HIERARCHICAL ELEMENTS, LOCAL MAPPINGS AND THE *H-P* VERSION OF THE FINITE ELEMENT METHOD (II)*

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Abstract

This is the second half of the article. The rate of convergence for the h-p version with geometric meshes is discussed.

§ 3. C^0-compatible Local Mappings and Geometric Meshes

In this section we discuss two C^0-compatible local mappings and the geometric meshes which utilize these mappings.

3.1. The Bilinear Mapping

The simplest mapping which maps the standard element $D=[-1, 1]\times[-1, 1]$ to an arbitrary quadrilateral element E is the bilinear mapping

$$\begin{cases} x=a_1\xi+b_1\eta+c_1\xi\eta+d_1, \\ y=a_2\xi+b_2\eta+c_2\xi\eta+d_2. \end{cases} \tag{3.1.1}$$

Suppose the vertices of the quadrilaterals E and D are numbered counterclockwise as shown in Fig. 3.1.1:

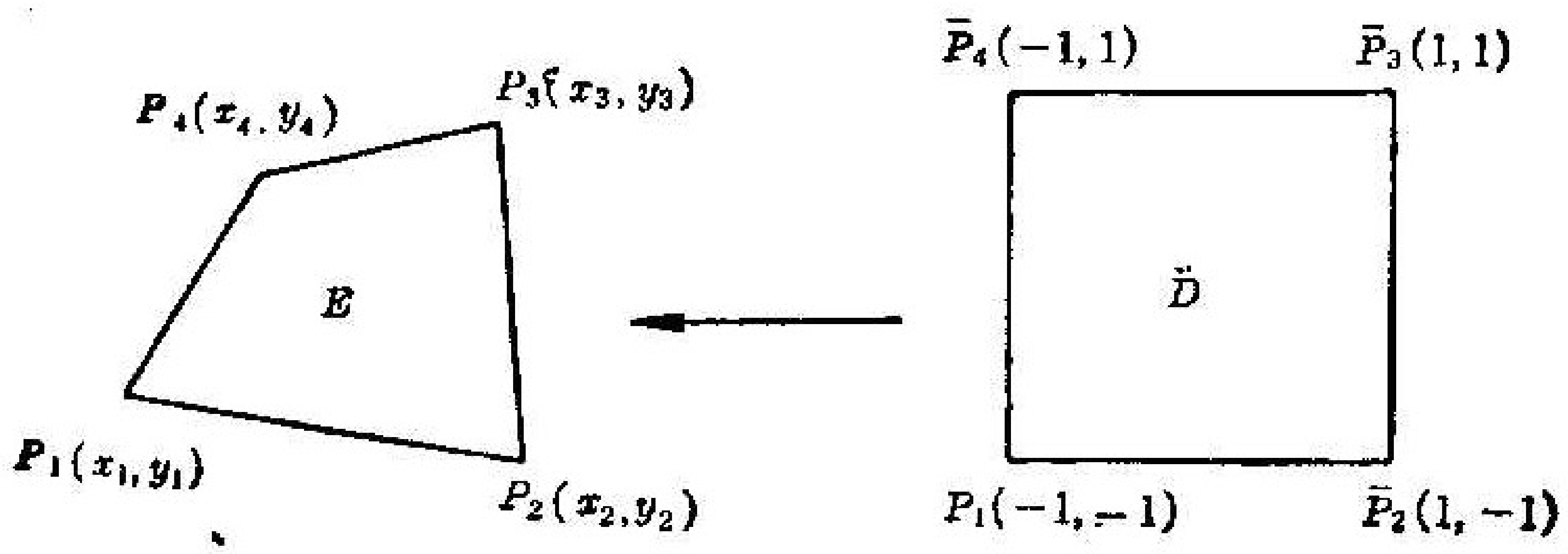

Fig. 3.1.1

Then we have

$$a_1=\frac{1}{4}(-x_1+x_2+x_3-x_4), \quad a_2=\frac{1}{4}(-y_1+y_2+y_3-y_4),$$

$$b_1=\frac{1}{4}(-x_1-x_2+x_3+x_4), \quad b_2=\frac{1}{4}(-y_1-y_2+y_3+y_4),$$

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$$c_1=\frac{1}{4}(x_1-x_2+x_3-x_4), \qquad c_2=\frac{1}{4}(y_1-y_2+y_3-y_4),$$

$$d_1=\frac{1}{4}(x_1+x_2+x_3+x_4), \qquad d_2=\frac{1}{4}(y_1+y_2+y_3+y_4). \tag{3.1.2}$$

The Jacobian of this mapping is

$$\frac{\partial(x, y)}{\partial(\xi, \eta)}=A\xi+B\eta+C, \tag{3.1.3}$$

where

$$A=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=\frac{1}{4}(S_{123}-S_{124}),$$

$$B=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=\frac{1}{4}(S_{134}-S_{124}),$$

$$C=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=\frac{1}{4}S_{1234}$$

in which S_{ijk} is the area of the triangle $P_iP_jP_k$, and S_{1234} is the area of the quadrilateral $E=P_1P_2P_3P_4$.

