NUMERICAL SOLUTION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH SINGULAR PERTURBATION CONSIDERATIONS*1)

HUANG LAN-CHIEH (黄兰洁) SHEN JIAN-XIONG (沈建雄)
(Computing Center, Academia Sinica, Beijing, China)

Abstract

A numerical method for coarse grids is proposed for the numerical solution of the incompressible Navier-Stokes equations. From singular perturbation considerations, we obtain partial differential equations and boundary conditions for the outer solution and the boundary layer correction. The former problem is solved with the finite difference method and the latter with the approximate method. Numerical experiments show that accurate outer flow and boundary flux result with little computational effort.

§1. Introduction

It is well known that partial differential equations exhibit multiple scale phenomena. To resolve all the different scales in numerical computation presents quite a challenge to the computational fluid dynamic community, see Chin, Hedstrom, and Howes [3]. Even for the numerical solution of boundary layer problems, affordable uniform grids are often too coarse to resolve such layers. Usually fine grids are used in the boundary layers for small scale effects represented by large gradients in the solutions, while coarse grids are used in most of the region for large scale phenomena described by smooth solutions, see e.g. MacCormack and Lomax [14]. Or, the viscous and inviscid equations can be coupled and solved iteratively, see e.g. Van Dalsem and Steger [19]. Both approaches need great computational effort. Often for practical purposes detailed resolution of the boundary layers is not necessary; only the boundary fluxes (shearing stress, heat flux, etc.) in terms of normal derivatives at the boundaries are needed. This will allow the use of more efficient numerical schemes.

The present paper is concerned with the numerical solution of the incompressible Navier-Stokes equations in primitive variables with large Reynolds numbers. It presents a method with singular perturbation considerations; concepts of outer solution, boundary layer correction, etc. are taken over, but not the solution procedures which consists of finding the outer and inner solutions of various orders successively. Our method is, first of all, to decompose the solution into two parts, an outer part which is smooth and a boundary layer correction part with large gradients, coupled essentially through suitable boundary conditions at the fixed boundary. Thus different methods can be applied to the two different problems,

* Received January 5, 1987.
1) The Project Supported by the National Natural Science Fundation of China.

giving numerical solution to the complete Navier-Stokes equations if necessary.

In this paper, for smooth flow in the main part of the computational domain, a finite difference methods on a coarse grid is used for its numerical solution. At the boundaries where the solution has large gradients, if the analytic behavior is known, then efficient numerical schemes can be developed, as the exponential schemes for two-point boundary value problems, see e.g. Doolan, Miller and Schilders [5]. For two-dimensional scalar partial differential equations, the analytic behavior of the solution near characteristic boundaries has been investigated by Eckhaus and de Jager [7], Howes [11], etc. But for our nonlinear system of partial differential equations, to the authors' knowledge, no simple analytic function can capture the essential behavior of the solution in the boundary layers. Hence approximate methods are applied directly to a simplified system of equations for the boundary layers. This system is to be formulated in terms of boundary layer correction, since the authors find it mathematically more tractable than that of outer and inner solutions with matched asymptotic expansion. However, the class of approximate methods for Prandtl's boundary layer equations due to von Karman and Pohlhausen, see Schlichting [17, Chap. 10], can be adapted to the numerical solution of the boundary layer correction equations. In this sense, our method is similar to the coupled inviscid integral-boundary-layer algorithm quoted in [19]. But here the complete outer solution can be obtained, not just the first order inviscid solution; and then corrections are added at the boundaries, with no matching because of our formulation. Numerical solution with boundary layer correction is also given in Flaherty and O'Mallay [8] for one dimensional problems, but with semi-analytic considerations and a different solution procedure.

In the following, Section 2 presents the basic idea of our method with a simple linear scalar differential equation. Section 3 focuses on the incompressible Navier-Stokes equations. The outer system of equations and its boundary conditions are discussed, and the boundary layer correction equations and the corresponding boundary conditions are derived. Then in Section 4 the method of solution is presented, and finally in Section 5 test results of our method on incompressible flow past a semi-infinite plate are given. These preliminary numerical experiments prove the feasibility of our present approach.

§ 2. The Basic Idea

We present the basic idea of our numerical method with a simple example from O'Malley [15, Chap. 1]——a linear scalar differential equation with constant coefficients

$$\varepsilon \frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0 \tag{2.1}$$

for $0\leqslant x\leqslant 1$, with boundary conditions

$$y(0)=0, \tag{2.2}$$

$$y(1)=1. \tag{2.3}$$

Its exact solution is

$$y(x)=\frac{e^{\mu_+ x}-e^{\mu_- x}}{e^{\mu_+}-e^{\mu_-}},\quad \mu_\pm=\frac{-1\pm\sqrt{1-4\varepsilon}}{2\varepsilon}. \tag{2.4}$$

We suppress notation for dependence on ε. For $\varepsilon=0.01$, it is shown in Fig. 1. We see that there is a boundary layer at $x=0$ with thickness decreasing with the small parameter ε. Our interest will be centered in the case of small ε and our goal is to obtain the numerical solution to this problem with the boundary flux, but without resolving the entire boundary layer if possible. This will be done with ideas and motivation (not solution procedure) from singular perturbation problems. For clarity, we review some of its terminology briefly.

For $0<x\leqslant 1$, the "outer solution" is considered. It is defined by (2.1) and (2.3). For, let the outer expansion

$$Y(x)\sim a_0(x)+\varepsilon a_1(x)+\varepsilon^2 a_2(x)+\cdots$$

satisfy (2.1) and (2.3) asymptotically; substituting the expansion into them and equating the coefficient of each ε^j yield

$$\begin{aligned}&\frac{da_0}{dx}+a_0=0, && a_0(1)=1,\\ &\frac{da_1}{dx}+a_1=-\frac{d^2a_0}{dx^2}, && a_1(1)=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned} \tag{2.5}$$

(2.5) is called the "reduced problem" and has solution

$$a_0(x)=e^{1-x} \tag{2.6}$$

it is the first order outer solution. Similarly $a_i(x)\,(i=1,\ 2,\ \cdots)$ can also be found, thus the outer expansion is determined, see [15].

In the neighborhood of $x=0$, the "inner solution" is considered. Let the inner expansion

$$\tilde{Y}(\xi)\sim b_0(\xi)+\varepsilon b_1(\xi)+\varepsilon^2 b_2(\xi)+\cdots$$

satisfy (2.1) and (2.2) asymptotically; the "stretched variable" ξ is defined as

$$\xi=\frac{x}{\varepsilon}. \tag{2.7}$$

We obtain as above

$$\begin{aligned}&\frac{d^2b_0}{d\xi^2}+\frac{db_0}{d\xi}=0, && b_0(0)=0,\\ &\frac{d^2b_1}{d\xi^2}+\frac{db_1}{d\xi}=-b_0, && b_1(0)=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

Working with just the first order solutions, for example, we find

$$b_0(\xi)=\beta(1-e^{-\xi}),$$

where constant β is determined by "matching",

$$\lim_{\xi\to\infty} b_0(\xi)=a_0(0),$$

hence $\beta=e$ and

$$b_0(\xi)=b_0\left(\frac{x}{\varepsilon}\right)=e\,(1-e^{-\frac{x}{\varepsilon}}). \tag{2.8}$$

See [15] for further discussion. For $\varepsilon=0.01$, (2.6) and (2.8) are shown in Fig. 2.

It is also possible to obtain a composite solution by adding a modifying term to the combination of outer and inner solutions. However the authors find the boundary layer correction formulation mathematically more tractable, and for our example, it is as follows:

First, extend the outer solution to $x=0$. Let the asymptotic solution of our problem be written as

$$y(x)=Y(x)+\overline{Y}\left(\frac{x}{\varepsilon}\right), \tag{2.9}$$

where $\overline{Y}(\xi)$ is the "boundary layer correction (solution)". Equation (2.1) being linear, $\overline{Y}$ satisfies the same equation, but its boundary conditions are

$$\overline{Y}(0)=-Y(0) \tag{2.10}$$

and by definition of boundary layer correction

$$\lim_{\xi\to\infty}\overline{Y}(\xi)=0. \tag{2.11}$$

For all practical purposes, this will ensure that y satisfy boundary condition (2.3), since for ε small, the boundary layer is thin. Let $\overline{Y}$ be expanded as

$$\overline{Y}(\xi)\sim c_0(\xi)+\varepsilon c_1(\xi)+\varepsilon^2 c_2(\xi)+\cdots$$

we have

$$\frac{d^2c_0}{d\xi^2}+\frac{dc_0}{d\xi}=0, \quad c_0(0)=-a_0(0), \quad \lim_{\xi\to\infty}c_0(\xi)=0,$$

$$\frac{d^2c_1}{d\xi^2}+\frac{dc_1}{d\xi}=-c_0, \quad c_1(0)=-a_1(0), \quad \lim_{\xi\to\infty}c_1(\xi)=0.$$

Then

$$c_0(\xi)=c_0\left(\frac{x}{\varepsilon}\right)=-a_0(0)e^{-\frac{x}{\varepsilon}} \tag{2.12}$$

and similarly $c_i(\xi)$ ($i=1, 2, \cdots$) can be found. Hence the boundary layer correction (solution) is defined by (2.1) for $\overline{Y}$, (2.10), and (2.11). For $\varepsilon=0.01$, the first order asymptotic solution is shown in Fig. 2. For later use we note that also

