$$

$$Q_n(f)=\sum_{k=1}^{n}H_k f(t_k),$$

$$R_n(f)=\int_0^{2\pi} w(t)r_{n-1}(t_1, t_2, \cdots, t_n, f; t)dt,$$

where the nodes are the zeros of $\Delta_n(t)$ *in* $[0, 2\pi)$; $\Delta_n(t)=\Pi_n\left(\frac{1}{2}t\right)$, $\Pi_n(t)$ *is the nodal trigonometric polynomial which is the orthogonal trigonometric polynomial of order n with respect to the weight* $w(2t)$; *the coefficients*

$$H_k=\Delta^*_n(t_k)/2\Delta'_n(t_k),\quad \Delta^*_n(z)=\int_0^{2\pi} w(t)\Delta_n(t)\cos^{n-2}\frac{1}{2}(t-z)\operatorname{ctg}\frac{1}{2}(t-z)dt$$

is the associated function of $\Delta_n(t)$;

$$r_{n-1}(t_1, t_2, \cdots, t_n, f; t)=f(t)-T_{n-1}(t_1, t_2, \cdots, t_n, f; t);$$

$T_{n-1}(t_1, t_2, \cdots, t_n, f; t)\in H^T_{n-1}$, *which takes the same values as* $f(t)$ *at* $\{t_k\}^n_1$.

The proof of this theorem is obtained directly from Theorems 1 and 2.

Corollary 1. The quadrature formula (2.13) possesses trigonometric precision of order $n-1$.

Two important properties for the associated function are given in the following.

Corollary 2.

$$\Delta^*_n(z)=\int_0^{2\pi} w(t)\Delta_n(t)\cos^m\frac{1}{2}(t-z)\operatorname{ctg}\frac{1}{2}(t-z)dt,\quad -1\leqslant m<n,\ m\equiv n\ (\mathrm{mod}\ 2). \tag{2.14}$$

In general, if $f(t)=\prod_{r=1}^{k+1}\sin\frac{1}{2}(t-\lambda_r)$, $-1\leqslant k, m, j<n$, $k+j+1\equiv m\equiv n\ (\mathrm{mod}\ 2)$, then

$$\int_0^{2\pi} w(t)\Delta_n(t)f(t)\cos^j\frac{1}{2}(t-z)\operatorname{ctg}\frac{1}{2}(t-z)dt$$
$$=f(z)\int_0^{2\pi} w(t)\Delta_n(t)\cos^m\frac{1}{2}(t-z)\operatorname{ctg}\frac{1}{2}(t-z)dt=f(z)\Delta^*_n(z). \tag{2.15}$$

It suffices to prove the first equality of (2.15). It follows from

$$\left[\Delta_n(t)\,f(t)\cos^l\frac{1}{2}(t-z)-f(z)\,\Delta_n(t)\cos^m\frac{1}{2}(t-z)\right]\operatorname{ctg}\frac{1}{2}(t-z)\in H^T_{n-1}$$

and (2.13).

Corollary 3.

$$(\Delta_n^*)'(t_k)=\frac{1}{2}\int_0^{2\pi}w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin\frac{1}{2}(t-t_j)\cos^{n-1}\frac{1}{2}(t-t_k)\operatorname{ctg}\frac{1}{2}(t-t_k)\,dt. \qquad (2.16)$$

This follows from (2.9), (2.13) and the orthogonality of $\Pi_n(t)$.

Corollary 4. The coefficients H_k in (2.13) may also be given by

$$H_k=\begin{cases}\dfrac{1}{2\Delta_n'(t_k)}\displaystyle\int_0^{2\pi}w(t)\,\Delta_n(t)\operatorname{ctg}\frac{1}{2}(t-t_k)\,dt, & n\equiv 0\ (\mathrm{mod}\ 2),\\[2ex] \dfrac{1}{2\Delta_n'(t_k)}\displaystyle\int_0^{2\pi}w(t)\,\Delta_n(t)\operatorname{cosec}\frac{1}{2}(t-t_k)\,dt, & n\equiv 1\ (\mathrm{mod}\ 2).\end{cases} \qquad (2.17)$$

Corollary 5. The coefficients H_k of (2.13) are positive:

$$H_k=\frac{1}{\prod\limits_{\substack{j=1\\j\neq k}}^{n}\sin^2\frac{1}{2}(t_k-t_j)}\int_0^{2\pi}w(t)\prod_{\substack{j=1\\j\neq k}}^{n}\sin^2\frac{1}{2}(t-t_j)\,dt,\quad k=1,2,\cdots,n. \qquad (2.18)$$

This follows from (2.15), or directly from (2.13).

§ 3. Quadrature Formulas of $H_n^T(\theta)$ Type

In the last section, we have established the quadrature formula of the highest trigonometric precision which is exact for any trigonometric polynomial in H^T_{n-1} and not so for a certain trigonometric polynomial of order n. It is worth noting that the quadrature formula (2.13) is not exact for all trigonometric polynomials of order n. In this regard, some authors had a wrong view[12]. In fact, it will be found that the trigonometric precision of (2.13) may be stated more exactly, namely, it is exact for a certain class $H_n^T(\theta)$.

It is interesting to note that there is an essential difference between the concepts of algebraic precision and of trigonometric precision. A quadrature formula having the algebraic precision of order $n-1$ is not exact for any algebraic polynomial of order n, but a quadrature formula having the trigonometric precision of order $n-1$ is exact for certain trigonometric polynomials of order n and not so for the rest. This will be verified later.