It is easy to show that the Jacobian evaluated at each vertex $\bar{P}_i$ of D equals half the area of the triangle which is determined by the corresponding vertex P_i of E with its two adjacent vertices. Thus we have (see (2.3.7) in [1])

$$C^2=\max\left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|=\frac{1}{2}\max\{S_{123}, S_{234}, S_{341}, S_{124}\}, \tag{3.1.4}$$

$$\delta^2=\min\left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|=\frac{1}{2}\min(S_{123}, S_{234}, S_{341}, S_{124}\}. \tag{3.1.5}$$

The only bilinear mapping with a constant Jacobian is the one which maps D to a parallelogram. In this case the mapping is

$$\begin{cases} x=a_1\xi+b_1\eta+d_1, \\ y=a_2\xi+b_2\eta+d_2 \end{cases} \tag{3.1.6}$$

with

$$\frac{\partial(x, y)}{\partial(\xi, \eta)}=C=\frac{1}{4}S_{1234}.$$

The simplest case that the mapping maps D to a rectangle was discussed in Theorem 2.3.2 (see [1]).

It is easy to show that the bilinear mapping on arbitrary quadrilateral meshes given by (3.1.1) is C^0-compatible.

3.2. The Polar Mapping

Using polar coordinates we can transform the polar net in (x, y)-plane to the rectangular net in (r, θ)-plane, then a linear mapping transforms the elements to the standard square. It is clear that this mapping is C^0-compatible.

The local mapping is the composition of the following:

$$\begin{cases} x=r\cos\theta, \\ y=r\sin\theta; \end{cases} \qquad \begin{cases} r=\dfrac{1}{2}(r_2-r_1)\xi+\dfrac{1}{2}(r_2+r_1), \\ \theta=\dfrac{1}{2}(\theta_2-\theta_1)\eta+\dfrac{1}{2}(\theta_2+\theta_1). \end{cases} \tag{3.2.1}$$

