THE DISCRETE-TIME FINITE ELEMENT METHODS FOR NONLINEAR HYPERBOLIC EQUATIONS AND THEIR THEORETICAL ANALYSIS*

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Abstract

In this paper, some discrete-time finite element methods for nonlinear hyperbolic equations are derived and their theoretical analysis is given. The stability and the convergence of the finite element method for linear and semi-linear hyperbolic equations have already been discussed[1-3].

Consider the initial boundary problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - [\nabla_x, \mathscr{A}(x, u)\nabla_x u] = B(x, u)\nabla_x u + f(x, u), & x \in \Omega,\ t \in [0, T], \\ \dfrac{\partial u}{\partial t}(x, 0) = 0, & x \in \Omega, \\ u(x, 0) = 0, & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega,\ t \in [0, T], \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n with a smooth boundary,

$$u(x, t) = (u_1(x, t), u_2(x, t) \cdots u_L(x, t))^T, \ [\nabla_x, \mathscr{A}(x, u)\nabla_x u]$$

denotes an L-tuple whose l-th component is $\sum_{k=1}^{L}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{l,j,k,i}(x, u)\frac{\partial u_k}{\partial x_i}\right)$, $B(x, u)\nabla_x u$ also denotes an L-tuple whose l-th component is $\sum_{k=1}^{L}\sum_{i=1}^{n} b_{l,k,i}(x, u)\frac{\partial u_k}{\partial x_i}$, and $f(x, u) = (f_1(x, u), f_2(x, u), \cdots, f_L(x, u))^T$.

We now make several assumptions which will be referred to as condition (A).

Condition (A) (i) For $(x, p) \in \Omega \times R^L$, the matrix $\mathscr{A}(x, p) = (a_{l,j,k,i}(x, p))$ is symmetric and uniformly positive definite. Any $a_{l,j,k,i}(x, p)$, $\frac{\partial a_{l,j,k,i}(x, p)}{\partial p}$, $\frac{\partial^2 a_{l,j,k,i}(x, p)}{\partial p^2}$, $\frac{\partial^3 a_{l,j,k,i}(x, p)}{\partial p^3}$ are locally bounded.

(ii) For the matrix $B(x, p) = (b_{l,k,i}(x, p))$, any $b_{l,k,i}(x, p) \in C^1$ for $(x, p) \in \Omega \times R^L$ and $\frac{\partial b_{l,k,i}(x, p)}{\partial p}$ are locally bounded.

(iii) For any component $f_l(x, p)$, $\frac{\partial f_l(x, p)}{\partial p}$ is locally bounded.

(iv) $u \in L_\infty(0, T; H^{k+1} \cap W^{1,\infty})$ is a unqiue solution to Problem (1).

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The matrix $\mathscr{A}=(a_{l,j,k,i})$ is uniformly positive definite in the sense that there exists a $c_0>0$ such that for all $\xi=(\xi_{lj})$

$$c_0\|\xi\|^2\leqslant\sum a_{l,j,k,i}\xi_{lj}\xi_{ki}, \tag{2}$$

where we have used the norm $\|\xi\|=[[\xi,\ \xi]]^{\frac{1}{2}}$.

Let $S_0=\{(l,\ j);\ 1\leqslant l\leqslant L,\ 1\leqslant j\leqslant n\}$. We shall use two "inner products" on the vector. Suppose $S=\{(l,\ j);\ 1\leqslant l\leqslant L,\ j\in S_l\}$; then for $U=(U_{lj})$ and $W=(W_{lj})$, define $[U,\ W]$ to be an L-tuple given by

$$[U,\ W]=([U,\ W]_1,\ [U,\ W]_2\cdots[U,\ W]_L)^T,$$

$$[U,\ W]_l=\sum_{j\in S_l}U_{lj}W_{lj},$$

$$[[U,\ W]]=\sum_{S}U_{lj}W_{lj}.$$

The variational problem corresponding to Problem (1) is

$$\begin{cases}\left\langle\dfrac{\partial^2 u}{\partial t^2},\ v\right\rangle+a(u;\ u,\ v)=\langle B(x,\ u)\nabla_x u,\ v\rangle+\langle f(x,\ u),\ v\rangle,\\ \left\langle\dfrac{\partial u}{\partial t}(0),v\right\rangle=0,\quad t\in[0,\ T],\ \forall v\in H_0^1,\\ \langle u(0),\ v\rangle=0,\qquad \forall v\in H_0^1,\\ u\in H_0^1(\Omega),\end{cases} \tag{3}$$

where $\langle w,\ v\rangle=(\langle w,\ v\rangle_1,\ \langle w,\ v\rangle_2\cdots\langle w,\ v\rangle_L)^T$, $\langle w,\ v\rangle_l=\int_\Omega w_l(x)v_l(x)dx$, $a(w;\ u,\ v)=\int_\Omega[\mathscr{A}(x,\ w(x))\nabla_x u(x),\ \nabla_x v]\,dx$, and the notation $u\in H_0^1(\Omega)$ means that each component is in $H_0^1(\Omega)$.

We shall approximate the solution to (3) by requiring that the approximate function space and the test function space lie in $\mu=\mu_1\times\mu_2\times\cdots\times\mu_L$, where μ_l is a finite-dimensional subspace of H_0^1.

Let $\mu_l=S_{k,j}^{h,0}(\Omega)\subset H^j(\Omega)\cap H_0^1(\Omega)$ which satisfies[2]:

(i) $\mathscr{P}_k(\Omega)\subset S_{k,j}^{h,0}(\Omega)$,

(ii) let $h\in(0,\ 1)$ and for any $U\in H^r(\Omega)\cap H_0^1(\Omega)$, $r\geqslant 0$, and $0\leqslant s\leqslant\min(r,\ j)$ there exists a constant c_1 independent of h and U such that

$$\inf_{\chi\in S_{k,j}^{h,0}}\|U-\chi\|_{H^s(\Omega)}\leqslant c_1h^\sigma\|U\|_{H^r(\Omega)},$$

where $\sigma=\min(k+1-s,\ r-s)$,

(iii) the inverse property:

$$\|\chi\|_{L_\infty(\Omega)}\leqslant K_0h^{-\frac{n}{2}}\|\chi\|_{L_2(\Omega)},\quad \forall\chi\in S_{k,j}^{h,0}(\Omega).$$

We employ a fully discrete approximation to (3) using finite elements in x and differences in t such that the following hold:

(i) The interval $[0,\ T]$ is partitioned into N equal subintervals

$$0=t_0<t_1<\cdots<t_N=T,\quad t_{j+1}-t_j=\Delta t.$$

(ii)

$$U^{j+\frac{1}{2}}=\frac{1}{2}(U^{j+1}+U^j),\quad U^{j-\frac{1}{2}}=\frac{1}{2}(U^j-U^{j-1}),$$

$$U^{j,\frac{1}{4}}=\frac{1}{2}(U^{j+1}+2U^j+U^{j-1})=\frac{1}{2}(U^{j+\frac{1}{2}}+U^{j-\frac{1}{2}}),\quad \partial_t U^{j+\frac{1}{2}}=\frac{U^{j+1}-U^j}{\Delta t},$$

$$\partial_t U^{j-\frac{1}{2}}=\frac{U^j-U^{j-1}}{\Delta t},\quad \partial_t U^j=\frac{U^{j+1}-U^{j-1}}{2\Delta t}=\frac{1}{\Delta t}(U^{j+\frac{1}{2}}-U^{j-\frac{1}{2}})=\frac{1}{2}(\partial_t U^{j+\frac{1}{2}}+\partial_t U^{j-\frac{1}{2}}),$$

$$\partial_{t^2} U^j=\frac{U^{j+1}-2U^j+U^{j-1}}{\Delta t^2}=\frac{1}{\Delta t}(\partial_t U^{j+\frac{1}{2}}-\partial_t U^{j-\frac{1}{2}}).$$

We propose two finite element schemes:

Scheme 1. Let U^{j-1}, U^j be given. Find $U^{j+1}\in\mu$ which satisfies:

$$\langle\partial_{t^2}U^j,\ v\rangle+a(U^j;\ U^{j,\frac{1}{4}},\ v)=\langle B(U^j)\nabla_x U^{j,\frac{1}{4}},\ v\rangle+\langle f(U^j),\ v\rangle,\quad \forall v\in\mu. \tag{4}$$

Scheme 2. Find $U^{j+1}\in\mu$ which satisfies:

$$\langle\partial_{t^2}U^j,\ v\rangle+a\left(U^j;\ \frac{U^{j+1}+U^{j-1}}{2},\ v\right)$$
$$=\left\langle B(U^j)\nabla_x\left(\frac{U^{j+1}+U^{j-1}}{2}\right),\ v\right\rangle+\langle f(U^j), v\rangle,\ \forall v\in\mu. \tag{5}$$

The initial approximation U^0, U^1 can be determined as follows:

$$\begin{cases} a(u^0;\ U^0,\ v)=a(u^0;\ u^0,\ v), & \forall v\in\mu,\\ a(u^*;\ U^1,\ v)=a(u^*;\ u^*,\ v), & \forall v\in\mu,\end{cases} \tag{6}$$

where

$$u^0=\theta,\quad u^*=u^0+\Delta t\frac{\partial u}{\partial t}(x,0)+\frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}(x,\ 0)=\frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}(x,\ 0)\quad\text{and}$$

$\frac{\partial^2 u}{\partial t^2}$ $(x,\ 0)$ can be found from equation (1).

§ 1. The Theoretical Analysis of Scheme 1

To determine the error estimates for discrete-time approximation we define $\tilde{u}(x,\ y)\in\mu$ by

$$a(u(x,\ t);\ (u-\tilde{u})(x,\ t), v)=0,\quad \forall v\in\mu. \tag{7}$$

Let $U-u=U-\tilde{u}+\tilde{u}-u=\xi+\eta$ where $\xi=U-\tilde{u}$, $\eta=\tilde{u}-u$.

For function $u(x,\ t)=(u_1(x,\ t), u_2(x,\ t)\cdots u_L(x,\ t))^T$ defind on $\Omega\times[0,\ T]$, let

$$\|u\|^2_{L_2(\Omega)}(t)=\|u\|^2_{L_2}=\int_\Omega\sum_{l=1}^{L}|u_l(x,\ t)|^2dx,$$

$$\|u\|^2_{H_0^1(\Omega)}(t)=\|u\|^2_{H_0^1}=\int_\Omega\sum_{s_0}\left|\frac{\partial u_l}{\partial x_j}\right|^2dx,$$

$$\|u\|^2_{L_2(0,T;L_2(\Omega))}=\|u\|^2_{L_2\times L_2}=\int_0^T\int_\Omega|u(x,\ t)|^2dx\,dt,$$

$$\|u\|^2_{L_2(0,T;H_0^1(\Omega))}=\|u\|^2_{L_2\times H_0^1}=\int_0^T\int_\Omega\sum_{s_0}\left|\frac{\partial u_l}{\partial x_j}(x,\ t)\right|^2dx\,dt,$$

$$\|\nabla_x u\|^2_{L_\infty(0,T;L_\infty(\Omega))}=\|\nabla_x u\|^2_{L_\infty\times L_\infty}=\sup_{\Omega\times[0,T]}\sum_{s_0}\left|\frac{\partial u_l}{\partial x_j}(x,\ t)\right|^2.$$

Because the matrix $\mathscr{A}(x, p)$ is positive definite, by variational form (6) and the auxiliary functional definition we obtain $\xi^0=U^0-\tilde{u}^0=\theta$,

$$\|\xi^0\|_{H_0^1}=0,\ \|\xi^1\|_{H_0^1},\ \|\partial_t\xi^{\frac{1}{2}}\|_{L_2}\leqslant c_2(\Delta t)^2. \tag{8}$$

For equation (3) average the differential equations at times t_{j-1}, t_j, t_{j+1} with weights $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, and we obtain

$$\left\langle\left(\frac{\partial^2 u}{\partial t^2}\right)^{j,\frac{1}{4}},\ v\right\rangle+\int_\Omega[\{\mathscr{A}(x,u)\nabla_x u\}^{j,\frac{1}{4}},\ \nabla_x v]dx$$
$$=\langle\{B(u)\nabla_x u\}^{j,\frac{1}{4}},\ v\rangle+\langle f^{j,\frac{1}{4}}(u),\ v\rangle,\quad \forall v\in H_0^1(\Omega). \tag{9}$$

Notice that $(u_{tt})^{j,\frac{1}{4}}=\partial_{t\bar{t}}u^j-r^j$, $\|r^j\|_{L_2}^2\leqslant c(\Delta t)^3\int_{t_{j-1}}^{t_{j+1}}\left\|\frac{\partial^4 u}{\partial\tau^4}\right\|^2 d\tau$ where c is a positive constant. For (9) we have

$$\langle\partial_{t\bar{t}}u^j,\ v\rangle+\int_\Omega[\{\mathscr{A}(x,u)\nabla_x u\}^{j,\frac{1}{4}},\ \nabla_x v]dx=\langle\{B(u)\nabla_x u\}^{j,\frac{1}{4}},\ v\rangle+\langle f^{j,\frac{1}{4}}(u),\ v\rangle$$
$$+\langle r^j,\ v\rangle,\quad \forall v\in H_0^1(\Omega). \tag{10}$$

Subtracting (9) from (4) and taking $v=\partial_t\xi^j$ we have

$$\langle\partial_{t\bar{t}}\xi^j,\ \partial_t\xi^j\rangle+\int_\Omega[\mathscr{A}(U^j)\nabla_x\xi^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]dx+\int_\Omega[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\tilde{u}^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]dx$$
$$+\int_\Omega[\mathscr{A}(u^j)\nabla_x\tilde{u}^{j,\frac{1}{4}}-\{\mathscr{A}(u)\nabla_x\tilde{u}\}^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]dx$$
$$=\langle B(U^j)\nabla_x U^{j,\frac{1}{4}}-\{B(u)\nabla_x u\}^{j,\frac{1}{4}},\ \partial_t\xi^j\rangle+\langle f(U^j)-f^{j,\frac{1}{4}}(u),\ \partial_t\xi^j\rangle$$
$$+\langle r^j,\ \partial_t\xi^j\rangle-\langle\partial_{t\bar{t}}\eta^j,\ \partial_t\xi^j\rangle. \tag{11}$$