If the quadrature formula (2.4) is exact for any trigonometric polynomial in $H_n^T(\theta)$, we say it is of $H_n^T(\theta)$ type. In this section, we discuss the quadrature formula of $H_n^T(\theta)$ type, which is important when applied to the theory of numerical solution of the singular integral equations with Hilbert kernel. Obviously, the quadrature formula of $H_n^T(\theta)$ type possesses the trigonometric precision of order $n-1$. Hence it must also be given by (2.13). We have seen that the quadrature formula (2.13) is not unique, since $\Pi_n(t)$ is not unique and exists in each class $H_n^T(\theta)$ $(0\leqslant\theta<\pi)$. Thus, we have a certain chance of choice in the construction of (2.13). We now discuss how to choose $\Pi_n(t)$ such that (2.13) is of $H_n^T(\theta)$ type.

In the first place, we mention an obvious and useful fact: whatever the choice of $\Pi_n(t)$ is, the quadrature formula (2.4) cannot be exact for both of the two different classes $H_n^T(\theta_1)$ and $H_n^T(\theta_2)$ $(\theta_1\neq\theta_2)$. In fact, if (2.13) is exact for both $H_n^T(\theta_1)$ and $H_n^T(\theta_2)$, then it is exact for both of the trigonometric polynomials of order n

$$\sin(nt+\theta_1)=\cos\theta_1\sin(nt)+\sin\theta_1\cos(nt)$$

and

$$\sin(nt+\theta_2)=\cos\theta_2\sin(nt)+\sin\theta_2\cos(nt).$$

Hence there exist k_1 and k_2 such that

$$\begin{pmatrix}\cos\theta_1 & \cos\theta_2\\ \sin\theta_1 & \sin\theta_2\end{pmatrix}\begin{pmatrix}k_1\\ k_2\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Thus, (2.13) is exact for

$$\cos(nt)=k_1\sin(nt+\theta_1)+k_2\sin(nt+\theta_2).$$

By analogy, it is exact for $\sin(nt)$. Therefore it is exact for any trigonometric polynomial of order n. This means it possesses the trigonometric precision of order n at least, which is impossible as shown above.

Lemma 10. *If (2.13) is exact for one trigonometric polynomial of order n in $H_n^T(\theta)$, then it is of $H_n^T(\theta)$ type.*

These facts reveal the characteristics of the quadrature formula of $H_n^T(\theta)$ type. In fact, the quadrature formula of $H_n^T(\theta)$ type is exact for all trigonometric polynomials in $H_n^T(\theta)$, and is not exact for any trigonometric polynomial of order n not in $H_n^T(\theta)$. From this point of view, it is reasonable that we specify the trigonometric precision of the quadrature formula by the class $H_n^T(\theta)$.

Moreover, we shall prove that any quadrature formula having the trigonometric precision of order $n-1$ is of a certain $H_n^T(\theta)$ type, in reverse, given a class $H_n^T(\theta)$, a quadrature formula may be constructed so that it is of $H_n^T(\theta)$ type.

We first establish several lemmas. Denote

$$\begin{aligned} r_n^c&=(\Pi_n^{\frac{\pi}{2}},\ \Pi_n^{\frac{\pi}{2}})=\int_0^{2\pi}w(2t)\,[\Pi_n^{\frac{\pi}{2}}(t)]^2dt,\\ r_n^s&=(\Pi_n^0,\ \Pi_n^0)=\int_0^{2\pi}w(2t)\,[\Pi_n^0(t)]^2dt,\\ r_n&=(\Pi_n^0,\ \Pi_n^{\frac{\pi}{2}})=\int_0^{2\pi}w(2t)\,\Pi_n^0(t)\Pi_n^{\frac{\pi}{2}}(t)dt.\end{aligned}\tag{3.1}$$

Lemma 11. *For any given $\alpha(0\leqslant\alpha<\pi)$, there exists a unique β, such that Π_n^α and Π_n^β are orthogonal with respect to the weight $w(2t)$.*

Proof.

$$\begin{aligned}&\int_0^{2\pi}w(2t)\,\Pi_n^\alpha(t)\sin(nt+\beta)dt\\ &=\cos\beta\int_0^{2\pi}w(2t)\,\Pi_n^\alpha(t)\sin(nt)dt+\sin\beta\int_0^{2\pi}w(2t)\,\Pi_n^\alpha(t)\cos(nt)dt\\ &=\cos\beta\int_0^{2\pi}w(2t)\,\Pi_n^\alpha(t)\,\Pi_n^0(t)dt+\sin\beta\int_0^{2\pi}w(2t)\,\Pi_n^\alpha(t)\,\Pi_n^{\frac{\pi}{2}}(t)dt\end{aligned}$$

$$= \cos\beta\cos\alpha\int_0^{2\pi} w(2t)\sin(nt)\Pi_n^0(t)\,dt + \cos\beta\sin\alpha\int_0^{2\pi} w(2t)\cos(nt)\Pi_n^0(t)\,dt$$

$$+ \sin\beta\cos\alpha\int_0^{2\pi} w(2t)\sin(nt)\Pi_n^{\frac{\pi}{2}}(t)\,dt + \sin\beta\sin\alpha\int_0^{2\pi} w(2t)\cos(nt)\Pi_n^{\frac{\pi}{2}}(t)\,dt$$

$$= \cos\alpha\cos\beta r_n^s + \sin(\alpha+\beta)r_n + \sin\alpha\sin\beta r_n^c.$$

Hence $\Pi_n^\alpha(t)$ and $\Pi_n^\beta(t)$ are orthogonal with respect to the weight $w(2t)$ iff

$$\cos\alpha\cos\beta r_n^s + \sin(\alpha+\beta)r_n + \sin\alpha\sin\beta r_n^c = 0, \tag{3.2}$$

i.e., iff

$$\beta = \begin{cases} \operatorname{arc\,ctg}\left(\dfrac{-\sin\alpha r_n^c - \cos\alpha r_n}{\cos\alpha r_n^s + \sin\alpha r_n}\right), & \text{for } \cos\alpha r_n^s + \sin\alpha r_n \neq 0, \\ 0, & \text{for } \cos\alpha r_n^s + \sin\alpha r_n = 0. \end{cases} \tag{3.3}$$

In the above proof, for the case $\cos\alpha r_n^s + \sin\alpha r_n = 0$, the uniqueness of the solution of (3.2) must be specified. In this case, we should prove $\sin\alpha r_n^c + \cos\alpha r_n \neq 0$ for the uniqueness of solution. In fact, if

$$\begin{cases} \cos\alpha r_n^s + \sin\alpha r_n = 0, \\ \sin\alpha r_n^c + \cos\alpha r_n = 0, \end{cases}$$

then

$$r_n^s r_n^c = (r_n)^2.$$

On the other hand, noting (3.1), by the Schwarz inequality, we get

$$r_n^s r_n^c \geq (r_n)^2 \tag{3.4}$$

in which the equality holds iff $\Pi_n^0(t)$ and $\Pi_n^{\frac{\pi}{2}}(t)$ are linearly independent.