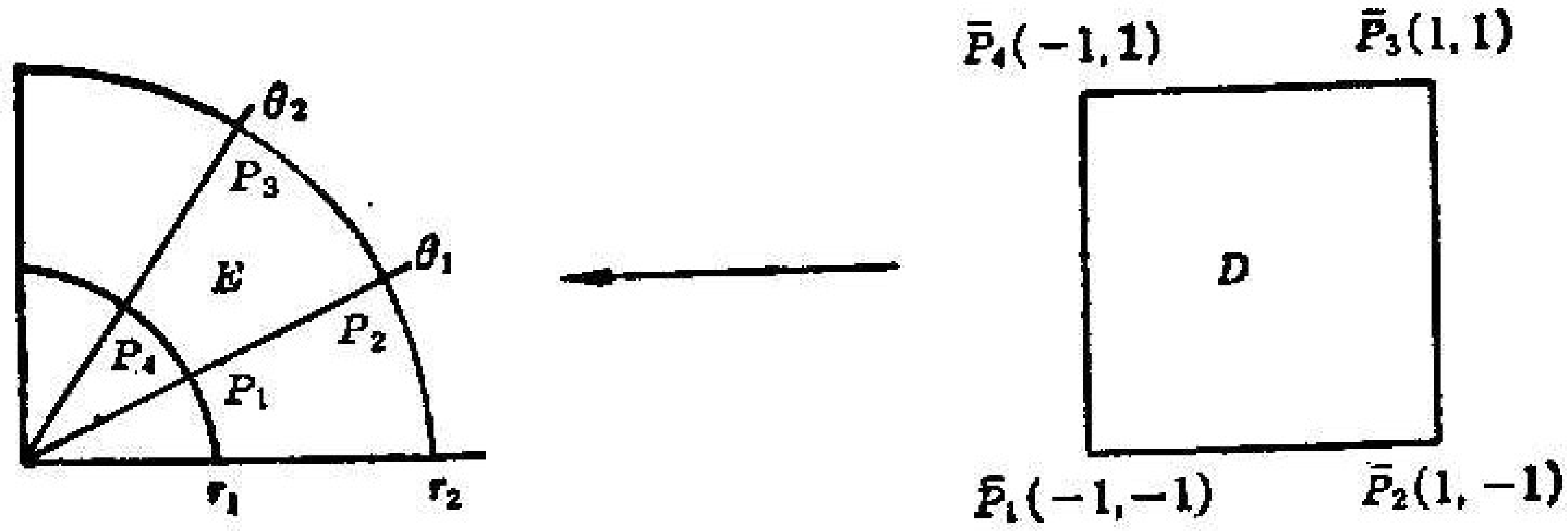

Fig. 3.2.1

The Jacobian is

$$\frac{\partial(x,y)}{\partial(\xi,\eta)}=\frac{r_2-r_1}{2}\,\frac{\theta_2-\theta_1}{2}\left[\frac{r_2-r_1}{2}\xi+\frac{r_2+r_1}{2}\right], \tag{3.2.2}$$

$$C=\frac{1}{4}(r_2-r_1)(\theta_2-\theta_1)\cdot r_2, \qquad \delta=\frac{1}{4}(r_2-r_1)(\theta_2-\theta_1)\cdot r_1. \tag{3.2.3}$$

3.3. The Geometric Meshes

We are interested in the finite element approximation to the functions which have singularities at the vertices of cornered domains. For simplicity we assume the domain $\Omega=[0,1]\times[0,1]$, and the function $u(x,y)$ has an α-type singularity at the vertex $(0,0)$, which can be expressed in the polar coordinates:

$$u(x,y)\equiv u(r,\theta)=r^{\alpha}g(\theta), \tag{3.3.1}$$

where $g(\theta)$ is analytic in θ with $|g(\theta)|\leqslant K$.

It should be pointed out that the singularity at a vertex of the solution for an elliptic problem on a cornered domain is related with the measure of the angle at the corner. Nevertheless, the idea can be applied to any angles.

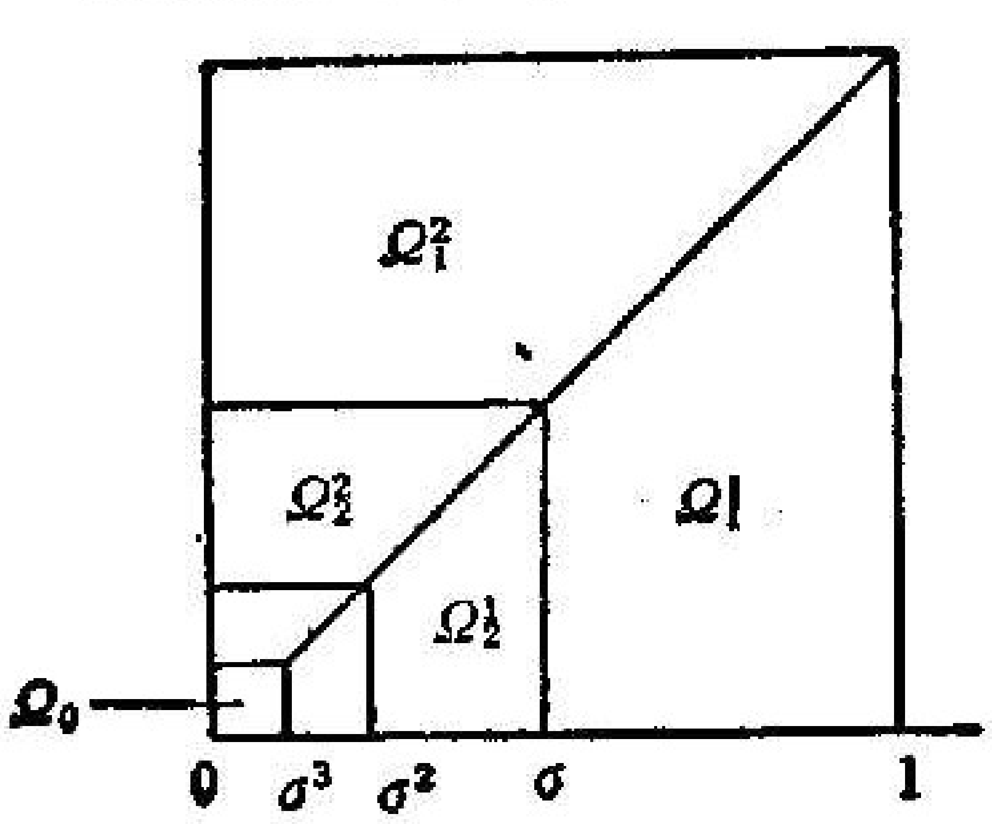

Fig. 3.3.1

We will discuss two kinds of geometric meshes.

The first kind is related with the bilinear mapping as shown by Fig. 3.3.1.