$$\lim_{\xi\to\infty}\frac{d\overline{Y}(\xi)}{d\xi}=0. \tag{2.13}$$

Now we discuss the numerical solution of (2.1), (2.2) and (2.3). We divide

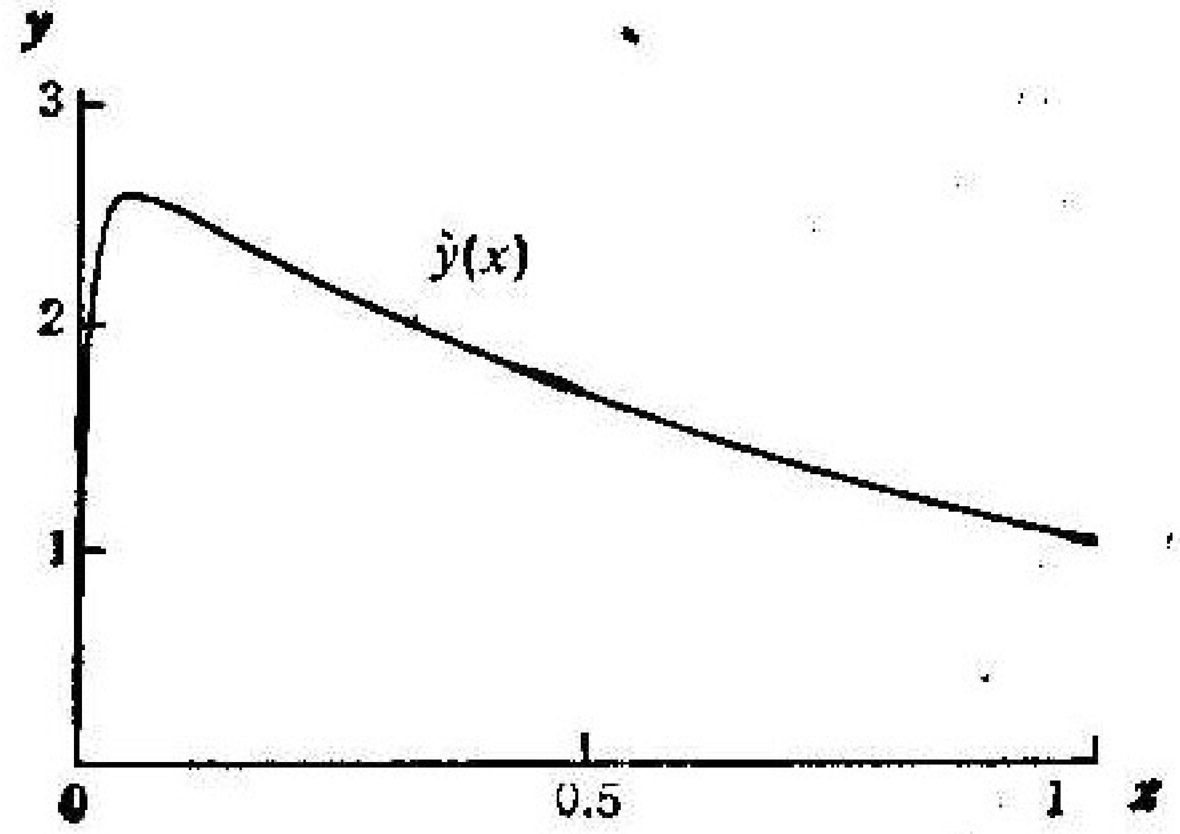

Fig. 1 Exact solution $y(x)$

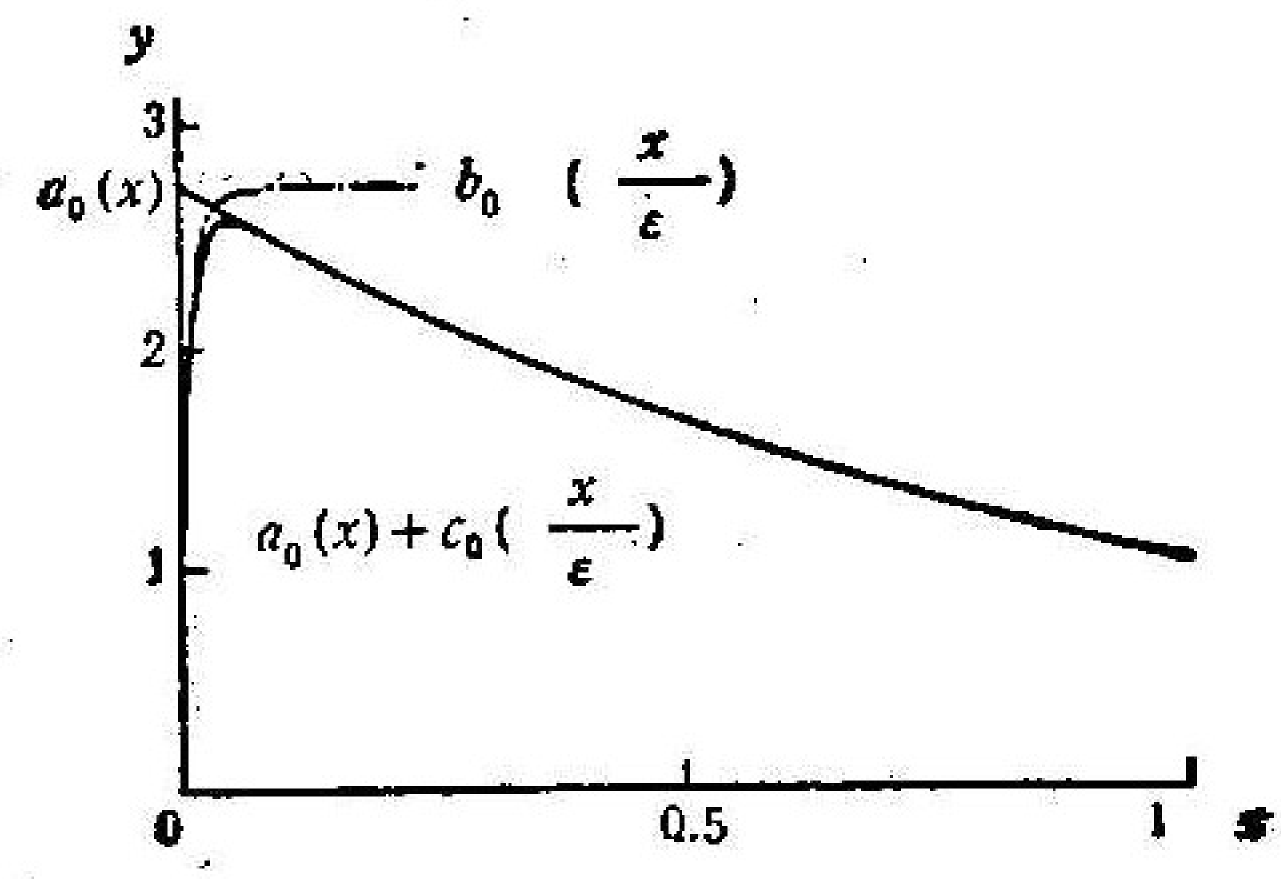

Fig. 2 Asymptotic solutions

the interval [0, 1] into J equal parts, with $\Delta x=1/J$. Since the outer solution is smooth, i.e. no large gradients for small ε, the classic finite difference method with a reasonable size Δx can be used for its numerical solution. For simplicity, let us consider the central difference scheme,

$$\varepsilon\frac{Y_{j+1}-2Y_j+Y_{j-1}}{\Delta x^2}+\frac{Y_{j+1}-Y_{j-1}}{2\Delta x}+Y_j=0 \tag{2.14}$$

or

$$\left(1-\frac{R}{2}\right)Y_{j-1}-\left(2-\frac{\Delta x^2}{\varepsilon}\right)Y_j+\left(1+\frac{R}{2}\right)Y_{j+1}=0,$$

where R is the cell Reynolds number

$$R=\frac{\Delta x}{\varepsilon}. \tag{2.15}$$

On the right from (2.3),

$$Y_J=1. \tag{2.16}$$

On the left we use, for example, parabolic extrapolation with

$$\left[\frac{dY}{dx}\right]_{x=0}=0, \tag{2.17}$$

so

$$Y_0=\frac{4}{3}Y_1-\frac{1}{3}Y_2. \tag{2.18}$$

Extrapolation can be considered as the extension of the outer solution to $x=0$, or (2.17) can be considered as an additional boundary condition for (2.1) with Y. That is, we solve in effect the following problem for an extended outer solution

$$\varepsilon\frac{d^2Y^e}{dx^2}+\frac{dY^e}{dx}+Y^e=0$$

at $x=0$, some boundary condition for extension, as $\left[\frac{dY^e}{dx}\right]_{x=0}=0$; at $x=1$, $Y^e(1)=1$. Let us still use the notation Y for such an extended outer solution.

We show that (2.14), (2.16) and (2.18) has a unique solution. For this we show that the corresponding homogeneous problem has only the trivial solution, at least for large R, which is of our interest. Substituting $Y_j=\lambda^j$ into (2.14), we get

$$\lambda_\pm=\frac{2\alpha\pm\sqrt{4\alpha^2-4+R^2}}{2+R},\quad \alpha=1-\frac{\Delta x^2}{2\varepsilon}.$$

Now $Y_j=c_+\lambda_+^j+c_-\lambda_-^j$, where c_+ and c_- are determined by (2.18) and $Y_J=0$, the homogeneous condition corresponding to (2.16). That is,

$$\left(1-\frac{4}{3}\lambda_++\frac{1}{3}\lambda_+^2\right)c_++\left(1-\frac{4}{3}\lambda_-+\frac{1}{3}\lambda_-^2\right)c_-=0,$$

$$\lambda_+^Jc_++\lambda_-^Jc_-=0. \tag{2.19}$$

For $R\to\infty$, $\lambda_\pm\to-\Delta x\pm\sqrt{1+\Delta x^2}$; in the case of small fixed Δx, it is easy to see that the determinant

$$\left(1-\frac{4}{3}\lambda_++\frac{1}{3}\lambda_+^2\right)\lambda_-^J-\left(1-\frac{4}{3}\lambda_-+\frac{1}{3}\lambda_-^2\right)\lambda_+^J\neq0. \tag{2.20}$$

So for large R, $c_+ = c_- = 0$ is the only solution.

We comment on the oscillation of the numerical solution often associated with the centered difference scheme. For $\alpha=1$, in the case of (2.1) without the Y term, $\lambda_+=1$, then from the first equation of (2.19) we obtain $c_-=0$, thus there will be no oscillation. For the present problem, for large R, $\lambda_+ \cong 1$, there will be negligible oscillation. But for moderate R, there can be noticible oscillation in the outer solution Y_j though slight compared to that in the solution y_j from central difference scheme applied to (2.1) with (2.2) and (2.3). See Fig. 3, note the turn at the left boundary is due to extrapolation with (2.17).