From this contradiction, the lemma follows.

Obviously, $\Pi_n^\alpha(t)$ and $\Pi_n^\beta(t)$ are orthogonal to each other, and thus are said to be a pair of conjugate orthogonometric polynomial. The condition (3.2) is called the conjugate condition.

Lemma 12. *If $f \in H_n^T(\theta_1)$, $g \in H_n^T(\theta_2)$, then $fg \in H_{2n}^T\left(\left[\theta_1+\theta_2+\frac{\pi}{2}\right]_\pi\right)$ where the notation $[x]_\pi$ denotes the number congruent to x in $[0, \pi)$ for the modulus π.*

This lemma is established by direct evaluation.

Corollary 6. $\Pi_n^\theta\left(\frac{1}{2}t\right)\sin\left(\frac{1}{2}nt+\theta_1\right) \in H_n^T\left(\left[\theta+\theta_1+\frac{1}{2}\pi\right]_\pi\right)$.

Theorem 4. *If the nodal trigonometric polynomial of the quadrature formula (2.13) is $\Pi_n^\alpha(t)$, whose conjugate orthogonal trigonometric polynomial is $\Pi_n^\beta(t)$, then the necessary and sufficient condition for (2.13) to be of $H_n^T(\theta)$ type is*

$$\theta = \left[\alpha+\beta+\frac{1}{2}\pi\right]_\pi. \tag{3.5}$$

Proof. Note

$$\int_0^{2\pi} w(2t)\Pi_n^\alpha(t)\sin(nt+\phi)\,dt = 2\int_0^{2\pi} w(t)\,\Pi_n^\alpha\left(\frac{1}{2}t\right)\sin\left(\frac{1}{2}nt+\phi\right)dt. \tag{3.6}$$

When $\phi = \left[\theta-\frac{1}{2}\pi-\alpha\right]_\pi$, by Lemma 12 and the above corollary, we know

$$\Pi_n^\alpha\left(\frac{1}{2}t\right)\sin\left(\frac{1}{2}nt+\phi\right) \in H_n^T(\theta).$$

Hence if (2.13) is of $H_n^T(\theta)$ type, then

$$\int_0^{2\pi} w(2t)\,\Pi_n^\alpha(t)\sin(nt+\phi)\,dt=0,$$

i.e.,

$$\int_0^{2\pi} w(2t)\,\Pi_n^\alpha(t)\,\Pi_n^\phi(t)\,dt=0.$$

It follows that $\Pi_n^\alpha(t)$ and $\Pi_n^\phi(t)$ are conjugate to each other. By Lemma 11 we get

$$\beta=\phi=\left[\theta-\alpha-\frac{1}{2}\pi\right]_\pi,$$

i.e., $\theta=\left[\alpha+\beta+\frac{1}{2}\pi\right]_\pi$.

In the second place, we show the sufficiency of the condition.

If we take $\phi=\beta$ in (3.6), then

$$\int_0^{2\pi} w(t)\,\Pi_n^\alpha\left(\frac{1}{2}t\right)\sin\left(\frac{1}{2}nt+\beta\right)dt=0.$$

It follows that (2.13) is exact for $\Pi_n^\alpha\left(\frac{1}{2}t\right)\sin\left(\frac{1}{2}nt+\beta\right)$. Again noting that this trigonometric polynomial is of order n and $\in H_n^T(\theta)$, by Lemma 10 we know that (2.13) is of $H_n^T(\theta)$ type.

Now, it is easy to state exactly the trigonometric precision of (2.13) as follows.

Theorem 5. *If the nodal trigonometric polynomial of* (2.13) *is* $\Pi_n^\alpha(t)$, *then* (2.13) *is of* $H_n^T\left(\left[\alpha+\beta+\frac{1}{2}\pi\right]_\pi\right)$ *type, where* β *is given as in* (3.3).

This theorem shows that (2.13) is of a definite type. The following theorem tells us how to construct the quadrature formula of $H_n^T(\theta)$ type for a given class $H_n^T(\theta)$.

Theorem 6. *There are only two quadrature formulae of* $H_n^T(\theta)$ *type. More precisely, if the nodal trigonometric polynomial of* (2.13) *is* $\Pi_n^\alpha(t)$, *then* (2.13) *is of* $H_n^T(\theta)$ *type iff*

$$\alpha=\left[\frac{1}{2}\theta+\frac{1}{2}\arcsin A\right]_\pi$$

or

$$\alpha=\left[\frac{1}{2}\pi+\frac{1}{2}\theta-\frac{1}{2}\arcsin A\right]_\pi,$$

where

$$A=[\sin\theta(r_n^s-r_n^c)-2\cos\theta r_n]/(r_n^s+r_n^c).$$

Proof. By Lemma 4 and the conjugate condition (3.2), we know that (2.13) is of $H_n^T(\theta)$ type iff α satisfies the following system of equations

$$\begin{cases}\theta=\left[\alpha+\beta+\frac{1}{2}\pi\right]_\pi,\\ \cos\alpha\cos\beta r_n^s+\sin(\alpha+\beta)r_n+\sin\alpha\sin\beta r_n^c=0, \quad 0\leqslant\alpha,\ \beta<\pi.\end{cases} \tag{3.7}$$