Because the auxiliary functions $\tilde{u}(x,t)$, $\nabla_x\tilde{u}$, $\frac{\partial\tilde{u}}{\partial t}$, $\nabla_x\frac{\partial\tilde{u}}{\partial t}$ are bounded[4,5], we set

$$Q=\|\tilde{u}\|_{L_\infty\times L_\infty}+\|\tilde{u}_t\|_{L_\infty\times L_\infty}+\|u\|_{L_\infty\times L_\infty}+\left\|\frac{\partial u}{\partial t}\right\|_{L_\infty\times L_\infty},$$

$$M(Q)=\sup_{\substack{1\leqslant i,j\leqslant n,p\in[-Q,Q]^L\\1\leqslant l,k\leqslant L}}\left\{\|a_{l,j,k,i}(x,p)\|_{L_\infty},\ \left\|\frac{\partial a_{l,j,k,i}(x,p)}{\partial p}\right\|_{L_\infty},\right.$$
$$\left\|\frac{\partial^2 a_{l,j,k,i}(x,p)}{\partial p^2}\right\|_{L_\infty},\ \left\|\frac{\partial^3 a_{l,j,k,i}(x,p)}{\partial p^3}\right\|_{L_\infty},\ \|b_{l,k,i}(x,p)\|_{L_\infty},$$
$$\left.\left\|\frac{\partial b_{l,k,i}(x,p)}{\partial p}\right\|_{L_\infty},\ \|f_l(x,p)\|_{L_\infty},\ \left\|\frac{\partial f_l(x,p)}{\partial p}\right\|_{L_\infty}\right\}.$$

Taking such Δt, h that satisfy $c_2k_0h^{-\frac{n}{2}}(\Delta t)^2\leqslant Q$ we have

$$\|\xi^{\frac{1}{2}}\|_{L_\infty}\leqslant Q,\quad \|\partial_t\xi^{\frac{1}{2}}\|_{L_\infty}\leqslant Q. \tag{12}$$

Let
$$\sup_{j=1,2,\cdots,R-1}\{\|\xi^{j-\frac{1}{2}}\|_{L_\infty},\ \|\partial_t\xi^{j-\frac{1}{2}}\|_{L_\infty}\}\leqslant Q.$$

Then,
$$\sup_{j=1,2,\cdots,R-1}\{\|U^{j-\frac{1}{2}}\|_{L_\infty},\ \|\partial_t U^{j-\frac{1}{2}}\|_{L_\infty}\}\leqslant 2Q.$$

Summing up all L components of equation (11) and noting that

$$\langle\langle \partial_{tt}\xi^j,\ \partial_t\xi^j\rangle\rangle = \frac{1}{2\Delta t}\{\|\partial_t\xi^{j+\frac{1}{2}}\|^2_{L_2} - \|\partial_t\xi^{j-\frac{1}{2}}\|^2_{L_2}\},$$

$$\int_\Omega [[\mathscr{A}(U^j)\nabla_x\xi^{j,\frac{1}{4}}, \nabla_x\partial_t\xi^j]]dx = \frac{1}{2\Delta t}\Big\{\int_\Omega [[\mathscr{A}(U^j)\nabla_x\xi^{j+\frac{1}{2}},\ \nabla_x\xi^{j+\frac{1}{2}}]]dx - \int_\Omega [[\mathscr{A}(U^j)\nabla_x\xi^{j-\frac{1}{2}},\ \nabla_x\xi^{j-\frac{1}{2}}]]dx$$

we obtain from (11)

$$\frac{1}{2\Delta t}\{\|\partial_t\xi^{j+\frac{1}{2}}\|^2_{L_2} - \|\partial_t\xi^{j-\frac{1}{2}}\|^2_{L_2}\} + \frac{1}{2\Delta t}\Big\{\int_\Omega [[\mathscr{A}(U^j)\nabla_x\xi^{j+\frac{1}{2}},\ \nabla_x\xi^{j+\frac{1}{2}}]]$$
$$-\int_\Omega [[\mathscr{A}(U^j)\nabla_x\xi^{j-\frac{1}{2}},\ \nabla_x\xi^{j-\frac{1}{2}}]]dx\Big\}$$
$$= -\int_\Omega [[\{\mathscr{A}(U^j) - \mathscr{A}(u^j)\}\nabla_x\tilde{u}^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]]dx$$
$$-\int_\Omega [[\mathscr{A}(u^j)\nabla_x\tilde{u}^{j,\frac{1}{4}} - \{\mathscr{A}(u)\nabla_x\tilde{u}\}^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]]dx$$
$$+\langle\langle B(U^j)\nabla_x U^{j,\frac{1}{4}} - \{B(u)\nabla_x u\}^{j,\frac{1}{4}},\ \partial_t\xi^j\rangle\rangle + \langle\langle f(U^j) - f^{j,\frac{1}{4}}(u),\ \partial_t\xi^j\rangle\rangle$$
$$+\langle\langle r^j,\ \partial_t\xi^j\rangle\rangle - \langle\langle \partial_{tt}\eta^j,\ \partial_t\xi^j\rangle\rangle. \tag{13}$$

Multiplying (13) by $2\Delta t$ and then summing for $j=1, 2, \cdots, R-1$, we have

$$\|\partial_t\xi^{R-\frac{1}{2}}\|^2_{L_2} - \|\partial_t\xi^{\frac{1}{2}}\|^2_{L_2} + \int_\Omega [[\mathscr{A}(U^{R-1})\nabla_x\xi^{R-\frac{1}{2}},\ \nabla_x\xi^{R-\frac{1}{2}}]]dx$$
$$-\int_\Omega [[\mathscr{A}(U^1)\nabla_x\xi^{\frac{1}{2}},\ \nabla_x\xi^{\frac{1}{2}}]]dx$$
$$= \sum_{j=2}^{R-1}\int_\Omega [[\{\mathscr{A}(U^j) - \mathscr{A}(U^{j-1})\}\nabla_x\xi^{j-\frac{1}{2}},\ \nabla_x\xi^{j-\frac{1}{2}}]]dx$$
$$-2\Delta t\sum_{j=1}^{R-1}\int_\Omega [[\{\mathscr{A}(U^j) - \mathscr{A}(u^j)\}\nabla_x\tilde{u}^{j,\frac{1}{4}}, \nabla_x\partial_t\xi^j]]dx$$
$$-2\Delta t\sum_{j=1}^{R-1}[[\mathscr{A}(u^j)\nabla_x\tilde{u}^{j,\frac{1}{4}} - \{\mathscr{A}(u)\nabla_x\tilde{u}\}^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]]dx$$
$$+2\Delta t\sum_{j=1}^{R-1}\{\langle\langle B(U^j)\nabla_x U^{j,\frac{1}{4}} - \{B(u)\nabla_x u\}^{j,\frac{1}{4}},\ \partial_t\xi^j\rangle\rangle$$
$$+\langle\langle f(U^j) - f^{j,\frac{1}{4}}(u),\ \partial_t\xi^j\rangle\rangle + \langle\langle r^j,\ \partial_t\xi^j\rangle\rangle - \langle\langle \partial_{tt}\eta^j, \partial_t\xi^j\rangle\rangle\}$$
$$= G_1 + G_2 + G_3 + G_4, \tag{14}$$