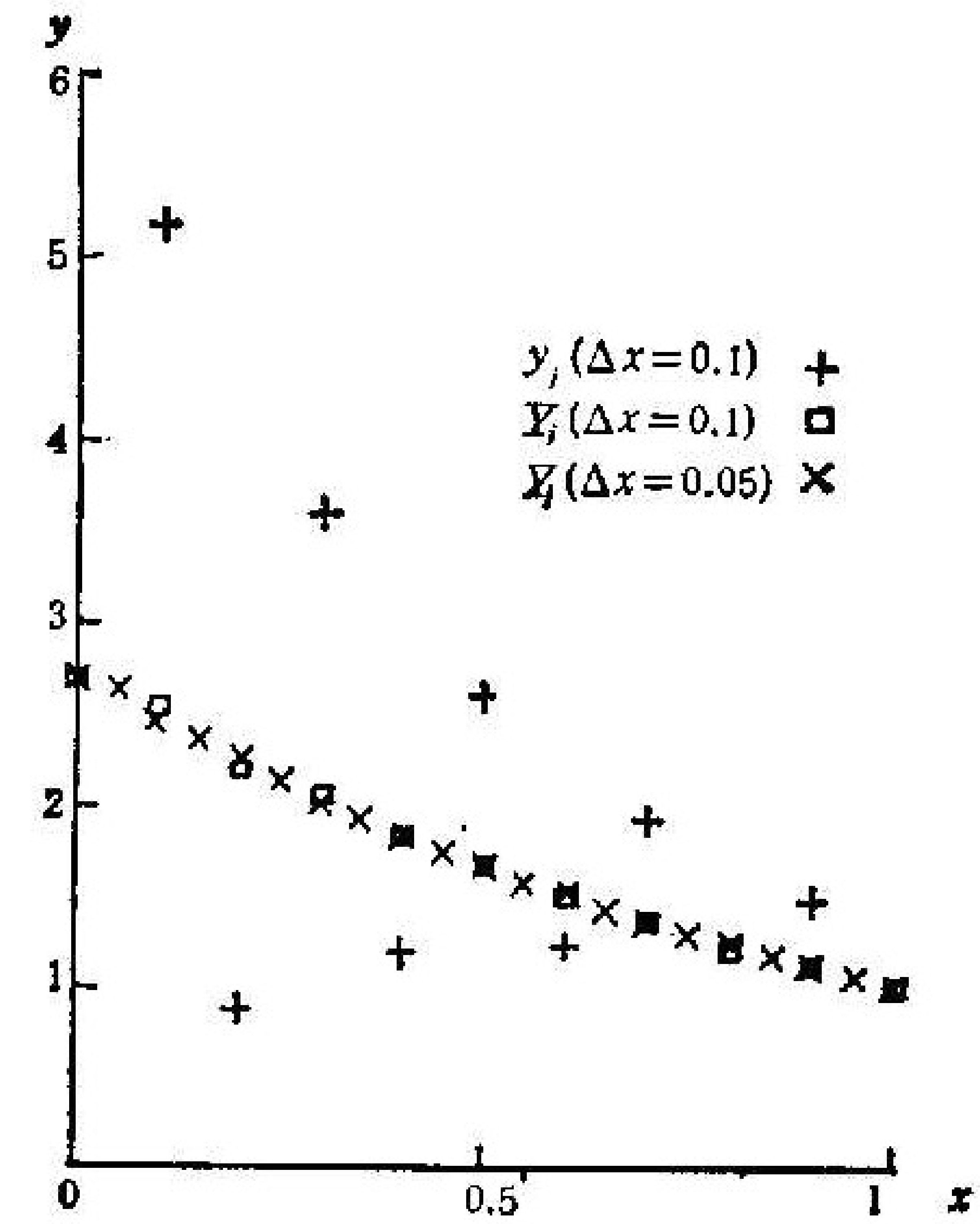

Fig. 3 Finite difference solutions Y, y

In general, to obtain the correct boundary flux, we need to obtain the boundary layer correction (or its $O(\varepsilon^m)$ approximation, some positive m), exactly when possible, with finite difference method when necessary, or by some simple approximate method when allowable. We will consider the last method for the present problem. First, from (2.1) for $\bar{Y}$, we have

$$\left[\varepsilon\frac{d\bar{Y}}{dx}+\bar{Y}\right]_{x=X}-\left[\varepsilon\frac{d\bar{Y}}{dx}+\bar{Y}\right]_{x=0}+\int_0^X \bar{Y}dx=0$$

for some X outside the boundary layer, and from (2.11) and (2.13), this becomes

$$\left[\varepsilon\frac{d\bar{Y}}{dx}+\bar{Y}\right]_{x=0}=\int_0^X \bar{Y}dx. \tag{2.21}$$

Now for demonstrative purposes, we simply approximate $\bar{Y}$ by a linear function $\bar{Y}=f(\varphi)=A+B\varphi$, where

$$\varphi=\frac{x}{\delta}$$

and δ is the boundary layer thickness, such that

$$f(0)=-Y_0,$$
$$f(1)=0$$

which correspond to (2.10), and (2.11). So

$$f(\varphi)=Y_0(\varphi-1). \tag{2.22}$$

Substituting into (2.21), we get

$$\frac{\varepsilon}{\delta}Y_0-Y_0=-\frac{Y_0}{2}\delta$$

and so

$$\delta=1-\sqrt{1-2\varepsilon}\cong\varepsilon.$$

The approximation for $\bar{Y}$ is completely determined and in particular the boundary layer correction flux $(2.21)=-Y_0\delta/2\cong -Y_0\varepsilon/2$. Finally, we have the boundary flux

$$\left[\varepsilon\frac{dY}{dx}+Y\right]_{x=0}+\left[\varepsilon\frac{d\overline{Y}}{dx}+\overline{Y}\right]_{x=0}\cong\left(1-\frac{1}{2}\varepsilon\right)Y_0 \tag{2.23}$$

Numerical results have been obtained with the present method for $\varepsilon=0.01$, $J=10$, 20 and 40. The numerical value of the outer solution at the midpoint of the interval and the boundary flux are given in Table 1. We see that with $\Delta x=0.05$, the order of accuracy of the centered difference scheme is of order 0.0025, and the error at the midpoint of the interval is 0.0004, while the error of the exact solution of the reduced equation ($a_0(0.5)=1.6487$) is 0.008. We see then the reason for solving the entire equation, i.e. retaining the ε term, rather than just the reduced equation. Also given in Table 1 are results from centered difference scheme applied to (2.1) with (2.2) and (2.3), for $J=100$, 200 and 400. We see that the boundary flux of the present method with 21 points in the whole interval is better than the result for the usual finite difference method with 401 point, i.e. 18 points in the boundary layer (we take its thickness to be 0.0465, for which $e^{\mu-x}=0.01$). We point out that approximation of the boundary layer correction is usually problem dependent, and for the present problem only its order is important, since the outer solution can be found independently and, neglecting $O(\varepsilon)$, the boundary flux is simply Y_0.

Table 1 Solution at mid-point of interval and boundary flux
($\varepsilon=0.01$, exact: $y(0.5)=1.6572$, $\varepsilon y'(0)=2.6907$)

Present Method			Central FDE (2.1) (2.2) (2.3)		
J	$Y_{\frac{J}{2}+1}$	$(1-\varepsilon/2)Y_0$	J	$y_{\frac{J}{2}+1}$	$\varepsilon(4y_1-y_2)/2\Delta x$
10	1.6810	2.6539	100	1.6571	2.3977
20	1.6568	2.6762	200	1.6572	2.5853
40	1.6571	2.6924	400	1.6572	2.6578

We conclude from this simple example that the present method is very efficient and yield accurate numerical outer solution and approximate boundary flux, and hence is worthwhile investigating for more complicated problems.

§ 3. Incompressible Navier-Stokes Equations

Now we consider our method for the incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.1a}$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial p}{\partial x}=\varepsilon^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{3.1b}$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial p}{\partial y}=\varepsilon^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{3.1c}$$

where

$$\varepsilon^2=\frac{1}{\mathrm{Re}}, \tag{3.2}$$

Re the Reynolds number. We state only the bottom boundary conditions

$$u(x, 0, t)=0, \tag{3.3}$$

$$v(x, 0, t)=0. \tag{3.4}$$

Let

$$u(x, y, t)=U(x, y, t)+\bar{U}(x, \eta, t),$$
$$v(x, y, t)=V(x, y, t)+\bar{V}(x, \eta, t), \qquad (3.5)$$
$$p(x, y, t)=P(x, y, t)+\bar{P}(x, \eta, t),$$

where U, V, P constitute the outer solution and $\bar{U}, \bar{V}, \bar{P}$ the boundary layer correction, and

$$\eta=\frac{y}{\varepsilon}. \qquad (3.6)$$

For $y>0$, let the outer expansion be

$$U(x, y, t)\sim U_1(x, y, t)+\varepsilon U_2(x, y, t)+\cdots,$$
$$V(x, y, t)\sim V_1(x, y, t)+\varepsilon V_2(x, y, t)+\cdots,$$
$$P(x, y, t)\sim P_1(x, y, t)+\varepsilon P_2(x, y, t)+\cdots,$$

see van Dyke [20]. From the asymptotic point of view, U, V and P are determined by the original partial differential equations, i.e. (3.1) with U, V, P, and the boundary condition

$$V(x, 0, t)=V_B(x, t), \qquad (3.7)$$

where V_B is to be specified. For, substituting the outer expansion into (3.1) with U, V, P, we obtain for first order solution the reduced equations

$$\frac{\partial U_1}{\partial x}+\frac{\partial V_1}{\partial y}=0,$$
$$\frac{\partial U_1}{\partial t}+\frac{\partial U_1^2}{\partial x}+\frac{\partial U_1V_1}{\partial y}+\frac{\partial P_1}{\partial x}=0,$$
$$\frac{\partial V_1}{\partial t}+\frac{\partial U_1V_1}{\partial x}+\frac{\partial V_1^2}{\partial y}+\frac{\partial P_1}{\partial y}=0$$

also known as the Euler equations, which takes the boundary condition

$$V_1(x, 0, t)=0.$$

We obtain for second order solution the equations

$$\frac{\partial U_2}{\partial x}+\frac{\partial V_2}{\partial y}=0,$$
$$\frac{\partial U_2}{\partial t}+U_1\frac{\partial U_2}{\partial x}+V_1\frac{\partial U_2}{\partial y}+\frac{\partial P_2}{\partial x}=-\frac{\partial U_1}{\partial x}\cdot U_2-\frac{\partial U_1}{\partial y}\cdot V_2,$$
$$\frac{\partial V_2}{\partial t}+U_1\frac{\partial V_2}{\partial x}+V_1\frac{\partial V_2}{\partial y}+\frac{\partial P_2}{\partial y}=-\frac{\partial V_1}{\partial x}\cdot U_2-\frac{\partial V_1}{\partial y}\cdot V_2.$$

The equations are linear and have the same characteristic, $dx:dy:dt=U_1:V_1:1$, as the reduced equations, hence we assume the same type of boundary condition, i.e. impose $V_2(x, 0, t)$, see also [20]. It turns out that equations of all the following orders are linear and have these characteristics, hence the outer solution is indeed determined by (3.1) with U, V, P and (3.7).