Substituting the first equality into the conjugate condition, we get

$$\sin(2\alpha-\theta)=[\sin\theta(r_n^s-r_n^c)-2\cos\theta r_n]/(r_n^s+r_n^c)=A.$$

Noting (3.4) (not using the equality), we have

$$|A|^2 \leqslant [(r_n^s - r_n^c)^2 + (2r_n)^2]/(r_n^s + r_n^c)^2 < [(r_n^s - r_n^c)^2 + 4r_n^s r_n^c]/(r_n^s + r_n^c)^2 = 1.$$

Finally we obtain

$$\begin{cases} \alpha = \left[\frac{1}{2}\theta + \frac{1}{2}\arcsin A\right]_\pi, \\ \beta = \left[\frac{1}{2}\pi + \frac{1}{2}\theta - \frac{1}{2}\arcsin A\right]_\pi, \end{cases} \tag{3.8}$$

or

$$\begin{cases} \alpha = \left[\frac{1}{2}\pi + \frac{1}{2}\theta - \frac{1}{2}\arcsin A\right]_\pi, \\ \beta = \left[\frac{1}{2}\theta + \frac{1}{2}\arcsin A\right]_\pi. \end{cases} \tag{3.9}$$

The two systems of solutions are symmetrical and different. In fact, α and β satisfy the conjugate condition (3.2). Hence $\alpha \neq \beta$, i.e.,

$$\left[\frac{1}{2}\theta + \frac{1}{2}\arcsin A\right]_\pi \neq \left[\frac{1}{2}\pi + \frac{1}{2}\theta - \frac{1}{2}\arcsin A\right]_\pi.$$

The theorem is proved.

We take $w(t) = 1$, for illustration.

In this case, the nodal trigonometric polynomial is $\Pi_n^\alpha(t) = \sin(nt + \alpha)$; the nodes are $t_k = h + 2k\pi/n$, $k = 0, 1, \cdots, n-1$, $h = 2(\pi - \alpha)/n$; the coefficients are $H_k = 2\pi/n$. When $n = 1$ this is obvious, when $n > 1$, the evaluation is as below:

$$\int_0^{2\pi} \sin(nt)\operatorname{ctg}\frac{1}{2}(t-z)\,dt = 2\pi\cos(nz),$$

$$\int_0^{2\pi} \cos(nt)\operatorname{ctg}\frac{1}{2}(t-z)\,dt = -2\pi\sin(nz), \quad z \in [0, 2\pi).$$

Hence, we have, when $n = 2j$,

$$\Delta_n^*(z) = \int_0^{2\pi} \sin(jt + \alpha)\operatorname{ctg}\frac{1}{2}(t-z)\,dt = 2\pi\cos\alpha\cos jz - 2\pi\sin\alpha\sin(jz)$$

$$= 2\pi\cos(jz + \alpha) = 2\pi\cos\left(\frac{1}{2}nz + \alpha\right);$$

and when $n = 2j+1$,

$$\Delta_n^*(z) = \int_0^{2\pi} \sin\left[\left(j + \frac{1}{2}\right)t + \alpha\right]\cos\frac{1}{2}(t-z)\operatorname{ctg}\frac{1}{2}(t-z)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\sin\left(jt + \alpha + \frac{1}{2}z\right) + \sin\left((j+1)t + \alpha - \frac{1}{2}z\right)\right]\operatorname{ctg}\frac{1}{2}(t-z)\,dt$$

$$\doteq 2\pi\cos\left(\frac{1}{2}nz + \alpha\right).$$

Thereby, in any case, we have

$$\Delta_n^*(z) = 2\pi\cos\left(\frac{1}{2}nz + \alpha\right), \quad z \in [0, 2\pi). \tag{3.10}$$

Finally, from (2.9) we have

$$H_k = \Delta_n^*(t_k)/2\Delta_n'(t_k) = 2\pi/n.$$

Thus, we obtain the quadrature formula

$$\int_0^{2\pi} f(t)\,dt \approx \frac{2\pi}{n}\sum_{k=0}^{n-1} f(t_k), \quad t_k = \frac{2k\pi}{n} + \frac{2(\pi-\alpha)}{n}, \tag{3.11}$$

which is just the classical quadrature formula for the periodic functions[12]. Here it follows from (3.3) and Theorem 5 that (3.11) is of $H_n^T([2\alpha]_\pi)$ type.

Particularly, taking $\alpha=0$, we have

$$\int_0^{2\pi} f(t)dt \approx \frac{2\pi}{n}\sum_{k=0}^{n-1} f(2k\pi/n), \tag{3.12}$$

which is of $H_n^T(0)$ type and is quoted by Ioakimidis[8].

§ 4. Quadrature Formulae for Singular Integrals

In this section, we establish the quadrature formulae for singular integrals with Hilbert kernel (1.1).

We assume $f \in C'_{2\pi}$. Let

$$F(t, x) = \begin{cases} [f(t) - f(x)]\operatorname{ctg}\frac{1}{2}(t-x), & t \not\equiv x \pmod{2\pi}, \\ 2f'(x), \ t \equiv x \pmod{2\pi}. \end{cases} \tag{4.1}$$

It is obvious that $F(t, x)$ is a continuous and periodic function of t and x with period 2π. Sometimes we treat x as a parameter and write $F(t, x)$ as $F_x(t)$.

If $f \in H_n^T(\theta)$, we have the following lemma.