where

$$G_1 = \sum_{j=2}^{R-1}\int_\Omega [[\{\mathscr{A}(U^j) - \mathscr{A}(U^{j-1})\}\nabla_x\xi^{j-\frac{1}{2}},\ \nabla_x\xi^{j-\frac{1}{2}}]]dx,$$

$$G_2 = -2\Delta t\sum_{j=1}^{R-1}\int_\Omega [[\{\mathscr{A}(U^j) - \mathscr{A}(u^j)\}\nabla_x\tilde{u}^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]]dx,$$

$$G_3 = -2\Delta t\sum_{j=1}^{R-1}[[\mathscr{A}(u^j)\nabla_x\tilde{u}^{j,\frac{1}{4}} - \{\mathscr{A}(u)\nabla_x\tilde{u}\}^{j,\frac{1}{4}},\ \nabla_x\partial_t\xi^j]]dx,$$

$$G_4=2\Delta t\sum_{j=1}^{R-1}\langle\langle B(U^j)\nabla_x U^{j,\frac{1}{4}}-\{B(u)\nabla_x u\}^{j,\frac{1}{4}},\ \partial_t\xi^j\rangle\rangle$$
$$+2\Delta t\sum_{j=1}^{R-1}\{\langle\langle f(U^j)-f^{j,\frac{1}{4}}(u),\ \partial_t\xi^j\rangle\rangle+\langle\langle r^j,\ \partial_t\xi^j\rangle\rangle-\langle\langle\partial_{t^*}\eta^j,\ \partial_t\xi^j\rangle\rangle\}.$$

First we estimate G_1:

$$|G_1|\le c\Delta t\sum_{j=2}^{R-1}\|\xi^{j-\frac{1}{2}}\|^2_{H^1}. \tag{15.1}$$

For G_2 we have

$$G_2=-2\int_\Omega[[\{\mathscr{A}(U^{R-1})-\mathscr{A}(u^{R-1})\}\nabla_x\tilde{u}^{R-1,\frac{1}{4}},\ \nabla_x\xi^{R-\frac{1}{2}}]]dx$$
$$+2\int_\Omega[[\{\mathscr{A}(U^1)-\mathscr{A}(u^1)\}\nabla_x\tilde{u}^{1,\frac{1}{4}},\ \nabla_x\xi^{\frac{1}{2}}]]dx$$
$$+2\sum_{j=2}^{R-1}\int_\Omega[[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\tilde{u}^{j,\frac{1}{4}}-\{\mathscr{A}(U^{j-1})$$
$$-\mathscr{A}(u^{j-1})\}\nabla_x\tilde{u}^{j-1,\frac{1}{4}},\ \nabla_x\xi^{j-\frac{1}{2}}]]dx$$
$$=-2\int_\Omega[[\{\mathscr{A}(U^{R-1})-\mathscr{A}(u^{R-1})\}\nabla_x\tilde{u}^{R-1,\frac{1}{4}},\ \nabla_x\xi^{R-\frac{1}{2}}]]dx$$
$$+2\int_\Omega[[\{\mathscr{A}(U^1)-\mathscr{A}(u^1)\}\nabla_x\tilde{u}^{1,\frac{1}{4}},\ \nabla_x\xi^{\frac{1}{2}}]]dx$$
$$+2\Delta t\sum_{j=2}^{R-1}\int_\Omega\Big[\Big[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\Big(\frac{1}{4}\partial_t\tilde{u}^{j+\frac{1}{2}}+\frac{1}{2}\partial_t\tilde{u}^{j-\frac{1}{2}}+\frac{1}{4}\partial_t\tilde{u}^{j-\frac{3}{2}}\Big),\ \nabla_x\xi^{j-\frac{1}{2}}\Big]\Big]dx$$
$$+2\Delta t\sum_{j=2}^{R-1}\int_\Omega\Big[\Big[\Big\{\frac{\mathscr{A}(U^j)-\mathscr{A}(U^{j-1})}{\Delta t}-\frac{\mathscr{A}(u^j)-\mathscr{A}(u^{j-1})}{\Delta t}\Big\}\nabla_x\tilde{u}^{j-1,\frac{1}{4}},\ \nabla_x\xi^{j-\frac{1}{2}}\Big]\Big]dx.$$

Notice that $\mathscr{A}(U^j)=(a_{i,j,k,l}(U^j))$ and let $a(U^j)=a_{i,j,k,l}(U^j)$,

$$a(U^j)=a(U^{j-\frac{1}{2}})+a'_u(U^{j-\frac{1}{2}})(U^j-U^{j-\frac{1}{2}})+a''_{u^2}(U^{j-\frac{1}{2}})\frac{1}{2!}(U^j-U^{j-\frac{1}{2}})^2$$
$$+a'''_{u^3}(\theta_1U^j+(1-\theta_1)U^{j-\frac{1}{2}})\frac{1}{3!}(U^j-U^{j-\frac{1}{2}})^3,$$

where $0<\theta_1<1$,

$$a'_u(U^{j-\frac{1}{2}})(U^j-U^{j-\frac{1}{2}})=\sum_{i=1}^{L}\frac{\partial a}{\partial u_i}(U^{j-\frac{1}{2}})(U^j-U^{j-\frac{1}{2}})_i,$$
$$a''_{u^2}(U^{j-\frac{1}{2}})(U^j-U^{j-\frac{1}{2}})^2=\sum_{i,l=1}^{L}\frac{\partial^2a}{\partial u_i\partial u_l}(U^{j-\frac{1}{2}})(U^j-U^{j-\frac{1}{2}})_i(U^j-U^{j-\frac{1}{2}})_l,$$
$$a'''_{u^3}(\theta_1U^j+(1-\theta_1)U^{j-\frac{1}{2}})(U^j-U^{j-\frac{1}{2}})^3$$
$$=\sum_{i,l,k=1}^{L}\frac{\partial^3a}{\partial u_i\partial u_l\partial u_k}(\theta_1U^j+(1-\theta_1)U^{j-\frac{1}{2}})(U^j-U^{j-\frac{1}{2}})_i(U^j-U^{j-\frac{1}{2}})_l(U^j-U^{j-\frac{1}{2}})_k.$$

Similarly, we have

$$a(U^{j-1})=a(U^{j-\frac{1}{2}})+a'_u(U^{j-\frac{1}{2}})(U^{j-1}-U^{j-\frac{1}{2}})+a''_{u^2}(U^{j-\frac{1}{2}})\frac{1}{2!}(U^{j-1}-U^{j-\frac{1}{2}})^2$$
$$+a'''_{u^3}(\theta_2U^{j-1}+(1-\theta_2)U^{j-\frac{1}{2}})\frac{1}{3!}(U^{j-1}-U^{j-\frac{1}{2}})^3,$$

where $0<\theta_2<1$.