Let us now extend U and P to $y=0$. Subtracting (3.1) with U, V, P from (3.1) with u, v, p, we obtain the following boundary layer correction equations

$$\frac{\partial \bar{U}}{\partial x}+\frac{\partial \bar{V}}{\partial y}=0, \qquad (3.8a)$$

$$\frac{\partial\overline{U}}{\partial t}+\frac{\partial}{\partial x}(2U\overline{U}+\overline{U}^2)+\frac{\partial}{\partial y}(U\overline{V}+\overline{U}V+\overline{U}\overline{V})+\frac{\partial\overline{P}}{\partial x}=\varepsilon^2\left(\frac{\partial^2\overline{U}}{\partial x^2}+\frac{\partial^2\overline{U}}{\partial y^2}\right), \quad (3.8b)$$

$$\frac{\partial\overline{V}}{\partial t}+\frac{\partial}{\partial x}(U\overline{V}+\overline{U}V+\overline{U}\overline{V})+\frac{\partial}{\partial y}(2V\overline{V}+\overline{V}^2)+\frac{\partial\overline{P}}{\partial y}=\varepsilon^2\left(\frac{\partial^2\overline{V}}{\partial x^2}+\frac{\partial^2\overline{V}}{\partial y^2}\right). \quad (3.8c)$$

From fluid dynamics considerations, see Schlichting [17, Chap. 7], $v=O(\varepsilon)$ in the boundary layer, the boundary layer thickness $\delta=O(\varepsilon)$. At the boundary, V_1 being 0, $V=O(\varepsilon)$, and it remains so in the boundary layer, hence $\overline{V}=O(\varepsilon)$. So we expand as follows

$$\overline{U}(x,\eta,t)\sim\overline{U}_1(x,\eta,t)+\varepsilon\overline{U}_2(x,\eta\ t)+\cdots,$$
$$\overline{V}(x,\eta,t)\sim\varepsilon\overline{V}_1(x,\eta,t)+\varepsilon^2\overline{V}_2(x,\eta,t)+\cdots,$$
$$\overline{P}(x,\eta,t)\sim\overline{P}_1(x,\eta,t)+\varepsilon\overline{P}_2(x,\eta,t)+\cdots.$$

Substituting into (3.8) we get from (3.8c) $\frac{\partial\overline{P}}{\partial\eta}=O(\varepsilon^2)$, then with $O(\varepsilon^2)$ accuracy

$$\frac{\partial\overline{P}}{\partial\eta}=0$$

for which we apply the boundary layer correction condition

$$\lim_{\eta\to\infty}\overline{P}(x,\eta,t)=0$$

and obtain

$$\overline{P}=0. \quad (3.9)$$

With this simplification, (3.8a) and (3.8b) become, also with $O(\varepsilon^2)$ accuracy

$$\frac{\partial\overline{U}}{\partial x}+\frac{\partial\overline{V}}{\partial y}=0, \quad (3.10a)$$

$$\frac{\partial\overline{U}}{\partial t}+\frac{\partial}{\partial x}(2U\overline{U}+\overline{U}^2)+\frac{\partial}{\partial y}\left(U\overline{V}+\overline{U}V+\overline{U}\overline{V}-\varepsilon^2\frac{\partial\overline{U}}{\partial y}\right)=0 \quad (3.10b)$$

two equations for two unknowns $\overline{U}$ and $\overline{V}$. Hence the solution of the boundary layer correction equations can be obtained more easily than with (3.8). A total of three boundary conditions should be imposed on the top and the bottom boundaries. Considering (3.3) and (3.4), we impose

$$\overline{U}(x,0,t)=-U(x,0,t), \quad (3.11)$$
$$\overline{V}(x,0,t)=-V(x,0,t) \quad (3.12)$$

and the boundary layer correction condition for $\overline{U}$

$$\lim_{\eta\to\infty}\overline{U}(x,\eta,t)=0. \quad (3.13)$$

From the discussion of boundary layer problems for system of equations in Abrahamson, Keller and Kreiss [1], and also Li [12], we see that the boundary condition for the outer solution indeed should not be the original one, i.e. $V(x,0,t)=0$. For, then the solution in the interior of the region would be influenced, and $\overline{V}$ would not vanish outside the boundary layer. From a suggestion by Li [12], we specify $V_B(x,t)$ in (3.7) such that the boundary layer correction condition does hold,

$$\lim_{\eta\to\infty}\overline{V}(x,\eta,t)=0. \quad (3.14)$$

So the outer solution U, V, P with boundary condition (3.7) and the boundary layer correction $\overline{U}$ and $\overline{V}$, or its $O(\varepsilon^2)$ approximation, with boundary conditions (3.11)—(3.14) are coupled essentially by boundary conditions at $y=0$, since U varies only slightly in the boundary layer. We insert here an additional condition from fluid dynamic considerations, see [17, Chap. 10],

$$\lim_{\eta\to\infty}\frac{\partial\overline{U}(x,\eta,t)}{\partial\eta}=0. \tag{3.15}$$

Before discussing the method of solution, we put the boundary layer correction equations into integral form. For a fixed small ε, let Y be sufficiently large, in particular larger than the boundary layer thickness, outside of which $\overline{U}$ and $\overline{V}$ are sufficiently small. Integrating (3.10) over $x_l\leqslant x\leqslant x_r$ and $0\leqslant y\leqslant Y$, see Fig. 4, applying the divergence theorem, considering conditions (3.11)—(3.15), and noting

$$\frac{\partial}{\partial x}\int_0^Y\int_{x_l}^{x_r}\overline{U}\left(x,\frac{y}{\varepsilon},t\right)dx\,dy=\int_0^Y\int_{x_l}^{x_r}\frac{\partial\overline{U}\left(x,\frac{y}{\varepsilon},t\right)}{\partial t}dx\,dy$$

since the limits of integration are either fixed or such that $\overline{U}=0$, we get

$$\int_0^Y\overline{U}\left(x_r,\frac{y}{\varepsilon},t\right)dy-\int_0^Y\overline{U}\left(x_l,\frac{y}{\varepsilon},t\right)dy-\int_{x_l}^{x_r}\overline{V}(x,0,t)dx=0,$$

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^Y\int_{x_l}^{x_r}\overline{U}\left(x,\frac{y}{\varepsilon},t\right)dx\,dy&+\int_0^Y\left[2U(x_r,y,t)\overline{U}\left(x_r,\frac{y}{\varepsilon},t\right)+\overline{U}^2\left(x_r,\frac{y}{\varepsilon},t\right)\right]dy\\&-\int_0^Y\left[2U(x_l,y,t)\overline{U}\left(x_l,\frac{y}{\varepsilon},t\right)+\overline{U}^2\left(x_l,\frac{y}{\varepsilon},t\right)\right]dy\\&-\int_{x_l}^{x_r}\left(U(x,0,t)\overline{V}(x,0,t)-\varepsilon^2\left[\frac{\partial\overline{U}\left(x,\frac{y}{\varepsilon},t\right)}{\partial y}\right]_{y=0}\right)dx\\&=0.\end{aligned}$$

For all practical purposes, this is

$$\int_0^{\delta_r}\overline{U}_r\,dy-\int_0^{\delta_l}\overline{U}_l\,dy+\int_{x_l}^{x_r}V_B\,dx=0, \tag{3.16a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^{\delta(t)}\int_{x_l}^{x_r}\overline{U}\,dx\,dy&+\int_0^{\delta_r}(2U_r\overline{U}_r+\overline{U}_r^2)dy-\int_0^{\delta_l}(2U_l\overline{U}_l+\overline{U}_l^2)dy\\&+\int_{x_l}^{x_r}\left[U_BV_B+\varepsilon^2\left(\frac{\partial\overline{U}}{\partial y}\right)_B\right]dx=0,\end{aligned} \tag{3.16b}$$

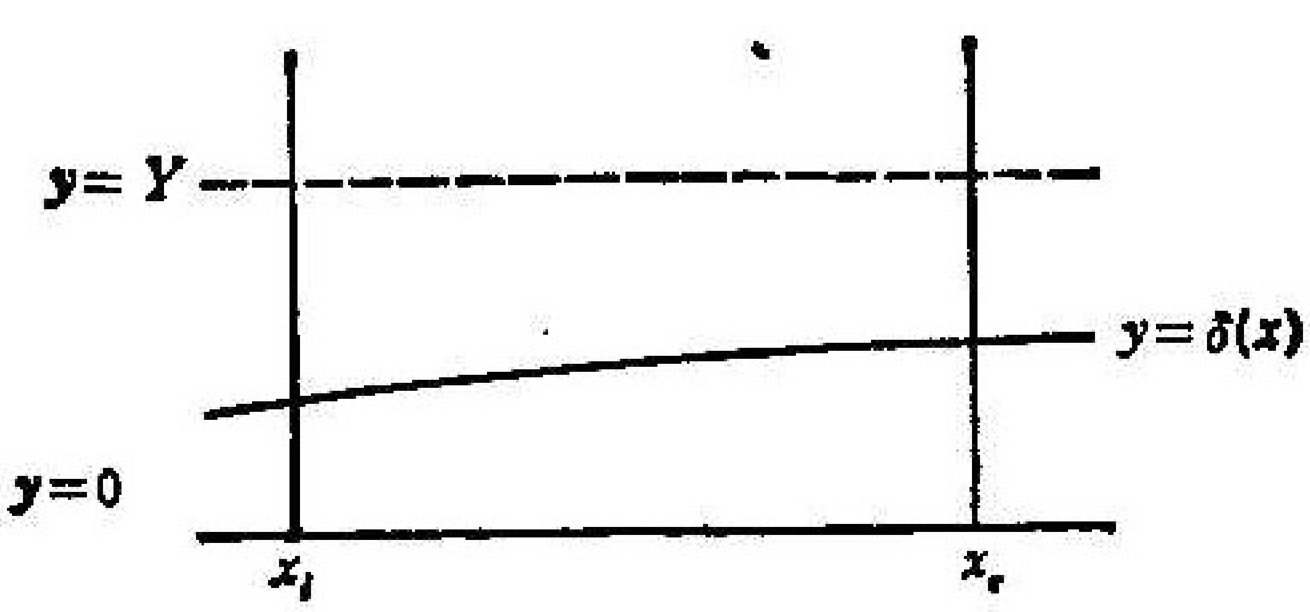

Fig. 4 Bottom boundary

where subscripts l and r denote $x=x_l$ and $x=x_r$ respectively, and B denote the bottom boundary.

We remark here that the boundary conditions for (3.1) satisfy

$$\oint_{\partial\Omega}\underline{v}\cdot\underline{n}\,ds=0 \tag{3.17}$$

where $\partial\Omega$ is the boundary of Ω our region of solution, s is the arc length, and $\underline{n}$ the unit normal vector. This is obvious from (3.1a). The boundary conditions of the outer solution also satisfy such a condition

$$\oint_{\partial\Omega} \underline{V} \cdot \underline{n}\, ds = 0. \tag{3.18}$$

This is ensured through (3.17) and (3.16a) which determines V_B for the outer solution. We note also that projection of a vector $\underline{V}$ onto a space with zero divergence in Ω and zero normal component on $\partial\Omega$, as discussed in Chorin and Marsden [14], can be easily extended to projection onto a space with zero divergence and condition (3.18).