Lemma 13. *If $f \in H_n^T(\theta)$, then $F_x(t) \in H_n^T\left(\left[\frac{1}{2}\pi + \theta\right]_\pi\right)$.*

Proof. We should only show the case $f = \cos(nt)$, $\sin(nt)$. Obviously,

$$[\cos(nt) - \cos(nx)]\operatorname{ctg}\frac{1}{2}(t-x) \in H_n^T.$$

Hence

$$\begin{aligned}
&[\cos(nt) - \cos(nx)]\operatorname{ctg}\frac{1}{2}(t-x) \\
&= \cos(n-1)t(\cos t - \cos x)\operatorname{ctg}\frac{1}{2}(t-x) - \sin(n-1)t(\sin t - \sin x)\operatorname{ctg}\frac{1}{2}(t-x) \\
&\quad + [\cos(n-1)t - \cos(n-1)x]\cos(x)\operatorname{ctg}\frac{1}{2}(t-x) \\
&\quad - [\sin(n-1)t - \sin(n-1)x]\sin(x)\operatorname{ctg}\frac{1}{2}(t-x) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

$I_3, I_4 \in H_{n-1}^T$, and

$$I_1 = \frac{1}{2}[\sin(n-2)t - \sin(nt)] - \sin(x)\cos(n-1)t,$$

$$I_2 = \frac{1}{2}[-\sin(n-2)t - \sin(nt)] - \cos(x)\sin(n-1)t,$$

so that

$$[\cos(nt) - \cos(nx)]\operatorname{ctg}\frac{1}{2}(t-x) = -\sin(nt) + T_{n-1}(t), \quad T_{n-1} \in H_{n-1}^T.$$

By analogy, we get

$$[\sin(nt)-\sin(nx)]\operatorname{ctg}\frac{1}{2}(t-x)=\cos(nt)+T^*_{n-1}(t),\quad T^*_{n-1}\in H^T_{n-1}.$$

Note 1. From the above proof, it is seen that if f is a trigonometric polynomial of order n, then $F_x(t)$ is also a trigonometric polynomial of order n.

Note 2. If f is a trigonometric polynomial of order n and not in $H^T_n(\theta)$, then $F_x(t)$ does not belong to $H^T_n\left(\left[\frac{1}{2}\pi+\theta\right]_x\right)$ either. In fact, if $f\in H^T_n(\theta_1)$ $(\theta_1\neq\theta)$, then by Lemma 13, $F_x(t)\in H^T_n\left(\left[\frac{1}{2}\pi+\theta_1\right]_x\right)$. But $H^T_n\left(\left[\frac{\pi}{2}+\theta\right]_x\right)$ and $H^T_n\left(\left[\frac{1}{2}\pi+\theta_1\right]_x\right)$ are two different classes; hence $F_x(t)$ is not in $H^T_n\left(\left[\frac{\pi}{2}+\theta\right]_x\right)$.

By the method of separation of singular Point, $I(f, x)$ can be divided into two parts. This thought is proposed by Professor Chien-ke Lu in the discussion of the approximate evaluation for singular integrals with Cauchy kernel[2]. We proceed as follows:

$$\begin{aligned}I(f,x)&=\int_0^{2\pi}w(t)f(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt\\&=\int_0^{2\pi}w(t)F(t,x)dt+f(x)\int_0^{2\pi}w(t)\operatorname{ctg}\frac{1}{2}(t-x)dt.\end{aligned}$$

In the above, the first integral is a proper integral in which $F(t, x)$ is as in (4.1). Using the quadrature formula (2.13) established before, we obtain the quadrature formula for the singular integral (1.1):

$$I(f,x)\approx Q_n(f,x)=Q_n(F_x)+f(x)\int_0^{2\pi}w(t)\operatorname{ctg}\frac{1}{2}(t-x)dt,\tag{4.2}$$

which is stated in detail in the following theorem.

Theorem 7. *If $f\in C'_{2\pi}$, then*

$$I(f,x)=Q_n(f,x)+R_n(f,x),\tag{4.3}$$

$$Q_n(f,x)=\begin{cases}\sum\limits_{k=1}^{n}H_k\operatorname{ctg}\frac{1}{2}(t_k-x)f(t_k)+\frac{\Delta^*_n(x)}{\Delta_n(x)}f(x), & x\neq t_k,\quad k=1,2,\cdots,n,\\ \sum\limits_{k=1}^{n}{}'H_k\operatorname{ctg}\frac{1}{2}(t_k-t_s)f(t_k)+K_sf(t_s)+2H_sf'(t_s), & x=t_s,\ 1\leqslant s\leqslant n,\end{cases}$$

$$R_n(f,x)=\int_0^{2\pi}w(t)r_n(t_1,t_2,\cdots,t_n,x,f;t)\operatorname{ctg}\frac{1}{2}(t-x)dt,\tag{4.4}$$

where $\Delta_n(t)=\Pi_n\left(\frac{1}{2}t\right)$, $\Pi_n(t)$ is the orthogonal trigonometric polynomial of order n with respect to the weight $w(2t)$, the nodes t_k are the zeros of $\Delta_n(t)$ in $[0, 2\pi)$, the coefficients $H_k=\Delta^*_n(t_k)/2\Delta'_n(t_k)$,

$$\Delta^*_n(z)=\int_0^{2\pi}w(t)\Delta_n(t)\cos^{n-2}\frac{1}{2}(t-z)\operatorname{ctg}\frac{1}{2}(t-z)\,dt$$

is the associated function of $\Delta_n(t)$ and

$$K_s=(\Delta^*_n)'(t_s)/\Delta'_n(t_s)-H_s\Delta''_n(t_s)/\Delta'_n(t_s),$$

with $(\Delta^*_n)'(t_s)$ given by (2.16).