Hence we obtain

$$\frac{a(U^j)-a(U^{j-1})}{\Delta t}=a'_u(U^{j-\frac{1}{2}})\frac{(U^j-U^{j-1})}{\Delta t}+\frac{a'''_{u^3}(\theta U^j+(1-\theta)U^{j-1})}{24}\frac{(U^j-U^{j-1})^3}{\Delta t},$$

$$\frac{a(u^j)-a(u^{j-1})}{\Delta t}=a'_u(u^{j-\frac{1}{2}})\frac{(u^j-u^{j-1})}{\Delta t}+\frac{a'''_{u^3}(\bar\theta u^j+(1-\bar\theta)u^{j-1})}{24}\frac{(u^j-u^{j-1})^3}{\Delta t},$$

where $0<\theta,\ \bar\theta<1$.

Finally we obtain the estimate of G_2:

$$G_2\leqslant\varepsilon\|\xi^{R-\frac{1}{2}}\|^2_{H^1_0}+c\,\Delta t\sum_{j=2}^{R-1}\{\|\partial_t\xi^{j-\frac{1}{2}}\|^2_{L_2}+\|\xi^{j-\frac{1}{2}}\|^2_{H^1_0}\}+c\{h^{2(k+1)}+(\Delta t)^4\}. \tag{15.2}$$

Similarly, we have

$$\begin{aligned}G_3=&-2\int_\Omega[[\mathscr{A}(u^{R-1})(\nabla_x\tilde u)^{R-1,\frac{1}{4}}-\{\mathscr{A}(u)\nabla_x\tilde u\}^{R-1,\frac{1}{4}},\ \nabla_x\xi^{R-\frac{1}{2}}]]dx\\&+2\int_\Omega[[\mathscr{A}(u^1)(\nabla_x\tilde u)^{1,\frac{1}{4}}-\{\mathscr{A}(u)\nabla_x\tilde u\}^{1,\frac{1}{4}},\ \nabla_x\xi^{\frac{1}{2}}]]dx\\&+2\sum_{j=2}^{R-1}\int_\Omega[[\mathscr{A}(u^j)(\nabla_x\tilde u)^{j,\frac{1}{4}}-\{\mathscr{A}(u)\nabla_x\tilde u\}^{j,\frac{1}{4}}\\&-\{\mathscr{A}(u^{j-1})(\nabla_x\tilde u)^{j-1,\frac{1}{4}}-(\mathscr{A}(u)\nabla_x\tilde u)^{j-1,\frac{1}{4}}\},\ \nabla_x\xi^{j-\frac{1}{2}}]]dx\\&\leqslant\varepsilon\|\xi^{R-\frac{1}{2}}\|^2_{H^1_0}+c\,\Delta t\sum_{j=2}^{R-1}\|\xi^{j-\frac{1}{2}}\|^2_{H^1_0}+c(\Delta t)^4,\end{aligned} \tag{15.3}$$

$$G_4\leqslant\varepsilon\|\xi^{R-\frac{1}{2}}\|^2_{H^1_0}+c\,\Delta t\sum_{j=2}^{R}\{\|\xi^{j-\frac{1}{2}}\|^2_{H^1_0}+\|\partial_t\xi^{j-\frac{1}{2}}\|^2_{L_2}\}+c\{h^{2(k+1)}+(\Delta t)^4\}. \tag{15.4}$$

By (14), (15) we obtain

$$\|\partial_t\xi^{R-\frac{1}{2}}\|^2_{L_2}+c_0\|\xi^{R-\frac{1}{2}}\|^2_{H^1_0}\leqslant\varepsilon\|\xi^{R-\frac{1}{2}}\|^2_{H^1_0}+c_3\Delta t\sum_{j=2}^{R}\{\|\xi^{j-\frac{1}{2}}\|^2_{H^1_0}+\|\partial_t\xi^{j-\frac{1}{2}}\|^2_{L_2}\}+c_3\{h^{2(k+1)}+(\Delta t)^4\}. \tag{16}$$

We take $\varepsilon=\frac{c_0}{4}$, $c_3\Delta t\leqslant\min\left\{\frac{c_0}{4},\frac{1}{2}\right\}$ and obtain

$$\|\partial_t\xi^{R-\frac{1}{2}}\|^2_{L_2}+c_0\|\xi^{R-\frac{1}{2}}\|^2_{H^1_0}\leqslant2c_3\,\Delta t\sum_{j=2}^{R-1}\{\|\xi^{j-\frac{1}{2}}\|^2_{H^1_0}+\|\partial_t\xi^{j-\frac{1}{2}}\|^2_{L_2}\}+2c_3\{h^{2(k+1)}+(\Delta t)^4\}, \tag{17}$$

where c_3 is a positive constant which depends on $M(2Q)$.

Using Gronwall's Lemma and the inverse estimate we obtain

$$\sup_{j=1,2,\cdots,R}\{\|\partial_t\xi^{j-\frac{1}{2}}\|_{L_2},\ \|\xi^{j-\frac{1}{2}}\|_{H^1_0}\}\leqslant c_4\{h^{k+1}+(\Delta t)^2\}, \tag{18.1}$$

$$\sup_{j=1,2,\cdots,R}\{\|\partial_t\xi^{j-\frac{1}{2}}\|_{L_\infty},\ \|\xi^{j-\frac{1}{2}}\|_{L_\infty}\}\leqslant c_5h^{-\frac{n}{2}}\{h^{k+1}+(\Delta t)^2\}, \tag{18.2}$$

where $c_5=K_0c_4$, and Δt and h should satisfy:

$$c_5h^{-\frac{n}{2}}\{h^{k+1}+(\Delta t)^2\}\leqslant Q. \tag{19}$$

We now prove by reduction to absurdity Δt and h satisfy (12) and (19) we have

$$\sup_{j=1,2,\cdots,N-1}\{\|\partial_t\xi^{j-\frac{1}{2}}\|_{L_\infty},\quad \|\xi^{j-\frac{1}{2}}\|_{L_\infty}\}<Q, \tag{20.1}$$

$$\sup_{j=1,2,\cdots,N-1}\{\|\partial_t U^{j-\frac{1}{2}}\|_{L_\infty},\ \|U^{j-\frac{1}{2}}\|_{L_\infty}\}<2Q. \tag{20.2}$$

Suppose there exist $\bar{h}$ and $\Delta\bar{t}$ which satisfy (12) and (19) such that

$$\sup_{j=1,2,\cdots,N-1}\max\{\|\partial_t\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty},\ \|\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty}\}>Q. \tag{21}$$

Let

$$j^*=\min\{j\in[1,\ 2,\ \cdots,\ N-1],\ \max[\|\partial_t\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty},\ \|\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty}]>Q\}, \tag{22}$$

$$\sup_{j=1,2,\cdots,j^*-1}\{\|\partial_t\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty},\ \|\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty}\}<Q,$$

$$\sup_{j=1,2,\cdots,j^*-1}\{\|\partial_t\bar{U}^{j-\frac{1}{2}}\|_{L_\infty},\ \|\bar{U}^{j-\frac{1}{2}}\|_{L_\infty}\}<2Q.$$

By the same estimation for (14)—(17) only and changing R into j^* we have

$$\sup_{j=1,2,\cdots,j^*}\{\|\partial_t\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty},\ \|\bar{\xi}^{j-\frac{1}{2}}\|_{L_\infty}\}<Q,$$

contradictory to (22).