§ 4. Method of Solution

For the outer solution, we again can use finite difference methods on relatively coarse grids, assuming the solution to have no large gradients for large Re (small ε). We write the finite difference equations for (3.1) with U, V, P in the following form

$$\frac{\delta_x U}{\Delta x} + \frac{\delta_y V}{\Delta y} = D = 0, \tag{4.1a}$$

$$\frac{\Delta U}{\Delta t} + \frac{\Delta_x F_x}{\Delta x} + \frac{\Delta_y F_y}{\Delta y} + \frac{\delta_x P}{\Delta x} = 0, \tag{4.1b}$$

$$\frac{\Delta V}{\Delta t} + \frac{\Delta_x G_x}{\Delta x} + \frac{\Delta_y G_y}{\Delta y} + \frac{\delta_y P}{\Delta y} = 0, \tag{4.1c}$$

where

$$\begin{aligned} F_x &= U^2 - \alpha^2 \frac{\Delta_x U}{\Delta x}, \quad F_y = UV - \alpha^2 \frac{\Delta_y U}{\Delta y}, \\ G_x &= UV - \alpha^2 \frac{\Delta_x V}{\Delta x}, \quad G_y = V^2 - \alpha^2 \frac{\Delta_y V}{\Delta y}. \end{aligned} \tag{4.2}$$

In the above, $\alpha^2 = \varepsilon^2$, $\delta_x(\delta_y)$ denotes first order centered difference in $x(y)$, Δ_x (Δ_y) denotes any first order difference in the $x(y)$, and Δ denote any first order difference in t. We use the staggered mesh as shown in Fig. 5. As all the projection or MAC methods, see Peyret and Taylor [16, Chap. 6] and Harlow and Welch [9], we obtain the finite difference equation for pressure by combining (4.1b) for r and l and (4.1c) for a and b,

$$\frac{1}{\Delta x}\left[\left(\frac{\delta_x P}{\Delta x}\right)_r - \left(\frac{\delta_x P}{\Delta x}\right)_l\right] + \frac{1}{\Delta y}\left[\left(\frac{\delta_y P}{\Delta y}\right)_a - \left(\frac{\delta_y P}{\Delta y}\right)_b\right] = -RHS, \tag{4.3}$$

where

$$\begin{aligned} RHS = {} & \frac{\Delta D}{\Delta t} + \frac{1}{\Delta x}\left[\left(\frac{\Delta_x F_x}{\Delta x}\right)_r - \left(\frac{\Delta_x F_x}{\Delta x}\right)_l\right] + \frac{1}{\Delta x}\left[\left(\frac{\Delta_y F_y}{\Delta y}\right)_r - \left(\frac{\Delta_y F_y}{\Delta y}\right)_l\right] \\ & + \frac{1}{\Delta y}\left[\left(\frac{\Delta_x G_x}{\Delta x}\right)_a - \left(\frac{\Delta_x G_x}{\Delta x}\right)_b\right] + \frac{1}{\Delta y}\left[\left(\frac{\Delta_y G_y}{\Delta y}\right)_a - \left(\frac{\Delta_y G_y}{\Delta y}\right)_b\right] \\ = {} & \frac{\Delta D}{\Delta t} + (R_x)_r - (R_x)_l + (R_y)_a - (R_y)_b. \end{aligned} \tag{4.4}$$

On the bottom boundary, as Easton [6], we simply drop all the terms on B, so in (4.3), we set

$$\left(\frac{\delta_y P}{\Delta y}\right)_B = 0 \tag{4.5}$$

and

$$RHS=\frac{\Delta D}{\Delta t}+\frac{\Delta V_B}{\Delta t\,\Delta y}+(R_x)_r-(R_x)_l+(R_y)_a \tag{4.6}$$

in which V_B is determined with the boundary layer correction solution.

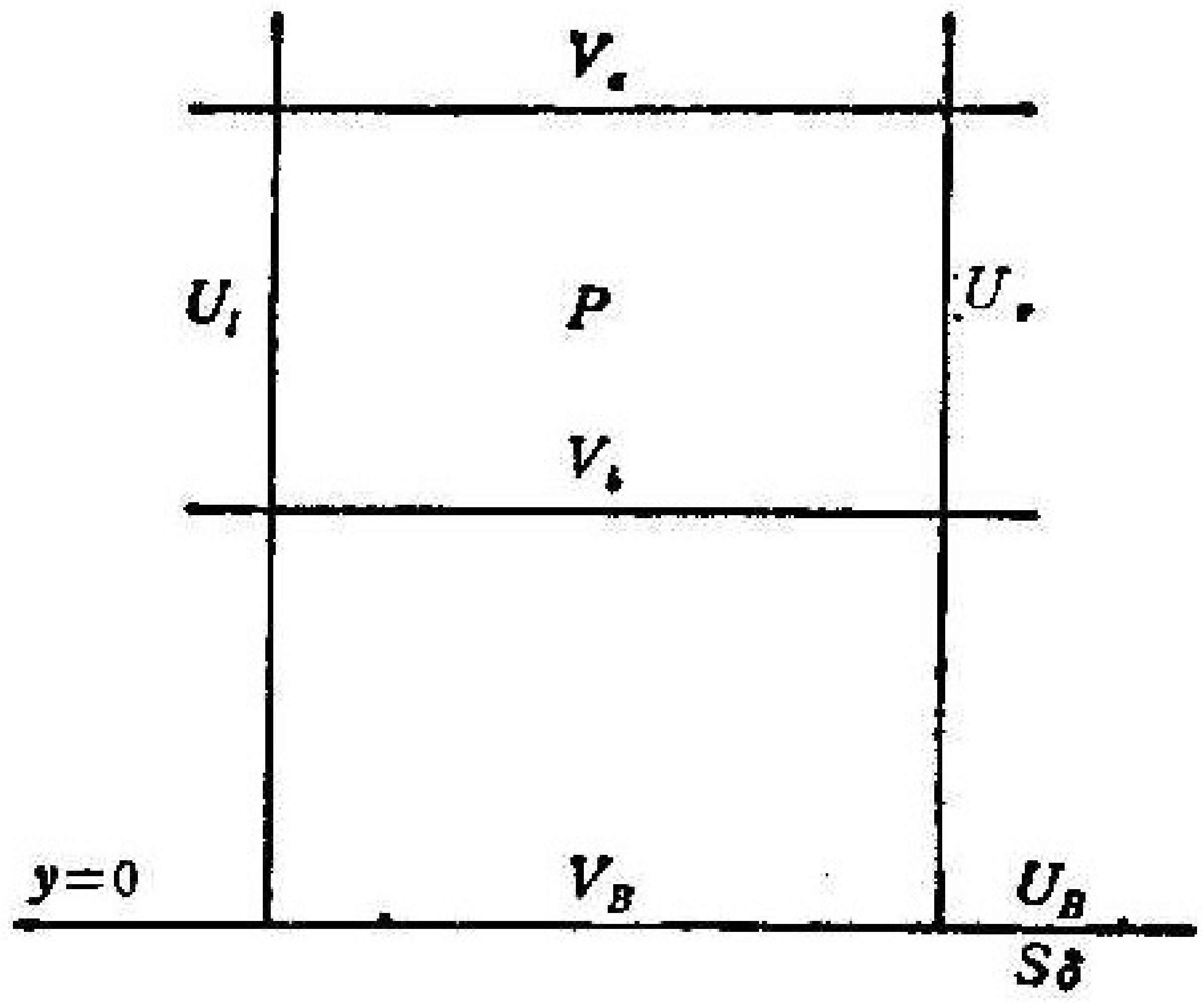

Fig. 5 The staggered mesh

As before, we can extend U by extrapolation with

$$\left[\frac{\partial U}{\partial y}\right]_{y=0}=0. \tag{4.7}$$

This can also be considered as an additional boundary condition to (3.7) for (3.1) with U, V, P; that is, again we really deal with a extended outer solution. Also the resulting extrapolation formula can be considered as an additional numerical boundary condition to (3.7) for (4.1), (4.2).

Suppose the accuracy of the difference scheme is $O(\Delta x^m)$ ($\Delta y=q\Delta x$, q a constant), and Δx is chosen to attain the desired accuracy. Now, for high Re flow the exact viscosity outside the boundary layer is often not important. So if $\varepsilon^2\ll O(\Delta x^m)$, then the ε^2 term can not be resolved; one might as well take α^2 to be $O(\Delta x^m)$ and regard the second order partial derivative terms as artificial viscosity frequently used for improving stability, linear or nonlinear.

We remark that the solution P of (4.3), (4.4) with the corresponding boundary treatment (4.5), (4.6), exists and has one degree of freedom since the sum of RHS over the computational region is zero. Indeed, RHS is of discrete divergence form, and all the boundary terms are dropped. However, in practice (4.6) is, say,

$$RHS=\frac{D^{n+1}-D^n}{\Delta t}+\frac{V_B^{n+1}-V_B^n}{\Delta t\,\Delta y}+(R_x)_r-(R_x)_l+(R_y)_a,$$

where D^{n+1} is set to zero to ensure (4.1a), and D^n is retained to keep the iteration error of P from accumulating, see [9]. In the sum of RHS, there remain the V_B terms and those on other boundaries, which all add up to zero within discretization error, due to (3.18).

Now we come to the numerical solution of the boundary layer correction equations. Let

$$\begin{aligned} S&=-\int_0^\delta \overline{U}\,dy,\\ \bar{S}&=-\int_0^\delta (U\overline{U}+\overline{U}^2)\,dy, \end{aligned} \tag{4.8}$$

and

$$F_B=\varepsilon^2\left(\frac{\partial\overline{U}}{\partial y}\right)_B$$

which represent the shearing stress since $\varepsilon^2(\partial u/\partial y)_B=\varepsilon^2(\partial\overline{U}/\partial y)_B$ due to (4.7). Then on our mesh, (3.16) yields difference equations of form

$$\frac{\Delta_x S}{\Delta x}-V_B=0, \tag{4.9a}$$

$$\frac{\Delta S}{\Delta t}+\frac{\Delta_x \tilde{S}}{\Delta x}+\frac{\Delta_x U_B S}{\Delta x}-U_B V_B-F_B=c\alpha^2\frac{\delta_x^2 S}{\Delta x^2}. \tag{4.9b}$$

Artificial viscosity has been added on the right hand side of (4.9b) for stability, with coefficient $c\alpha^2$ within error tolerance. We have assumed only for convenience that the boundary layer is thin so that U is approximated by U_B. We can adapt the approximate methods for the classic boundary layer equations, see Schlichting [17, Chap. 10], to the boundary layer correction equations. For example, let $\overline{U}$ be approximated by a third degree polynomial

$$\overline{U}=f(\varphi)=A+B\varphi+C\varphi^2+D\varphi^3, \tag{4.10}$$

where

$$\varphi=\frac{y}{\delta},$$

the coefficients A, B, C, and D are determined by the conditions

$$f(0)=-U_0, \tag{4.11a}$$

$$\left[\frac{d^2 f}{d\varphi^2}\right]_{\varphi=0}=-\frac{\delta^2}{\varepsilon^2}\left(\frac{\Delta U}{\Delta t}+U\frac{\Delta_\sigma U}{\Delta x}\right)_B=\delta^2 a(U_B), \tag{4.11b}$$

$$f(1)=0, \tag{4.11c}$$

$$\left[\frac{df}{d\varphi}\right]_{\varphi=1}=0. \tag{4.11d}$$

Here the second condition is obtained from (3.10b), (4.11a) and (4.7). Substituting (4.10) into (4.8) and assuming U_B and S to be known, results one equation in one unknown δ, which we write as

$$a(U_B)\delta^3+c(U_B)\delta+d\cdot S=0. \tag{4.12}$$

Once δ is found, $\overline{U}$ is determined.