Let the elements be numbered as in the figure. The elements having the same sizes are said to be in one level, the number of levels is denoted by $m=m(\Delta)$. The levels are numbered from larger elements to smaller ones. Therefore the square Ω_0 at the corner of $(0,0)$ has the size σ^m. In our mesh the i-th level has two trapezoids Ω_i^1 and Ω_i^2. The typical element Ω_i^1 has the following vertices:

$$P_1(\sigma^i, 0),\ P_2(\sigma^{i-1}, 0),\ P_3(\sigma^{i-1}, \sigma^{i-1}),\ P_4(\sigma^i, \sigma^i). \tag{3.3.2}$$

By Section 3.1, the corresponding mapping for this element is

$$\begin{cases} x=\frac{1}{2}(1-\sigma)\sigma^{i-1}\left[\xi+\frac{1+\sigma}{1-\sigma}\right], \\ y=\frac{1}{4}(1-\sigma)\sigma^{i-1}\left[\xi+\frac{1+\sigma}{1-\sigma}\right](1+\eta). \end{cases} \tag{3.3.3}$$

Thus the Jacobian of the mapping is

$$\frac{\partial(x, y)}{\partial(\xi, \eta)}=\frac{1}{8}[(1-\sigma)\sigma^{i-1}]^2\left[\xi+\frac{1+\sigma}{1-\sigma}\right]. \tag{3.3.4}$$

And

$$\frac{1}{4}\sigma(1-\sigma)[\sigma^{i-1}]^2\leqslant\frac{\partial(x, y)}{\partial(\xi, \eta)}\leqslant\frac{1}{4}(1-\sigma)[\sigma^{i-1}]^2, \tag{3.3.5}$$

$$\left[\frac{\partial x}{\partial\xi}\right]^2+\left[\frac{\partial y}{\partial\xi}\right]^2=\left[\frac{(1-\sigma)\sigma^{i-1}}{4}\right]^2[4+(1+\eta)^2]\leqslant\frac{1}{2}[(1-\sigma)\sigma^{i-1}]^2, \tag{3.3.6}$$

$$\left[\frac{\partial x}{\partial\eta}\right]^2+\left[\frac{\partial y}{\partial\eta}\right]^2=\left[\frac{(1-\sigma)\sigma^{i-1}}{4}\right]^2\left[\xi+\frac{1+\sigma}{1-\sigma}\right]^2\leqslant\frac{1}{4}[\sigma^{i-1}]^2. \tag{3.3.7}$$

Moreover, function (3.3.1) becomes

$$U(\xi, \eta)=\left[\frac{(1-\sigma)\sigma^{i-1}}{2}\right]^\alpha\left[\xi+\frac{1+\sigma}{1-\sigma}\right]^\alpha[4+(1+\eta)^2]^{\frac{\alpha}{2}}G(\eta), \tag{3.3.8}$$

where

$$G(\eta)=g\left[\arctan\frac{y}{x}\right]=g\left[\arctan\frac{1+\eta}{2}\right]$$

which is analytic (with two isolated singularities at $-1\pm2i$).

By Lemma 2.3.1 (see [1]) we obtain

$$\|Du\|_{L_2(\Omega_i^1)\times L_2(\Omega_i^1)}\leqslant C_1(\sigma)\|DU\|_{L_2(D)\times L_2(D)}, \tag{3.3.9}$$

$$\|u\|_{L_2(\Omega_i^1)}\leqslant C_2(\sigma)\cdot\sigma^i\|U\|_{L_2(D)}, \tag{3.3.10}$$

where

$$C_1(\sigma)=\max\left[\sqrt{\frac{2(1-\sigma)}{\sigma}},\ \frac{1}{\sqrt{\sigma(1-\sigma)}}\right],$$

$$C_2(\sigma)=\frac{1}{2\sigma}\sqrt{1-\sigma}.$$

Note that C_1, C_2 only depend on σ.

The second kind of geometric mesh is related with the polar mapping (as shown by Fig 3.3.2). For simplicity we assume that the domain Ω is a sector of degree $\frac{\pi}{2}$.

The elements are arranged in levels with similar numbering as before. The number of levels is denoted by $m=m(\Delta)$. The typical element Ω_i^1 is related with the local mapping