Proof. From the quadrature formula (2.13) we have

$$Q_n(f,x)=Q_n(F_x)+f(x)\int_0^{2\pi}w(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt$$
$$=\sum_{k=1}^{n}H_kF_x(t_k)+f(x)\int_0^{2\pi}w(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt.$$

Using Lemma 13 and noting the trigonometric precision of (2.13), we get, when $n>1$,

$$\begin{aligned}\Delta_n^*(x)&=\int_0^{2\pi}w(t)\,\Delta_n(t)\cos^{n-2}\frac{1}{2}(t-x)\operatorname{ctg}\frac{1}{2}(t-x)\,dt\\&=\int_0^{2\pi}w(t)\Big[\Delta_n(t)\cos^{n-2}\frac{1}{2}(t-x)-\Delta_n(x)\Big]\operatorname{ctg}\frac{1}{2}(t-x)\,dt\\&\quad+\Delta_n(x)\int_0^{2\pi}w(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt\\&=-\sum_{k=1}^{n}H_k\Delta_n(x)\operatorname{ctg}\frac{1}{2}(t_k-x)+\Delta_n(x)\int_0^{2\pi}w(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt;\end{aligned}$$

and, when $n=1$,

$$\begin{aligned}\Delta_n^*(x)&=\int_0^{2\pi}w(t)\sin\frac{1}{2}(t-t_1)\operatorname{cosec}\frac{1}{2}(t-x)\,dt\\&=\int_0^{2\pi}w(t)\cos\frac{1}{2}(x-t_1)\,dt+\int_0^{2\pi}w(t)\sin\frac{1}{2}(x-t_1)\operatorname{ctg}\frac{1}{2}(t-x)\,dt\\&=-H_1\Delta_n(x)\operatorname{ctg}\frac{1}{2}(t_1-x)+\Delta_n(x)\int_0^{2\pi}w(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt.\end{aligned}\tag{4.5}$$

Thus, furthermore, we obtain

$$Q_n(f,x)=\sum_{k=1}^{n}H_k\operatorname{ctg}\frac{1}{2}(t_k-x)f(t_k)+[\Delta_n^*(x)/\Delta_n(x)]f(x).$$

If $x=t_s$, we evaluate

$$\lim_{x\to t_s}\Big[\sum_{k=1}^{n}H_k\operatorname{ctg}\frac{1}{2}(t_k-x)\,f(t_k)+\frac{\Delta_n^*(x)}{\Delta_n(x)}f(x)\Big]$$
$$=\sum_{k=1}^{n}{}'\,H_k\operatorname{ctg}\frac{1}{2}(t_k-x)f(t_k)+K_sf(t_s)+2H_sf'(t_s),$$

where

$$\begin{aligned}K_s&=\lim_{x\to t_s}\Big[H_s\operatorname{ctg}\frac{1}{2}(t_s-x)-\Delta_n^*(x)/\Delta_n(x)\Big]\\&=\lim_{x\to t_s}\Big[H_s\cos\frac{1}{2}(t_s-x)-\Delta_n^*(x)/\Delta_n^s(x)\Big]\Big/\sin\frac{1}{2}(t_s-x)=2[\Delta_n^*(x)/\Delta_n^s(x)]'_{x=t_s}\\&=(\Delta_n^*)'(t_s)/\Delta_n'(t_s)-H_s\Delta_n''(t_s)/\Delta_n(t_s),\end{aligned}$$

in which $\Delta_n^s(x)=\Delta_n(x)\Big/\sin\frac{1}{2}(x-t_s)$.

Noting (4.2), we know

$$R_n(f,x)=I(f,x)-Q_n(f,x)=R_n(F_x).\tag{4.6}$$

From the above equality and Lemma 13, taking the trigonometric precision of (2.13) into account, we know that the quadrature formula (4.3) is exact for any trigonometric polynomial in H_{n-1}^T. Therefore

$$R_n(f, x)=\int_0^{2\pi} w(t)f(t)\operatorname{ctg}\frac{1}{2}(t-x)dt-Q_n(f, x)=\int_0^{2\pi} w(t)f(t)\operatorname{ctg}\frac{1}{2}(t-x)dt$$

$$-\int_0^{2\pi} w(t)T_{n-1}(t_1, t_2, \cdots, t_n, x, f; t)\operatorname{ctg}\frac{1}{2}(t-x)dt$$

$$=\int_0^{2\pi} w(t)r_{n-1}(t_1, t_2, \cdots, t_n, x, f; t)\operatorname{ctg}\frac{1}{2}(t-x)dt.$$

Note 1. When $x=t_s$,

$$T_{n-1}(t_1, t_2, \cdots, t_n, x, f; t_k)=f(t_k), \quad k=1, 2, \cdots, n,$$

$$T'_{n-1}(t_1, t_2, \cdots, t_n, x, f; t_s)=f'(t_s).$$

Note 2. When $n=1$, in fact we have not given the remainder, since in general we cannot find the trigonometric polynomial of order 0 such that its values are the same as those of $f(t)$ at $\{t_1, x\}$. This shortcoming indicates just that the trigonometric precision of (4.3) must be stated more exactly.

Imitating the definition of the quadrature formula for proper integrals, we say the quadrature formula (4.2) has the trigonometric precision of order m if it is exact for any trigonometric polynomial in H_m^T and not so for a certain trigonometric polynomial of order $m+1$.

We have shown that (4.3) is exact for H_{n-1}^T. Hence it possesses trigonometric precision of order $n-1$ at least. Moreover, we may further prove that the trigonometric precision of (4.3) is just of order $n-1$. Note the following theorem.

Theorem 8. *If the nodal trigonometric polynomial of* (4.3) *is* $\Pi_n^\alpha(t)$ *whose conjugate orthogonal trigonometric polynomial is* $\Pi_n^\beta(t)$, *then it is exact for* $H_n^T([\alpha+\beta]_\pi)$ *and not so for trigonometric polynomials of order n not belonging to it.*

As before, we say that the quadrature formula (4.3) is of $H_n^T([\alpha+\beta]_\pi)$ type.

Proof. Just noting Lemma 13 and the note attached to it, by (4.6) we know that Theorem 8 is true.