Thus for each unknown depicted in Fig. 5, we have a corresponding equation: (4.1b) and (4.1c) for U and V, (4.3) for P, extrapolation formula for U_B, (4.9b) for S, and (4.12) for δ, and (4.9a) for V_B.

We now state the computational steps for explicit schemes. Suppose that up to U^n, V^n, U_B^n, V_B^n and δ^n (hence S^n, $\tilde{S}^n$ and F_B^n) are known, then:

(1) Compute S^{n+1} with (4.9b),
(2) Compute V_B^{n+1} with (4.9a),
(3) Calculate F_x, F_y, G_x, G_y with (4.2) and RHS with (4.4) or (4.6),
(4) Iterate for P with (4.3) and (4.5),
(5) Compute U^{n+1} and V^{n+1} with (4.1b) and (4.1c),
(6) Extrapolate for U_B^{n+1} with (4.7),
(7) Determine δ^{n+1} with (4.12),

which completes one time cycle. Note $(\Delta U/\Delta t)_B$ in (4.11b) is obtained lagging one time cycle.

§ 5. Numerical Example

Now we test our method on the classic problem of steady-state incompressible flow past a semi-infinite flat plate, see Fig. 6. The length of the plate for which we seek the solution and the free stream velocity are chosen as references. We have the

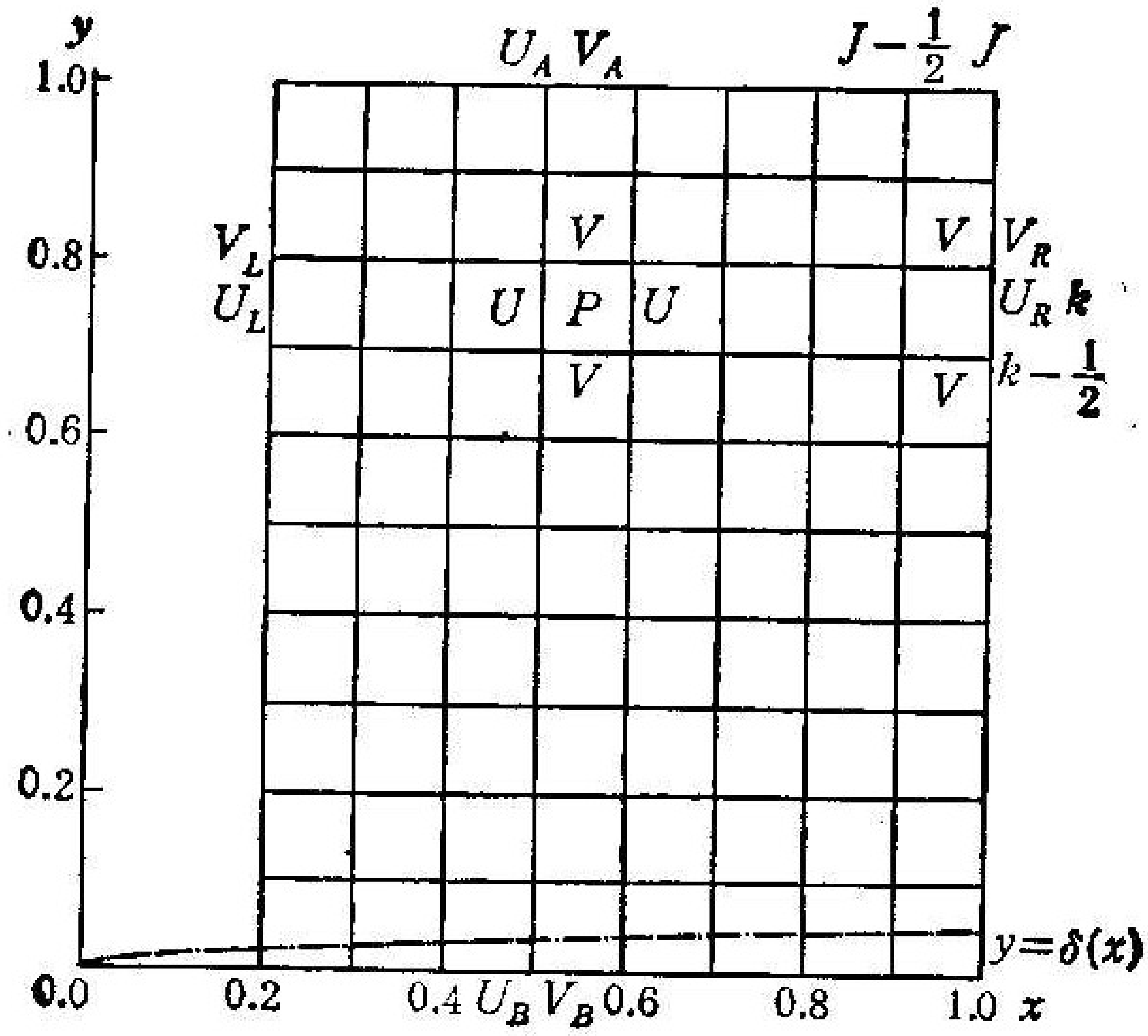

Fig. 6 Coarse mesh for flow past a semi-infinite plate

$O(\varepsilon^2)$ approximation of the outer solution in non-dimensional form

$$
\begin{aligned}
U^A(x, y) &= 1-\varepsilon\frac{\beta}{r}\sqrt{\frac{r-x}{2}},\\
V^A(x, y) &= \varepsilon\frac{\beta}{r}\sqrt{\frac{r+x}{2}},\\
P^A(x, y) &= \varepsilon\frac{\beta}{r}\sqrt{\frac{r-x}{2}},
\end{aligned}
\tag{5.1}
$$

where $r=\sqrt{x^2+y^2}$ and $\beta=0.860$, see Schlichting [17, Chap. 7]. We also know the Blasius solution to the classic Prandtl's boundary layer equations, which are the first order equations for inner asymptotic expansion. The solution to the second order equations happen to be zero, so the Blasius solution is also the $O(\varepsilon^2)$ approximation of the inner solution, again see [17, Chap. 7]. From it we have the non-dimensional shearing stress τ_0, displacement thickness δ_1, and momentum thickness δ_2,

$$
\begin{aligned}
\tau_0(x) &= \varepsilon\beta_0\sqrt{\frac{1}{x}},\\
\delta_1(x) &= \varepsilon\beta_1\sqrt{x},\\
\delta_2(x) &= \varepsilon\beta_2\sqrt{x},
\end{aligned}
\tag{5.2}
$$

where $\beta_0=0.332$, $\beta_1=1.721$ and $\beta_2=0.664$.

We seek numerical solution on the non-dimensional rectangle: $x_L=0.2\leqslant x\leqslant 1$, $0\leqslant y\leqslant y_A=1$, with staggered mesh shown in Fig. 6. To test the basic feasibility of our method, we use the FTCS (forward time centered space) differences in (4.1) for the outer solution because of its simplicity. We also use FTCS in (4.9b) and the third degree polynomial (4.10) for the boundary layer correction solution. Since our method is for coarse grids, the restriction on time steps should not be too severe.

The bottom boundary condition has been discussed in the last section. The left and the top boundary values of U and V are given as those of the asymptotic solution (5.1). The right boundary is left open i.e., we impose numerical boundary conditions compatible with the oncoming flow. We could use, for example, the non-reflecting boundary conditions as Hedstrom and Osterheld [10], but from the authors' numerical experiments on finite difference schemes and boundary treatment for the incompressible Navier–Stokes equations [18], we found the following to be the best from the practical point of view: U_R from $D=0$, and V_R from upwind difference scheme of the steady-state, non-viscous version of (3.1c) with U, V, P. So

$$\begin{aligned} U_{Rk} &= U_{J-\frac{1}{2}k} - \frac{\Delta x}{\Delta y}(V_{Jk+\frac{1}{2}} - V_{Jk-\frac{1}{2}}), \\ V_{Rk+\frac{1}{2}} &= V_{Jk+\frac{1}{2}} - \frac{\Delta x}{2\Delta y}\Big[\Big(\frac{V}{U}\Big)_{Jk+\frac{1}{2}}(V_{Jk+\frac{1}{2}} - V_{Jk-\frac{1}{2}}) + \frac{1}{U_{Jk+\frac{1}{2}}}\delta_y P_{Jk}\Big]. \end{aligned} \tag{5.3}$$

The initial values U^0 and V^0 are obtained from (5.1) by adding suitable terms

$$\begin{aligned} U^0(x, y) &= U^A(x, y) + r(x-x_L)^2(y-y_A), \\ V^0(x, y) &= V^A(x, y) - r(x-x_L)(y-y_A)^2, \end{aligned} \tag{5.4}$$

which satisfy the zero divergence condition and hence ensure (3.18) in the initial state. Note these satisfy the left and top boundary conditions.

The boundary and initial conditions for the boundary layer correction equations are as follows: On the left, S and $\tilde{S}$ are obtained from the Blasius solution as δ_1 and δ_2 respectively. The initial values of S are obtained from V_B with (4.9a). Equation (4.12) is then solved for δ, with simplifying assumptions which yield likely initial values of $\tilde{S}$ and F_B. On the right, we simply use upwind differences in (4.9b), dropping the artificial viscosity.

Now, for $\mathrm{Re}=10^4$ the boundary layer variables S, $\tilde{S}$ and F_B are of order $\varepsilon=0.01$. We use $\Delta x=\Delta y=0.05$ and $\alpha^2=\Delta x^2=0.0025$, so that the error of finite difference approximation may be within tolerance. We use $\Delta t=0.005$ to ensure stability for (4.1), (4.2), the strictest condition being $r_x^2\leqslant 2s_x$, see [18], i.e.

$$U^2\frac{\Delta t^2}{\Delta x^2}\leqslant 2\alpha^2\frac{\Delta t}{\Delta x^2}. \tag{5.5}$$

Putting (4.9b) into a difference equation for δ and dropping the lower order terms, we find (5.5) to be a sufficient condition for its stability also. In initial condition (5.4), we take $r=0.01$.

In our numerical experiments, we found the processes (4.9a) for V_B and (4.11b) for coefficient $a(U_B)$ to be destabilizing, since they are of numerical differentiation. Whereas we found the processes for the boundary layer correction variables S and $\tilde{S}$ to be quite stabilizing, since they are of integration; results of F_B determined in the same process is also very smooth. For the above mesh, c in (4.9b) was taken to be 4 and a strong filter was used for $a(U_B)$:

$$\tilde{a}_j=\frac{1}{4}(a_{j+1}+2a_j+a_{j-1}).$$

For a coaser mesh with $\Delta x=\Delta y=0.1$, as shown in Fig. 6, $\Delta t=0.02$, c was taken to be 1 and no filter was needed.