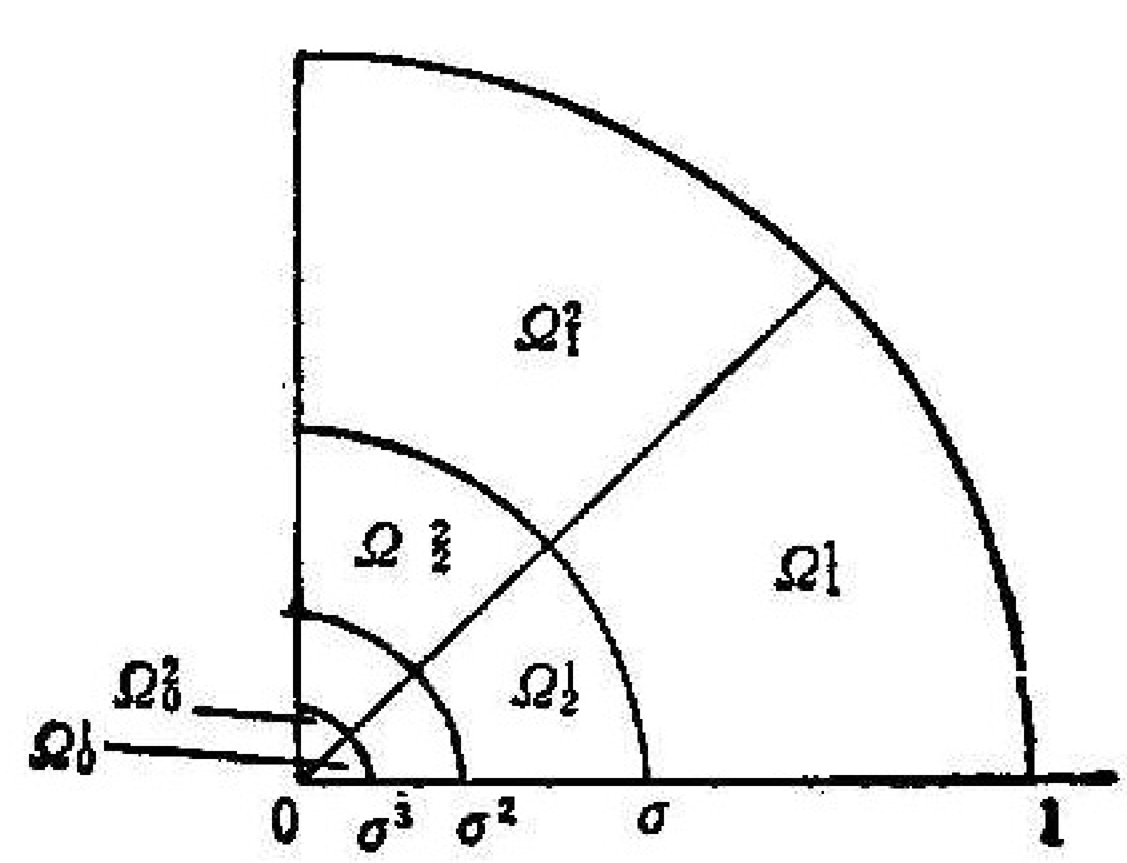

Fig. 3.3.2

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta; \end{cases} \quad \begin{cases} r = \dfrac{1}{2}(1-\sigma)\sigma^{i-1}\left[\xi + \dfrac{1+\sigma}{1-\sigma}\right], \\ \theta = \dfrac{\pi}{8}(1+\eta). \end{cases} \tag{3.3.11}$$

Thus the Jacobian of the mapping is

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\pi}{32}\left[(1-\sigma)\sigma^{i-1}\right]^2\left[\xi + \frac{1+\sigma}{1-\sigma}\right]. \tag{3.3.12}$$

And

$$\frac{\pi}{16}\sigma(1-\sigma)\left[\sigma^{i-1}\right]^2 \leqslant \frac{\partial(x, y)}{\partial(\xi, \eta)} \leqslant \frac{\pi}{16}(1-\sigma)\left[\sigma^{i-1}\right]^2, \tag{3.3.13}$$

$$\left[\frac{\partial x}{\partial \xi}\right]^2 + \left[\frac{\partial y}{\partial \xi}\right]^2 = \left[\frac{(1-\sigma)\sigma^{i-1}}{2}\right]^2, \tag{3.3.14}$$

$$\left[\frac{\partial x}{\partial \eta}\right]^2 + \left[\frac{\partial y}{\partial \eta}\right]^2 = \left[\frac{(1-\sigma)\sigma^{i-1}}{2}\right]^2\left[\frac{\pi}{8}\left[\xi + \frac{1+\sigma}{1-\sigma}\right]\right]^2 \leqslant \frac{\pi^2}{64}\left[\sigma^{i-1}\right]^2. \tag{3.3.15}$$

Moreover, function (3.3.1) becomes

$$U(\xi, \eta) = \left[\frac{(1-\sigma)\sigma^{i-1}}{2}\right]^{\alpha}\left[\xi + \frac{1+\sigma}{1-\sigma}\right]^{\alpha} G(\eta), \tag{3.3.16}$$

where

$$G(\eta) = g\left[\frac{\pi}{8}(\eta+1)\right]$$

which is analytic.