Theorem 1. *Let the solution u be such that $u(x,\ t)$, $\frac{\partial u}{\partial t}\in L_\infty(0,\ T;\ H^{k+1}(\Omega))$, $\frac{\partial^2 u}{\partial t^2}\in L_2(0,\ T;\ H^{k+1}(\Omega))$, $\frac{\partial^4 u}{\partial t^4}\in L_2(0,\ T;\ L_2(\Omega))$ and suppose h and Δt satisfy (12) and (19). Then the following estimate holds*

$$\sup_{j=1,2,\cdots,N}\{\|\partial_t(U-u)^{j-\frac{1}{2}}\|_{L_2}+\|(U-u)^{j-\frac{1}{2}}\|_s\}<c\{h^{k+1-s}+(\Delta t)^2\}, \tag{23}$$

where $s=0,\ 1$.

In (4) we take $v=\partial_t U^j$ and multiply both sides by $2\Delta t$. Then summing for $j=1,\ 2,\ \cdots,\ R$ and using (20) we obtain

$$\|\partial_t U^{R-\frac{1}{2}}\|^2_{L_2}+c_0\|U^{R-\frac{1}{2}}\|^2_{H^1_0}<c\{\|\partial_t U^{\frac{1}{2}}\|^2_{L_2}+\|U^{\frac{1}{2}}\|^2_{H^1_0}+\|f(0)\|^2_{L_2}\}$$
$$+c\Delta t\sum_{j=1}^{R}\{\|\partial_t U^{j-\frac{1}{2}}\|^2_{L_2}+\|U^{j-\frac{1}{2}}\|^2_{H^1_0}\}.$$

Taking Δt which satisfies $c\Delta t<\min\left\{\frac{1}{2},\ \frac{c_0}{2}\right\}$ and using Gronwall's Lemma we obtain

$$\sup_{j=1,2,\cdots,N}\{\|\partial_t U^{j-\frac{1}{2}}\|_{L_2}+\|U^{j-\frac{1}{2}}\|_{H^1_0}\}<c\{\|\partial_t U^{\frac{1}{2}}\|_{L_2}+\|U^{\frac{1}{2}}\|_{H^1_0}+\|f(0)\|^2_{L_2}\}. \tag{24}$$

Theorem 2. *Suppose the conditions of Theorem 1 hold and $c\Delta t<\min\left\{\frac{1}{2},\ \frac{c_0}{2}\right\}$. Then Scheme 1 is stable.*

§2. The Theoretical Analysis of Scheme 2

For equation (3), at time t_j we obtain

$$\left\langle\frac{\partial^2 u^j}{\partial t^2},\ v\right\rangle+a(u^j;\ u^j,\ v)=\langle B(u^j)\nabla_x u^j,\ v\rangle+\langle f(u^j),\ v\rangle,\ \forall v\in\mu. \tag{25}$$

Subtracting (25) from (5) and taking $v=\partial_t\xi^j$, we have

$$\langle\partial_{tt}\xi^j,\partial_t\xi^j\rangle+\int_\Omega\left[\mathscr{A}(U^j)\nabla_x\left(\frac{U^{j+1}+U^{j-1}}{2}\right),\ \nabla_x\partial_t\xi^j\right]dx-\int_\Omega[\mathscr{A}(u^j)\nabla_x u^j,\ \nabla_x\partial_t\xi^j]\,dx$$
$$=\left\langle B(U^j)\nabla_x\left(\frac{U^{j+1}+U^{j-1}}{2}\right),\ \partial_t\xi^j\right\rangle-\langle B(u^j)\nabla_x u^j,\ \partial_t\xi^j\rangle$$
$$+\langle f(U^j)-f(u^j),\ \partial_t\xi^j\rangle+\langle r^j,\ \partial_t\xi^j\rangle-\langle\partial_{tt}\eta^j,\ \partial_t\xi^j\rangle. \tag{26}$$

Notice that

$$\int_\Omega\left[\mathscr{A}(U^j)\nabla_x\left(\frac{U^{j+1}+U^{j-1}}{2}\right),\ \nabla_x\partial_t\xi^j\right]dx-\int_\Omega[\mathscr{A}(u^j)\nabla_x u^j,\ \nabla_x\partial_t\xi^j]\,dx$$
$$=\int_\Omega\left[\mathscr{A}(U^j)\nabla_x\left(\frac{\xi^{j+1}+\xi^{j-1}}{2}\right),\ \frac{1}{2\Delta t}\nabla_x(\xi^{j+1}-\xi^{j-1})\right]dx$$
$$+\int_\Omega\left[\mathscr{A}(U^j)\nabla_x\left(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}\right)-\mathscr{A}(u^j)\nabla_x\tilde{u}^j,\ \nabla_x\partial_t\xi^j\right]dx$$
$$=\frac{1}{4\Delta t}\left\{\int_\Omega[\mathscr{A}(U^j)\nabla_x\xi^{j+1},\ \nabla_x\xi^{j+1}]\,dx-\int_\Omega[\mathscr{A}(U^j)\nabla_x\xi^{j-1},\ \nabla_x\xi^{j-1}]\,dx\right\}$$
$$+\int_\Omega\left[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\left(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}\right)+\mathscr{A}(u^j)\nabla_x\left\{\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}-\tilde{u}^j\right\},\right.$$
$$\left.\frac{1}{2\Delta t}\nabla_x(\xi^{j+1}-\xi^{j-1})\right]dx. \tag{27}$$