The numerical results U, V of the outer solution are presented with the asymptotic solution U^A, V^A of (5.1) in Fig. 7; and numerical results of the shearing stress F_B, are presented with the Blasius τ_0 of (5.2) in Fig. 8. Noting the scale, we conclude that the results are quite good, especially the shearing stress—even with the coaser grid. Now with extrapolation with (4.7), U is also close to 1, so we expect the boundary layer quantities S and $\tilde{S}$ to be respectively close to δ_1 and δ_2 of (5.2). From Fig. 9, we see that they are indeed so. Also we expect V_B to be close to $V^A(x, 0)$ of (5.1), but now on a finite grid V_B no longer has clear physical meaning; Fig. 10 shows that the results are still quite acceptable.

The steady-state results for $\Delta x=\Delta y=0.05$ were obtained with 1000 time steps. $|U^{n+1}-U^n|_{max}$ and $|V^{n+1}-V^n|_{max}$ are of order 10^{-8} and $|D|_{max}$ is of order 10^{-8} which was initially of 10^{-4}. For graphic purposes, results with 600 time steps would have sufficed.

§ 6. Concluding Remarks

The numerical results of the previous section indicate that our method is very promising. It can predict accurately the large scale phenomena outside the boundary layer and the boundary flux within the boundary layer with little computational effort. Hence it can be extended to computation of real problems, such as natural convection in enclosures with thermal boundary layers. The finite difference scheme for the outer solution and the approximate method for the boundary layer correction have much room for improvement, and will be one aspect of our future work. For the latter, the recent zonal methods in compressible transonic flow calculations for aircraft, reviewed by Lock and Firmin [13] for example, will be carefully studied.

There remain open questions in the mathematical formulations as well as the numerical methods concerned. The present work is a step in the direction of decomposing the computational domain into two parts: one where convection is dominant and one where viscosity is prominant, coupled essentially only at the boundary, so that the stage be set for incorporating different simplifications and using different numerical methods for different types of flow.

We note that viscosity has not been accounted for in the $O(\varepsilon^2)$ approximation of the asymptotic solution (5.1); it is not important in our simple example. But it does play essential roles in other problems and hence it should be kept in the finite difference equations as (4.1), (4.2), in which α can be taken as ε if needed and if computation done with suitable mesh. Viscosity must be considered for internal layers, computation of these layers will be another topic of our future research.

To resolve more complicated phenomena, the numerical solution of the complete incompressible Navier-Stokes equation on local fine grids will be necessary. Hence adaptive mesh, as Berger and Oliger [2] for hyperbolic problems, must also be investigated and adapted to the present problem. It is the authors' belief that only with a combination of these techniques, can efficient algorithms be developed for the computation of viscous flow fields.