Again by Lemma 2.3.1 we obtain

$$\|Du\|_{L_2(\Omega_2^i)\times L_2(\Omega_2^i)} \leqslant C_1(\sigma)\|DU\|_{L_2(D)\times L_2(D)}, \tag{3.3.17}$$

$$\|u\|_{L_2(\Omega_2^i)} \leqslant C_2(\sigma)\cdot\sigma^i\|U\|_{L_2(D)}, \tag{3.3.18}$$

where C_1, C_2 only depend on σ.

§ 4. The *h*–*p* Version and Its Error Analysis

The convergence of the finite element method is obtained by increasing the number of degrees of freedom of the finite element space. There are three basic versions for the finite element method: the h–version, which only refines the mesh, the p–version, which fixes the mesh while increasing the degrees of elements, and the h–p version which refines the mesh and increases the degrees of elements concurrently.

The h–p version has been proven to have an exponential rate of convergence for a certain class of solutions (see [2], [3]). Here we will use the hierarchical elements with the local mappings to show this rate, but give a more detailed analysis. We will answer the questions about the optimal ratio for the geometric meshes and the best degree distributions.

4.1. Variable Degrees on 2–D Mesh Elements

If we use the C^0 hierarchical basis on the square (see Section 2, [1]) with local mappings to construct the 2–D C^0 hierarchical elements (on a curvilinear quadrilateral mesh), it is easy to see that the continuity on the common edges of adjacent mesh-elements can be obtained by choosing corresponding edge modes of the same degrees.

We discuss now the effects when different degrees of the modes are used. For simplicity, we assume that on one edge of the standard square D its degree is $p-1$, where p is the degree of the other edges and the internal modes. For example, we let the edge with lower degree be $\eta=-1$.

Denote the partial sum for the expansion of $u(\xi, \eta)$ by $u_s(\xi, \eta)$. Using the notation in Section 2.3 (see [1]) we then have

$$u_s(\xi, \eta)=u_{p,p}(\xi, \eta)-b_{p-1}Q_{p-1}(\xi)Q_0^{-1}(\eta) \tag{4.1.1}$$

with

$$b_{p-1}=\frac{2p-1}{2}\int_{-1}^{1}\frac{\partial u}{\partial \xi}(\xi,-1)P_{p-1}(\xi)d\xi.$$

Thus

$$\|u-u_s\|_{L_2(D)}\leqslant\|u-u_{p,p}\|_{L_2(D)}+\|b_{p-1}Q_{p-1}(\xi)Q_0^{-1}(\eta)\|_{L_2(D)}$$

$$=\|u-u_{p,p}\|_{L_2(D)}+\left\{\frac{1}{3\left(p-\frac{3}{2}\right)\left(p-\frac{1}{2}\right)\left(p+\frac{1}{2}\right)}\right\}^{\frac{1}{2}}|b_{p-1}|,$$

$$\|D_\xi(u-u_s)\|_{L_2(D)}\leqslant\|D_\xi(u-u_{p,p})\|_{L_2(D)}+\left\{\frac{2}{3\left(p-\frac{1}{2}\right)}\right\}^{\frac{1}{2}}|b_{p-1}|,$$

$$\|D_\eta(u-u_s)\|_{L_2(D)}\leqslant\|D_\eta(u-u_{p,p})\|_{L_2(D)}+\left\{\frac{1}{4\left(p-\frac{3}{2}\right)\left(p-\frac{1}{2}\right)\left(p+\frac{1}{2}\right)}\right\}^{\frac{1}{2}}|b_{p-1}|.$$

By these estimates one can show that there are no essential changes in the rate of convergence if only a few terms are adjusted on the edges.

4.2. Some Auxiliary Results

We now consider the error estimates for the following functions:

$$U(\xi)=\left[\xi+\frac{1+\sigma}{1-\sigma}\right]^{\alpha}, \tag{4.2.1}$$

$$V(\eta)=[4+(\eta+1)^2]^{\frac{1}{2}\alpha}g\left(\arctan\frac{\eta+1}{2}\right), \tag{4.2.2}$$

where $g(z)$ is analytic.

Lemma 4.2.1. For $\varepsilon\leqslant\sigma\leqslant1-\varepsilon$, $\varepsilon>0$, we have

$$\|U'-U_p'\|_{L_2(-1,1)}\leqslant C(\alpha, \varepsilon)p^{-\alpha}r^{p}, \tag{4.2.3}$$

$$\|U-U_p\|_{L_2(-1,1)}\leqslant C(\alpha, \varepsilon)\, p^{-\alpha-1}r^{p} \tag{4.2.4}$$

with

$$r=\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}. \tag{4.2.5}$$

Proof. It follows from (1.4.9) by setting $\mu=\frac{1+\sigma}{1-\sigma}$ in (1.4.6) (see [1]).