Note. Now, $r_{n-1}(t_1, t_2, \cdots, t_n, x, f; t)$ in (4.4) may be replaced by

$$r_n^\theta(t_1, t_2, \cdots, t_n, x, f; t)=f(t)-T_n^\theta(t_1, t_2, \cdots, t_n, x, f; t),$$

where $T_n^\theta\in H_n^T(\theta)$ $(\theta=[\alpha+\beta]_\pi)$, and its values at $\{x, t_k, k=1, 2, \cdots, n\}$ are the same as those of $f(t)$. Thus the problem suggested in Note 2 of Theorem 7 may be solved reasonably.

From the above discussions, the following theorem may also be obtained.

Theorem 9. *If the nodal trigonometric polynomial of* (4.3) *is* $\Pi_n^\alpha(t)$, *then for a given* $\theta(0\leqslant\theta<\pi)$, (4.3) *is of* $H_n^T(\theta)$ *type iff*

$$\alpha=\left[\frac{1}{2}\pi+\frac{1}{2}\theta+\frac{1}{2}\arccos\frac{\cos\theta(r_n^s-r_n^c)+2\sin\theta r_n}{r_n^s+r_n^c}\right]_\pi$$

or

$$\alpha=\left[\frac{1}{2}\pi+\frac{1}{2}\theta-\frac{1}{2}\arccos\frac{\cos\theta(r_n^s-r_n^c)+2\sin\theta r_n}{r_n^s+r_n^c}\right]_\pi.$$

We point out that if $f\in AP(B_r)$, i.e., f is a 2π-periodic function analytic on the rectangular domain $D_r=\{z, 0\leqslant\operatorname{Re}(z)\leqslant2\pi, -r\leqslant\operatorname{Im}(z)\leqslant r, r>0\}$ with the boundary B_r. Then the remainder (4.4) of the quadrature formula (4.3) may also be given in the form of contour integral, which is convenient for the asymptotic estimation.

Theorem 10. *If $f\in AP(B_r)$, then the remainder of (4.3) is*

$$R_n(f,x)=-\frac{1}{4\pi i}\int_{B_r}\frac{f(z)\Delta_n^*(z)}{\Delta_n(z)}\operatorname{ctg}\frac{1}{2}(z-x)\,dz,\quad n\geq 3.$$

We first prove the following lemma.

Lemma 14. *Suppose $f\in AP(B_r)$ and the $n+1$ points λ_k lie inside B_r. Write*

$$\omega_{n+1}(t)=\prod_{k=1}^{n+1}\sin\frac{1}{2}(t-\lambda_k)$$

and construct

$$T_{n-1}(f,t)=\frac{1}{4\pi i}\int_{B_r}f(z)\left[\omega_{n+1}(z)-\omega_{n+1}(t)\cos^j\frac{1}{2}(z-t)\right]\operatorname{ctg}\frac{1}{2}(z-t)\,\omega_{n+1}^{-1}(z)\,dz,$$

$$j+1=n\ (\mathrm{mod}\ 2),\ 0\leq j\leq n-3.$$

Then, $T_{n-1}(f,t)\in H_{n-1}^T$, its values at $\{\lambda_k\}_1^{n+1}$ are the same as those of f, and the remainder is

$$r_{n-1}(f,t)=f(t)-T_{n-1}(f,t)$$

$$=(4\pi i)^{-1}\int_{B_r}f(z)\,\omega_{n+1}(t)\cos^j\frac{1}{2}(z-t)\,\omega_{n+1}^{-1}(z)\operatorname{ctg}(z-t)\,dz.$$

Proof. By Lemma 13 we get

$$\left[\omega_{n+1}(z)-\omega_{n+1}(t)\cos^j\frac{1}{2}(z-t)\right]\operatorname{ctg}\frac{1}{2}(z-t)\in H_{n-1}^T,\quad \text{for } t.$$

Hence, $T_{n-1}(f,t)\in H_{n-1}^T$. As t is inside B_r, we have

$$f(t)=(4\pi i)^{-1}\int_{B_r}f(z)\operatorname{ctg}\frac{1}{2}(z-t)\,dz.$$

Thereby

$$r_{n-1}(f,t)=(4\pi i)^{-1}\int_{B_r}f(z)\,\omega_{n+1}(t)\cos^j\frac{1}{2}(z-t)\operatorname{ctg}\frac{1}{2}(z-t)\,\omega_{n+1}^{-1}(z)\,dz$$

$$=(4\pi i)^{-1}\omega_{n+1}(t)\int_{B_r}f(z)\cos^j\frac{1}{2}(z-t)\operatorname{ctg}\frac{1}{2}(z-t)\,\omega_{n+1}^{-1}(z)\,dz.$$

Obviously, it follows that

$$r_{n-1}(f,\lambda_k)=0.$$

Note. In the above lemma, some of the λ_k's are allowed to be the same. For example, if λ_s appears m times, then T_{n-1} possesses the same derivatives as f up to order m at λ_s.

The proof of Theorem 10:

Taking $\omega_{n+1}(t)=\Delta_n(t)\sin\frac{1}{2}(t-x)$ and constructing $T_{n-1}(f,t)$ as in Lemma 14, we get

$$r_{n-1}(f,t)=(4\pi i)^{-1}\int_{B_r}\frac{f(z)\Delta_n(t)\sin\frac{1}{2}(t-x)\cos^j\frac{1}{2}(z-t)}{\Delta_n(z)\sin\frac{1}{2}(z-x)}\operatorname{ctg}\frac{1}{2}(z-t)\,dz.$$

Then using (4.4) and noting the property of associated function (2.15) we obtain

$$R_n(f, x)=\int_0^{2\pi} w(t)\frac{1}{4\pi i}\left[\int_{B_r}\frac{f(z)\Delta_n(t)\sin\frac{1}{2}(t-x)\cos^j\frac{1}{2}(z-t)}{\Delta_n(z)\sin\frac{1}{2}(z-x)}\operatorname{ctg}\frac{1}{2}(z-x)dz\right]dt$$

$$=\frac{1}{4\pi i}\int_{B_r}\frac{f(z)}{\Delta_n(z)\sin\frac{1}{2}(z-x)}\left[\int_0^{2\pi} w(t)\Delta_n(t)\sin\frac{1}{2}(t-x)\cos^j\frac{1}{2}(z-t)\right.$$

$$\left.\times\operatorname{ctg}\frac{1}{2}(z-t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt\right]dz$$

$$=-(4\pi i)^{-1}\int_{B_r} f(z)\Delta_n^*(z)\operatorname{ctg}\frac{1}{2}(z-x)\Delta_n^{-1}(z)dz.$$

Note. In Theorem 10 the requirement $n\geqslant 3$ is taken just for simplicity and is not necessary.