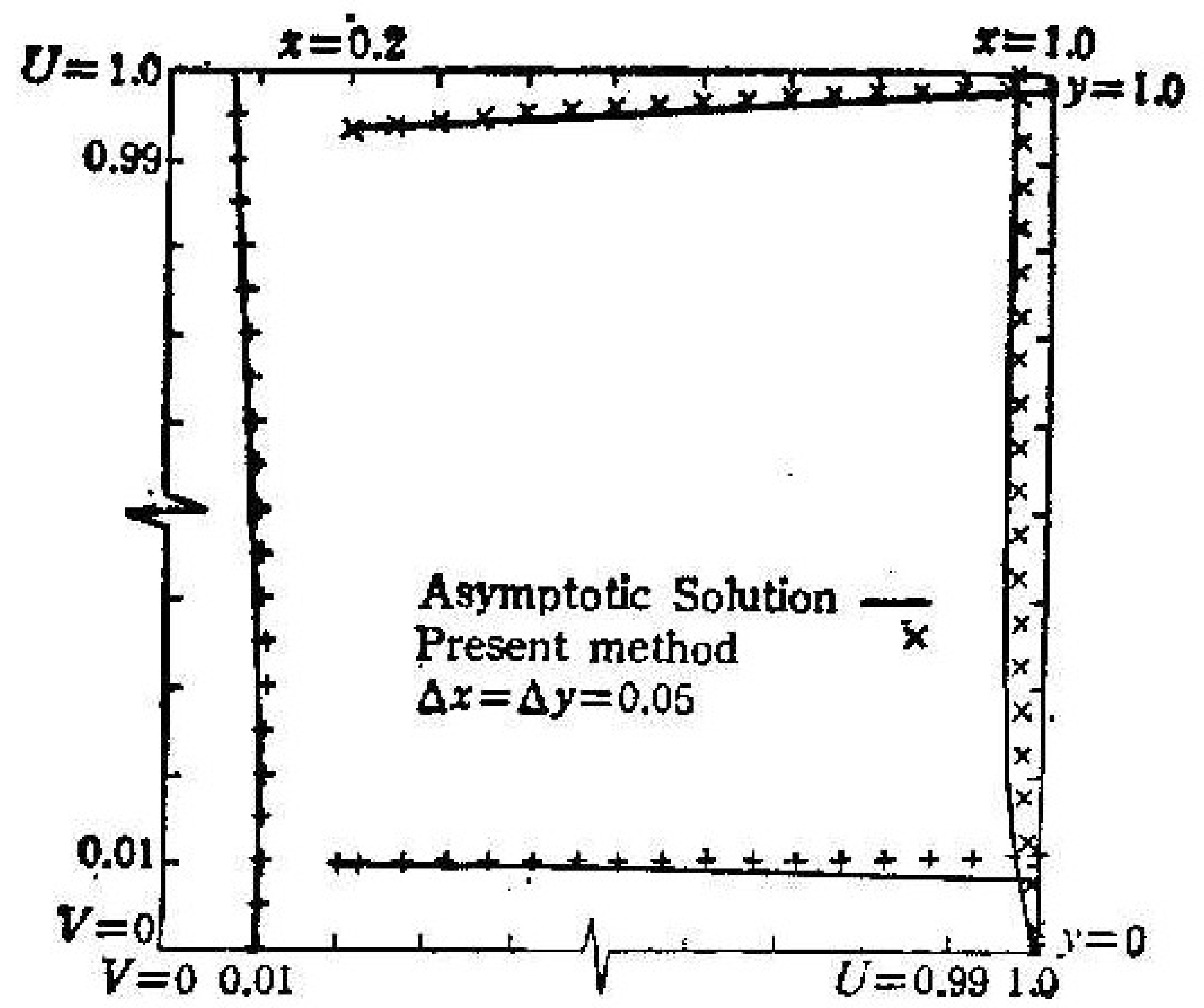

Fig. 7 Outer solution U and V

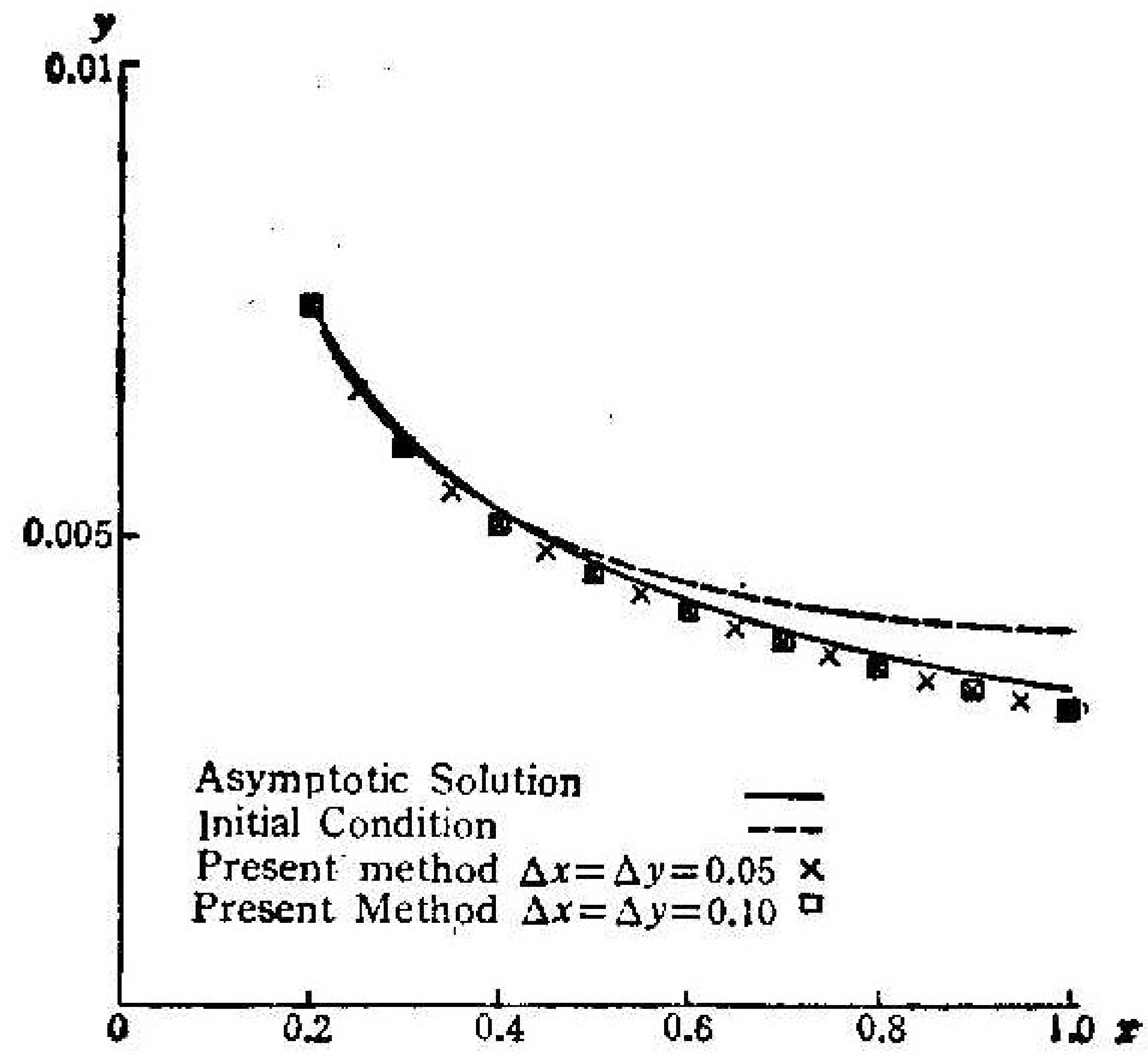

Fig. 8 Shearing stress F_s

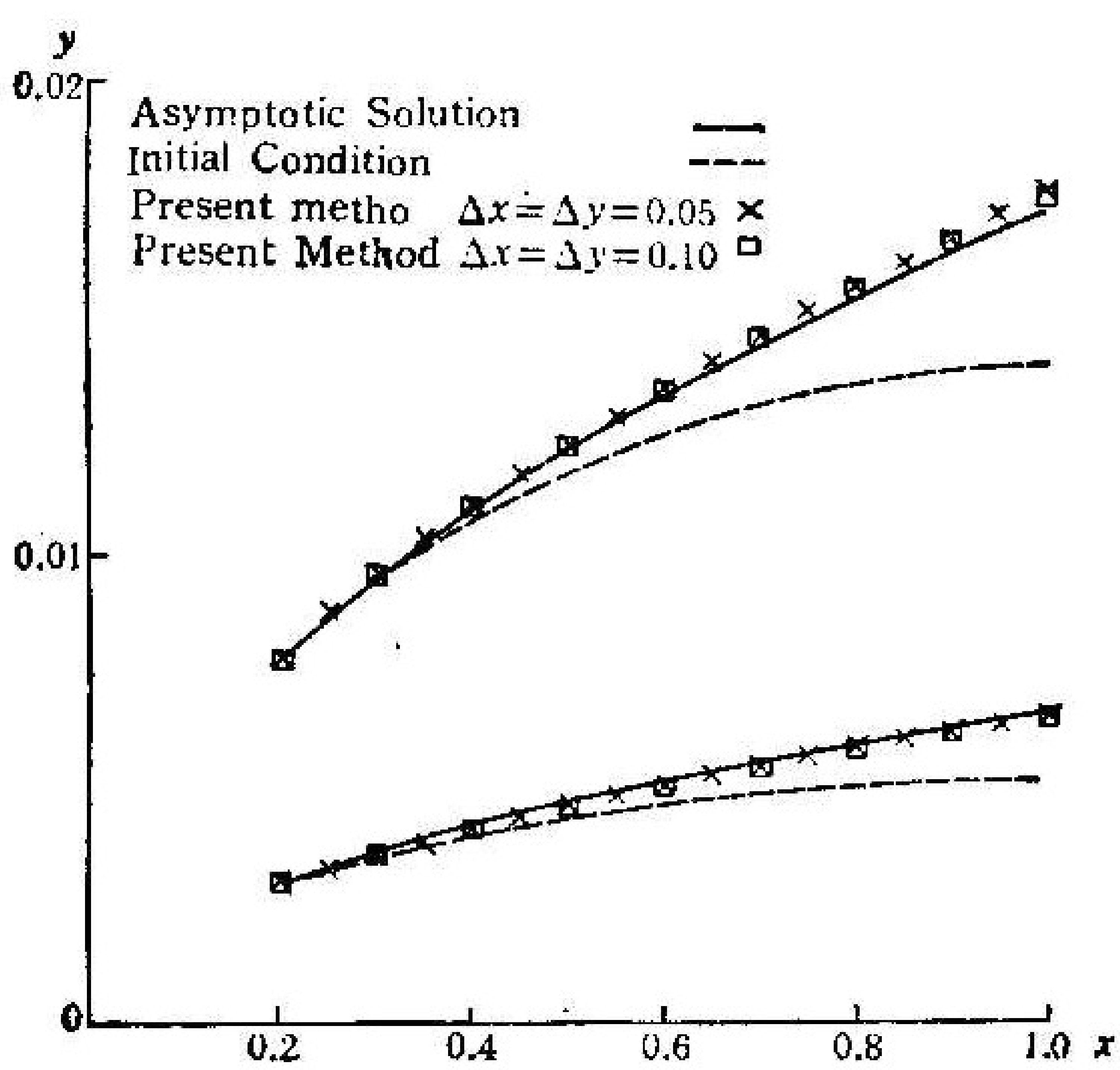

Fig. 9 Comparison of S and $\tilde{S}$ with δ_1 and δ_2

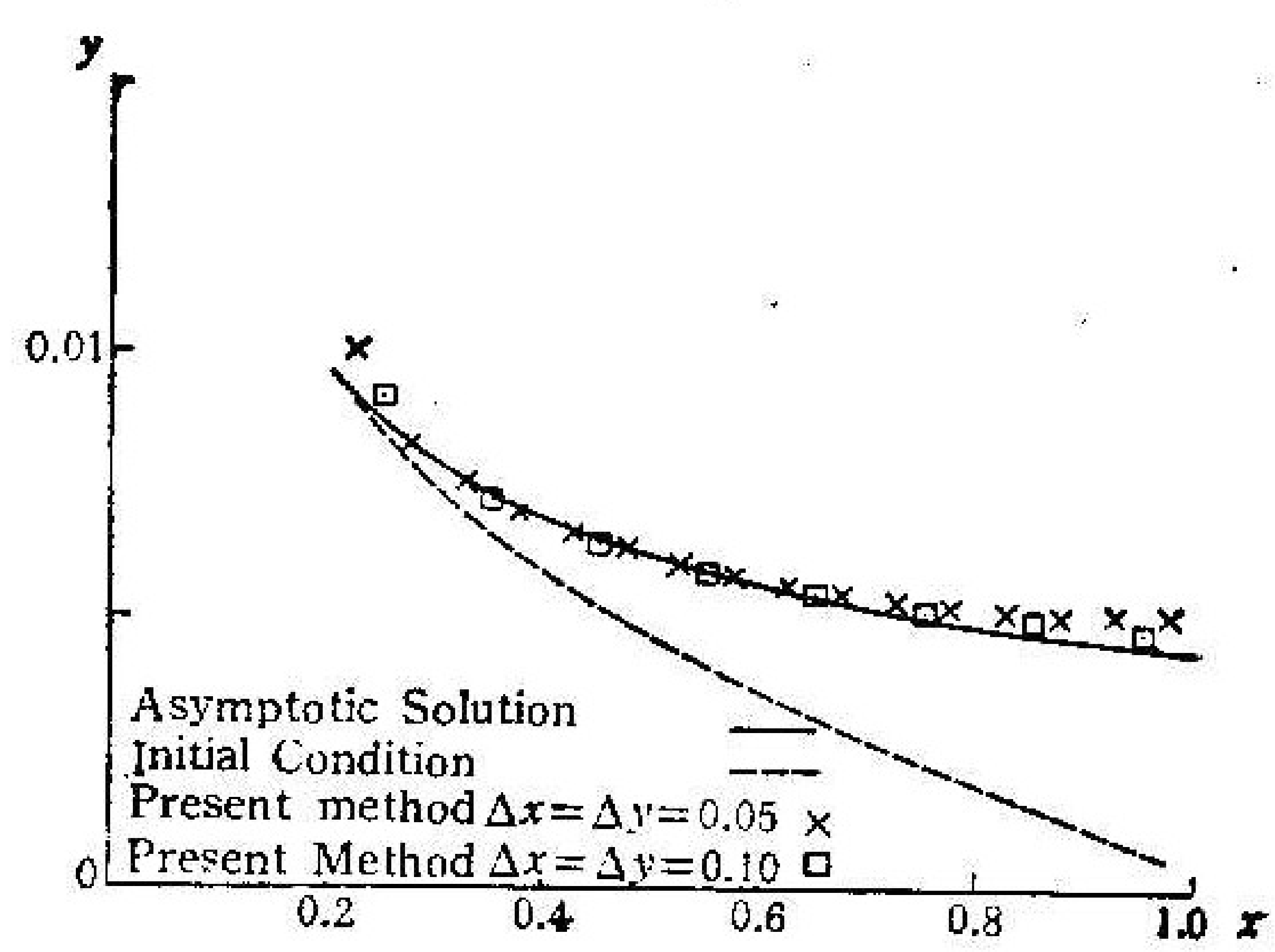

Fig. 10 Comparison of V_B with $V^A(x, 0)$

Acknowledgements. The authors wish to thank Prof. Y. G. Bian and Prof. L. Zhao of the Mechanics Institute and Prof. H. C. Huang of our Center for valuable discussions and helpful suggestions in the course of this work.

References

[1] Abrahamsson L. R.; Keller H. B.; Kreiss H. O.: Difference approximations for singular perturbations of system of ordinary differential equations, *Numer. Math.*, **22** (1974), 367—391.

[2] Berger M. J.; Oliger J.: Adaptive mesh refinement for hyperbolic partial differential equations, *J. Comput. Phys.*, **53** (1984), 484—512.

[3] Chin R. C. Y.; Hedstrom G. W.; Howes F. A.: Considerations on Solving Problems with Multiple Scales, Lawrence Livermore National Laboratory UCRL-90791, 1984.

[4] Chorin A. J.; Marsden J. E.: A Mathematical Introduction to Fluid Mechanics, Springer-Verlag, New York, 1979.

[5] Doolan E. P.; Miller J. J. H.; Schilders W. H. A.: Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.

[6] Easton C. R.: Homogeneous boundary condition for pressure in the MAC method, *J. Comput. Phys.*, **9** (1972), 375—379.

[7] Eckhaus W.; deJager E. M.: Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type, *Arch. Rational Mech. Anal.*, **23** (1966/67), 26—86.

[8] Flaherty J. E.; O'Malley Jr. R. E.: The numerical solution of boundary value problems for stiff differential equations, *Math. Comput.*, **31** (1977), 66—93.

[9] Harlow F. H.; Welsh J. E.: Numerical calculation of time dependent viscous incompressible flow with free surfaces, *Phys. Fluids*, **8** (1965), 2182—2189.

[10] Hedstrom G. W.; Osterheld A.: The effect of cell Reynolds number on the computation of a boundary layer, *J. Comput. Phys.*, **37** (1980), 399—421.

[11] Howes F. A.: Some Singularly Perturbed Nonlinear Boundary Value Problems of Elliptic Type, in Nonlinear Partial Differential Equations in Applied Science, ed. R. L. Sternberg, Marcel Dekker, New York, 1980, 150—165.

[12] Li J. X.: Comparison of numerical methods for singular perturbation boundary value problems, M. A. Thesis, Computing Center, Academia Sinica, 1987.

[13] Lock R. C.; Firmin M. C. P.: Survey of Techniques for Estimating Viscous Effects in External Aerodynamics, in Proc. IMA Conference on Numerical Methods in Aeronautical Fluid Dynamics, ed. P. L. Roe, Academic Press, 1982, 337—430.

[14] MacCormack R. W.; Lomax H.: Numerical solution of compressible viscous flow, *Annual Reviews of Fluid Mechanics*, **11** (1979), 289—316.

[15] O'Malley Jr. R. E.: Introduction to Singular Perturbation, Academic Press, New York, 1974.

[16] Peyret R.; Taylor T. D.: Computational Methods for Fluid Flow, Springer-Verlag, New York, 1983.

[17] Schlichting H.: Boundary-Layer Theory, McGraw-Hill Book Co., New York, Third English Edition, 1968.

[18] Shen J. X.; Huang L. C.: Improvements of the MAC Method, Technical Report, Computing Center, Academia Sinica, 1987.

[19] van Dalsem W. R.; Steger J. L.: Finite-difference simulation of transonic seperated flow using a full potential-boundary layer interaction approach, *AIAA*, Paper 83-1689, 1983.

[20] van Dyke M.: Higher approximation in boundary layer theory, Part I: General analysis, *J. Fluid Mech.*, **14** (1962), 161—177.