Lemma 4.2.2. For $\varepsilon>0$, $r_0=0.2168\cdots$, there is a constant $C=C(\alpha, g, \varepsilon)$ such that

$$\|V'-V'_p\|_{L_2(-1,1)}\leqslant C(r_0+\varepsilon)^p, \tag{4.2.6}$$

$$\|V-V_p\|_{L_2(-1,1)}\leqslant Cp^{-1}(r_0+\varepsilon)^p. \tag{4.2.7}$$

Proof. Note that $V(z)$ is analytic with isolated singularities at $z=-1\pm 2i$. The ellipse with foci ± 1 and passing through the singularities has a half major axis $a=\sqrt{2}+1=2.4142\cdots$ and a half minor axis $b=(2\sqrt{2}+2)^{\frac{1}{2}}=2.1973\cdots$. Thus $\rho=a+b=4.6115\cdots$ and (4.2.6) follows from Lemma 1.4.1 (see [1]) with $r_0=\rho^{-1}=0.2168453354\cdots$. ▮

The next lemma is concerned with the approximation to the function which is of the form $U(x)V(y)$.

Lemma 4.2.3. If $u(x, y)=U(x)V(y)$ with $U, V\in H^1(-1, 1)$, then the polynomial $u_{p,q}(x, y)$ defined in (2.3.1) is of the form

$$u_{p,q}(x, y)=U_p(x)\cdot V_q(y), \tag{4.2.8}$$

where $U_p(x)$ and $V_q(y)$ are the $1-D$ partial sums defined by (1.3.5) (see [1]). Moreover, we have

$$\begin{aligned}\|Du-Du_{p,q}\|_{L_2(D)\times L_2(D)} &\leqslant\|V\|\cdot\|U'-U'_p\|+\|U\|\cdot\|V'-V'_q\|\\ &+\|V'\|\cdot\|U-U_p\|+\|U'\|\cdot\|V-V_q\|\\ &+\|V-V_p\|\cdot\|U'-U'_p\|+\|U-U_p\|\cdot\|V'-V'_q\|,\end{aligned} \tag{4.2.9}$$

$$\|u-u_{p,q}\|_{L_2(D)}\leqslant\|V\|\cdot\|U-U_p\|+\|U\|\cdot\|V-V_q\|+\|U-U_p\|\cdot\|V-V_q\|, \tag{4.2.10}$$

where $\|\cdot\|$ *on the right-hand sides stands for the norm of* $L_2(-1, 1)$.

Proof. These relations follow from straight forward calculations. ▮

4.3. The h-p Version using the Geometric Mesh of the First Kind

We now give the error analysis for the solution of the form $r^\alpha g(\theta)$ when using the geometric mesh of first kind shown by Fig. 3.3.1. Let C be a generic constant which is independent of some quantities known easily by the context. Let u_s be the mapped function; on each specified element Ω^j_i it is obtained by the corresponding local mapping from a polynomial $u_{p_{i,j},q_{i,j}}(\xi, \eta)=U_{p_{i,j}}(\xi)\cdot V_{q_{i,j}}(\eta)$. Denote $e=u-u_s$.

As mentioned in 4.1, we can adjust the degrees on the edges to ensure the C^0 continuity. This will not affect the order of convergence in terms of the total number of degrees of freedom. Therefore, in the following we will simply consider different degrees on the elements.

First, the total error is denoted by

$$\begin{aligned}E(N)^2&=\|De\|^2_{L_2(\Omega)\times L_2(\Omega)}\\ &=\|De\|^2_{L_2(\Omega_0)\times L_2(\Omega_0)}+\sum_{i=1}^{m}\sum_{j=1}^{2}\|De\|^2_{L_2(\Omega^j_i)\times L_2(\Omega^j_i)}\\ &=E_0^2+\sum_{i=1}^{m}\sum_{j=1}^{2}E^2_{i,j}.\end{aligned} \tag{4.3.1}$$

On Ω_0 we use a very rough estimate which will not change the order. It is easy to see

$$\left|\frac{\partial u}{\partial x}\right|,\quad \left|\frac{\partial u}{\partial y}\right|<Cr^{\alpha-1}.$$

Thus

$$E_0^2 \leqslant C \int_0^{\sigma^m} \int_0^{\sigma^m} r^{2\alpha-2}\, dx\, dy \leqslant C\sigma^{2\alpha m}. \tag{4.3.2}$$

From (3.3.8), (3.3.9), Lemmas 4.2.1, 4.2.2 and 4.2.3, it follows that

$$E_{i,j}(p_{ij}, q_{ij}) \leqslant C(\varepsilon)\,[r^{p_{ij}} + (r_0 - \varepsilon)^{q_{ij}}]\,\sigma^{\alpha(i-1)} \tag{4.3.3}$$

with $\varepsilon > 0$, $r = \dfrac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}$, $r_0 = 0.2168\cdots$.