Summing all L components of (26) we obtain

$$\frac{1}{2\Delta t}\{\|\partial_t\xi^{j+\frac12}\|^2_{L_2}-\|\partial_t\xi^{j-\frac12}\|^2_{L_2}\}+\frac{1}{4\Delta t}\left\{\int_\Omega[[\mathscr{A}(U^j)\nabla_x\xi^{j+1},\ \nabla_x\xi^{j+1}]]\,dx\right.$$
$$\left.-\int_\Omega[[\mathscr{A}(U^j)\nabla_x\xi^{j-1},\ \nabla_x\xi^{j-1}]]\,dx\right\}$$
$$=-\int_\Omega\left[\left[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\left(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}\right)\right.\right.$$
$$\left.\left.+\mathscr{A}(u^j)\nabla_x\left(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}-\tilde{u}^j\right),\ \nabla_x\partial_t\xi^j\right]\right]dx$$
$$+\left\langle\left\langle B(U^j)\nabla_x\left(\frac{U^{j+1}+U^{j-1}}{2}\right)-B(u^j)\nabla_x u^j,\ \partial_t\xi^j\right\rangle\right\rangle+\langle\langle f(U^j)-f(u^j),\ \partial_t\xi^j\rangle\rangle$$
$$+\langle\langle r^j,\ \partial_t\xi^j\rangle\rangle-\langle\langle\partial_{tt}\eta^j,\ \partial_t\xi^j\rangle\rangle. \tag{28}$$

Multiplying (28) by $2\Delta t$ and then summing for $j=1,\ 2,\ \cdots,\ R-1$ we have

$$\|\partial_t\xi^{R-\frac12}\|^2_{L_2}-\|\partial_t\xi^{\frac12}\|^2_{L_2}+\frac12\int_\Omega[[\mathscr{A}(U^{R-1})\nabla_x\xi^R,\ \nabla_x\xi^R]]\,dx$$
$$-\frac12\int_\Omega[[\mathscr{A}(U^1)\nabla_x\xi^0,\ \nabla_x\xi^0]]\,dx+\frac12\int_\Omega[[\mathscr{A}(U^{R-2})\nabla_x\xi^{R-1},\ \nabla_x\xi^{R-1}]]\,dx$$
$$-\frac12\int_\Omega[[\mathscr{A}(U^2)\nabla_x\xi^1,\nabla_x\xi^1]]\,dx$$
$$=\frac12\sum_{j=2}^{R-1}\int_\Omega[[\{\mathscr{A}(U^j)-\mathscr{A}(U^{j-2})\}\nabla_x\xi^{j-1},\ \nabla_x\xi^{j-1}]]\,dx$$

$$-2\Delta t\sum_{j=1}^{R-1}\int_\Omega\Big[\Big[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\Big(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}\Big)$$
$$+\mathscr{A}(u^j)\nabla_x\Big(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}-\tilde{u}^j\Big),\ \nabla_x\partial_t\xi^j\Big]\Big]dx$$
$$+2\Delta t\sum_{j=1}^{R-1}\Big\{\Big\langle\Big\langle B(U^j)\nabla_x\Big(\frac{U^{j+1}+U^{j-1}}{2}\Big)-B(u^j)\nabla_x u^j,\ \partial_t\xi^j\Big\rangle\Big\rangle$$
$$+\langle\langle f(U^j)-f(u^j),\ \partial_t\xi^j\rangle\rangle+\langle\langle r^j,\ \partial_t\xi^j\rangle\rangle-\langle\langle\partial_{t^*}\eta^j,\ \partial_t\xi^j\rangle\rangle\Big\}$$
$$=G_1+G_2+G_3+G_4, \tag{29}$$

where

$$G_1=\sum_{j=2}^{R-1}\int_\Omega[[\{\mathscr{A}(U^j)-\mathscr{A}(U^{j-2})\}\nabla_x\xi^{j-1},\ \nabla_x\xi^{j-1}]]dx,$$
$$G_2=-2\Delta t\sum_{j=1}^{R-1}\int_\Omega\Big[\Big[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\Big(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}\Big),\ \nabla_x\partial_t\xi^j\Big]\Big]dx,$$
$$G_3=-2\Delta t\sum_{j=1}^{R-1}\int_\Omega\Big[\Big[\mathscr{A}(u^j)\nabla_x\Big(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}-\tilde{u}^j\Big),\ \nabla_x\partial_t\xi^j\Big]\Big]dx,$$
$$G_4=2\Delta t\sum_{j=1}^{R-1}\Big\{\Big\langle\Big\langle B(U^j)\nabla_x\Big(\frac{U^{j+1}+U^{j-1}}{2}\Big)-B(u^j)\nabla_x u^j,\ \partial_t\xi^j\Big\rangle\Big\rangle$$
$$+\langle\langle f(U^j)-f(u^j),\ \partial_t\xi^j\rangle\rangle+\langle\langle r^j,\partial_t\xi^j\rangle\rangle-\langle\langle\partial_{t^*}\eta^j,\ \partial_t\xi^j\rangle\rangle\Big\}.$$

Similarly as in Section 1, noticing that

$$a_{l,j,k,i}(U^j)-a_{l,j,k,i}(U^{j-2})=a'_u(\theta U^j+(1-\theta)U^{j-2})(U^j-U^{j-2}),$$
$$U^j-U^{j-2}=2\Delta t\partial_t U^{j-1}=\Delta t(\partial_t U^{j-\frac{1}{2}}+\partial_t U^{j-\frac{3}{2}})$$

we obtain

$$|G_1|\le c\,\Delta t\sum_{j=2}^{R-1}\|\xi^{j-1}\|^2_{H^1_0}. \tag{30.1}$$

As

$$G_2=-\sum_{j=1}^{R-1}\int_\Omega\Big[\Big[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\Big(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}\Big),\ \nabla_x(\xi^{j+1}-\xi^{j-1})\Big]\Big]dx$$
$$=-\int_\Omega\Big[\Big[\{\mathscr{A}(U^{R-1})-\mathscr{A}(u^{R-1})\}\nabla_x\Big(\frac{\tilde{u}^R+\tilde{u}^{R-2}}{2}\Big),\ \nabla_x\xi^R\Big]\Big]dx$$
$$-\int_\Omega\Big[\Big[\{\mathscr{A}(U^{R-2})-\mathscr{A}(u^{R-2})\}\nabla_x\Big(\frac{\tilde{u}^{R-1}+\tilde{u}^{R-3}}{2}\Big),\ \nabla_x\xi^{R-1}\Big]\Big]dx$$
$$+\int_\Omega\Big[\Big[\{\mathscr{A}(U^2)-\mathscr{A}(u^2)\}\nabla_x\Big(\frac{\tilde{u}^3+\tilde{u}^1}{2}\Big),\ \nabla_x\xi^1\Big]\Big]dx$$
$$+\int_\Omega\Big[\Big[\{\mathscr{A}(U^1)-\mathscr{A}(u^1)\}\nabla_x\Big(\frac{\tilde{u}^2+\tilde{u}^0}{2}\Big),\ \nabla_x\xi^0\Big]\Big]dx$$
$$-\sum_{j=1}^{R-3}\int_\Omega\Big[\Big[\{\mathscr{A}(U^j)-\mathscr{A}(u^j)\}\nabla_x\Big(\frac{\tilde{u}^{j+1}+\tilde{u}^{j-1}}{2}\Big)$$
$$-\{\mathscr{A}(U^{j+1})-\mathscr{A}(u^{j+1})\}\nabla_x\Big(\frac{\tilde{u}^{j+3}+\tilde{u}^{j+1}}{2}\Big),\ \nabla_x\xi^{j+1}\Big]\Big]dx$$