The following example is given for illustrated by taking $w(t)=1$.

Let $\Delta_n(t)=\sin\frac{1}{2}nt$. From (3.10) and (4.5), we get

$$\int_0^{2\pi} f(t)\operatorname{ctg}\frac{1}{2}(t-x)dt=Q_n(f, x)+R_n(f, x),$$

$$Q_n(f, x)=\begin{cases}\frac{2\pi}{n}\sum\limits_{k=0}^{n-1} f(t_k)\operatorname{ctg}\frac{1}{2}(t_k-x)+2\pi\operatorname{ctg}\frac{1}{2}(nx)f(x), & x\neq t_k,\ k=0, 1, \cdots, n-1,\\ \frac{2\pi}{n}\sum\limits_{k=1}^{n-1}{}' f(t_k)\operatorname{ctg}\frac{1}{2}(t_k-t_s)+\frac{4\pi}{n}f'(t_s), & x=t_s,\ 0\leqslant s\leqslant n-1,\end{cases}$$

where $t_k=2k\pi/n$.

If $f\in AP(B_r)$,

$$R_n(f, x)=-(4\pi i)^{-1}\int_{B_r} f(z)\Delta_n^*(z)\left(\sin\frac{1}{2}nz\right)^{-1}\operatorname{ctg}\frac{1}{2}(z-x)dz.$$

The result is just the same as that obtained by Chawla, Ramakrishnan and Ioakimidis[7,8]. It should be noted that the first two authors obtained this quadrature formula only for the case that n is even and Ioakimidis did not give the remainder.

§ 5. The Convergence of Quadrature Formulae

Theorem 11. *If $f\in C_{2\pi}$, then the quadrature formula (2.13) is convergent, i.e.,*

$$\lim_{n\to\infty} R_n(f)=0.$$

More precisely, we have

$$|R_n(f)|\leqslant 24\omega\left(f, \frac{1}{n-1}\right)\int_0^{2\pi} w(t)dt,$$

where $\omega(f, h)$ is the modulus of continuity of f.

Proof. Denote the best approximate trigonometric polynomial of order not greater than $n-1$ by T_{n-1}, then, from Jackson's theorem[13] it follows that

$$\max_{0<t<2\pi}|f(t)-T_{n-1}(t)|\leqslant 12\omega\left(f,\ \frac{1}{n-1}\right).$$

From the trigonometric precision of the quadrature formula (2.13) and noting that the coefficients are positive (see Corollary 5 of Theorem 3), we get

$$\begin{aligned}R_n(f)&=R_n(f-T_{n-1})\\&=\int_0^{2\pi}w(t)[f(t)-T_{n-1}(t)]dt-\sum_{k=1}^{n}H_k[f(t_k)-T_{n-1}(t_k)]\\&\leqslant 12\omega\left(f,\ \frac{1}{n-1}\right)\left[\int_0^{2\pi}w(t)dt+\sum_{k=1}^{n}H_k\right]\\&=24\omega\left(f,\ \frac{1}{n-1}\right)\int_0^{2\pi}w(t)dt.\end{aligned}$$

Theorem 12. *If $f\in C'_{2\pi}$, then the quadrature formula (4.3) is uniformly convergent, i.e.,*

$$\lim_{n\to\infty}R_n(f,\ x)=0$$

is uniformly true for $x\in[0,\ 2\pi)$. More precisely, we have

$$\|R_n(f,\ x)\|_\infty\leqslant 24\omega\left(F,\ \frac{1}{n-1}\right)\int_0^{2\pi}w(t)dt,$$

where $\omega(F,\ h)$ is the modulus of the function of two variables

$$F(t,\ x)=[f(t)-f(x)]\operatorname{ctg}\frac{1}{2}(t-x),\quad 0\leqslant t,\ x\leqslant 2\pi,$$

and $\|f\|_\infty$ denotes the Chebyshev norm of the function f:

$$\|f\|_\infty=\max_{0\leqslant t\leqslant 2\pi}|f(t)|.$$

Proof. Writing the partial modulus of continuity of $F(t,\ x)$ for t as

$$\omega(F_x,\ h)=\sup_{|t_1-t_2|<h}|F(t_1,\ x)-F(t_2,\ x)|,$$

from Theorem 11, (4.1) and (4.6), we have

$$|R_n(f,\ x)|\leqslant 24\omega\left(F_x,\ \frac{1}{n-1}\right)\int_0^{2\pi}w(t)\,dt\leqslant 24\omega\left(F,\ \frac{1}{n-1}\right)\int_0^{2\pi}w(t)dt.$$

Hence

$$\|R_n(f,\ x)\|_\infty\leqslant 24\omega\left(F,\ \frac{1}{n-1}\right)\int_0^{2\pi}w(t)dt.$$

Note

$$\lim_{n\to\infty}\omega\left(F,\ \frac{1}{n-1}\right)=0,$$

therefore,

$$\lim_{n\to\infty}\|R_n(f,\ x)\|_\infty=0.$$

The application of the quadrature formulae established in this paper to the method of numerical solutions of singular integral equations with Hilbert kernel will be given in another paper.

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