$$= -\int_\Omega \left[\left[\{\mathscr{A}(U^{R-1}) - \mathscr{A}(u^{R-1})\}\nabla_x\left(\frac{\tilde{u}^R + \tilde{u}^{R-2}}{2}\right),\ \nabla_x \xi^R\right]\right]dx$$
$$- \cdots + \int_\Omega [[\{\mathscr{A}(U^1) - \mathscr{A}(u^1)\}\nabla_x\left(\frac{\tilde{u}^2 + \tilde{u}^0}{2}\right),\ \nabla_x \xi^0]]dx$$
$$+ \frac{1}{2}\Delta t \sum_{j=1}^{R-3}\int_\Omega [[\{\mathscr{A}(U^j) - \mathscr{A}(u^j)\}\nabla_x(\partial_t \tilde{u}^{j+\frac{5}{2}} + \partial_t \tilde{u}^{j+\frac{3}{2}} + \partial_t \tilde{u}^{j+\frac{1}{2}} + \partial_t \tilde{u}^{j-\frac{1}{2}}),$$
$$\nabla \xi^{j+1}]]dx + \Delta t \sum_{j=1}^{R-3}\int_\Omega \left[\left[\left\{\frac{\mathscr{A}(U^{j+2}) - \mathscr{A}(U^j)}{\Delta t} - \frac{\mathscr{A}(u^{j+2}) - \mathscr{A}(u^j)}{\Delta t}\right\}\right.\right.$$
$$\left.\left.\times \nabla_x\left(\frac{\tilde{u}^{j+3} + \tilde{u}^{j+1}}{2}\right),\ \nabla_x \xi^{j+1}\right]\right]dx$$

we obtain

$$|G_2| \leqslant \varepsilon\{\|\xi^R\|^2_{H^1_0} + \|\xi^{R-1}\|^2_{H^1_0}\} + c\Delta t \sum_{j=1}^{R-2}\{\|\partial_t \xi^{j-\frac{1}{2}}\|^2_{L_2} + \|\xi^j\|^2_{H^1_0}\} + c\{h^{2(k+1)} + (\Delta t)^4\}. \tag{30.2}$$

Since

$$G_3 = -\int_\Omega \left[\left[\mathscr{A}(u^{R-1})\nabla_x\left(\frac{\tilde{u}^R + \tilde{u}^{R-2}}{2} - \tilde{u}^{R-1}\right),\ \nabla_x \xi^R\right]\right]dx$$
$$-\int_\Omega \left[\left[\mathscr{A}(u^{R-2})\nabla_x\left(\frac{\tilde{u}^{R-1} + \tilde{u}^{R-3}}{2} - \tilde{u}^{R-2}\right),\ \nabla_x \xi^{R-1}\right]\right]dx$$
$$+\int_\Omega \left[\left[\mathscr{A}(u^2)\nabla_x\left(\frac{\tilde{u}^3 + \tilde{u}^1}{2} - \tilde{u}^2\right),\ \nabla_x \xi^1\right]\right]dx$$
$$-\int_\Omega \left[\left[\mathscr{A}(u^1)\nabla_x\left(\frac{\tilde{u}^2 + \tilde{u}^0}{2} - \tilde{u}^1\right),\ \nabla_x \xi^0\right]\right]dx$$
$$-\sum_{j=1}^{R-3}\int_\Omega \left[\left[\mathscr{A}(u^j)\nabla_x\left(\frac{\tilde{u}^{j+1} + \tilde{u}^{j-1}}{2} - \tilde{u}^j\right)\right.\right.$$
$$\left.\left.-\mathscr{A}(u^{j+2})\nabla_x\left(\frac{\tilde{u}^{j+3} + \tilde{u}^{j+1}}{2} - \tilde{u}^{j+2}\right),\ \nabla_x \xi^{j+1}\right]\right]dx$$

we obtain

$$|G_3| \leqslant \varepsilon\{\|\xi^R\|^2_{H^1_0} + \|\xi^{R-1}\|^2_{H^1_0}\} + c\Delta t \sum_{j=1}^{R-2}\|\xi^j\|^2_{H^1_0} + c(\Delta t)^4. \tag{30.3}$$

Similarly we obtain

$$|G_4| \leqslant \varepsilon\{\|\xi^R\|^2_{H^1_0} + \|\xi^{R-1}\|^2_{H^1_0}\} + c\Delta t \sum_{j=1}^{R}\{\|\xi^j\|^2_{H^1_0} + \|\partial_t \xi^{j-\frac{1}{2}}\|^2_{L_2}\} + c\{h^{2(k+1)} + (\Delta t)^4\}. \tag{30.4}$$

By (29) and (30) we have

$$\|\partial_t \xi^{R-\frac{1}{2}}\|^2_{L_2} + c_0\{\|\xi^R\|^2_{H^1_0} + \|\xi^{R-1}\|^2_{H^1_0}\}$$
$$\leqslant \varepsilon\{\|\xi^R\|^2_{H^1_0} + \|\xi^{R-1}\|^2_{H^1_0}\} + c\Delta t \sum_{j=1}^{R}\{\|\xi^j\|^2_{H^1_0} + \|\partial_t \xi^{j-\frac{1}{2}}\|^2_{L_2}\} + c\{h^{2(k+1)} + (\Delta t)^4\}. \tag{31}$$

Taking $\varepsilon = \frac{c_0}{4}$ and $c\Delta t \leqslant \min\left\{\frac{c_0}{4},\ \frac{1}{2}\right\}$ we obtain

$$\|\partial_t \xi^{R-\frac{1}{2}}\|^2_{L_2} + c_0|\xi^R\|^2_{H^1_0} \leqslant 2c\Delta t \sum_{j=1}^{R-1}\{\|\xi^j\|^2_{H^1_0} + \|\partial_t \xi^{j-\frac{1}{2}}\|^2_{L_2}\} + 2c\{h^{2(k+1)} + (\Delta t)^4\}. \tag{32}$$

Theorem 3. *Suppose the conditions of Theorem 1 hold. Then the following estimate holds*

$$\sup_{j=1,2,\cdots,N}\{\|\partial_t(U-u)^{j-\frac{1}{2}}\|_{L_2}+\|(U-u)^j\|_s\}\leqslant c\{h^{k+1-s}+(\Delta t)^2\}, \tag{33}$$

where $s=0,\ 1$.

Theorem 4. *Suppose the conditions of Theorem 3 hold. Then for* Δt *properly small, Scheme 2 is stable and the following estimate holds*

$$\sup_{j=1,2,\cdots,N}\{\|\partial_t U^{j-\frac{1}{2}}\|_{L_2}+\|U^j\|_{H_0^1}\}\leqslant c\{\|\partial_t U^{\frac{1}{2}}\|_{L_2}+\|U^0\|_{H_0^1}+\|U^1\|_{H_0^1}+\|f(0)\|_{L_2}\}. \tag{34}$$

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