From (4.3.1), (4.3.2) and (4.3.3) we see first that the degrees on the same level should be equal:

$$p_{i1} = p_{i2} \equiv p_i, \quad q_{i1} = q_{i2} \equiv q_i.$$

If we choose varied degrees on different levels, the best choice is to let (for simplicity we write r_0 instead of $(r_0 - \varepsilon)$):

$$r^{p_i} = r_0^{q_i} = \sigma^{\alpha(m-i+1)}.$$

Thus

$$p_i = \frac{\alpha \ln \sigma}{\ln r}(m-i+1),$$

$$q_i = \frac{\alpha \ln \sigma}{\ln r_0}(m-i+1).$$

The number of degrees of freedom is

$$N \sim \frac{2\alpha^2 (\ln \sigma)^2}{\ln r \cdot \ln r_0} \frac{m^3}{3}.$$

Therefore

$$E(N) \leqslant C\sqrt{m}\,\sigma^{\alpha m} \leqslant C N^{\frac{1}{6}} e^{-[\frac{1}{2}\ln r \cdot \ln \sigma \cdot |\ln r_0|]^{\frac{1}{3}} (3\alpha N)^{\frac{1}{3}}}.$$

As proved by R. DeVore and K. Scherer in [4], the function $\ln r \cdot \ln \sigma$ as a function of σ has a unique maximum at $\sigma = (\sqrt{2} - 1)^2 = 0.1715\cdots$. If we let σ take this value, then

$$p_i = 2\alpha \cdot (m-i+1),$$

$$q_i = 1.1532\alpha \cdot (m-i+1),$$

$$E(N) \leqslant C e^{-(1.058983169\cdots - \varepsilon)(3\alpha N)^{\frac{1}{3}}} \leqslant C e^{-(1.527245908\cdots - \varepsilon)(\alpha N)^{\frac{1}{3}}}.$$

If we choose the same p and q on different levels (p may differ from q), then we will have

$$E(N)^2 \leqslant C(\varepsilon)\,[\sigma^{2\alpha m} + r^{2p} + (r_0 - \varepsilon)^{2q}].$$

In this case the best choice is

$$r^p = r_0^q = \sigma^{\alpha m}.$$

Thus

$$p = \frac{\alpha \ln \sigma}{\ln r} m,$$

$$q = \frac{\alpha \ln \sigma}{\ln r_0} m.$$

The number of degrees of freedom is

$$N\sim\frac{2\alpha^2(\ln\sigma)^2}{\ln r\cdot\ln r_0}\,m^3.$$

Therefore

$$E(N)\leqslant C\sigma^{\alpha m}\leqslant Ce^{-\left[\frac{1}{2}\ln r\cdot\ln\sigma\cdot|\ln r_0|\right]^{\frac{1}{3}}[\alpha N]^{\frac{1}{3}}}.$$

If we take the best choice of $\sigma=(\sqrt{2}-1)^2=0.1715\cdots$, then

$$p=2\alpha\cdot m,$$

$$q=1.1532\alpha\cdot m,$$

$$E(N)\leqslant Ce^{-(1.058933169\cdots-\varepsilon)(\alpha N)^{\frac{1}{3}}}.$$

If we let the degrees be uniform on all elements and $q=p$, then we will have

$$E(N)^2\leqslant C(\varepsilon)\left[\sigma^{2\alpha m}+r^{2p}+(r_0-\varepsilon)^{2p}\right].$$

In this case the best choice is

$$r^p=r_0^p=\sigma^{\alpha m}.$$

Thus $r=r_0$; this gives

$$\sigma=\left[\frac{1-r_0}{1+r_0}\right]^2=\sqrt{2}-1=0.4142135622\cdots,$$

$$p=\frac{\alpha\ln\sigma}{\ln r_0}\,m=0.5766\alpha\cdot m.$$

The number of degrees of freedom is

$$N\sim\frac{2\alpha^2(\ln\sigma)^2}{(\ln r_0)^2}\,m^3.$$

Therefore

$$E(N)\leqslant C\sigma^{\alpha m}\leqslant Ce^{-\left[\frac{1}{2}|\ln\sigma|\cdot(\ln r_0)^2\right]^{\frac{1}{3}}[\alpha N]^{\frac{1}{3}}}.$$

Hence

$$E(N)\leqslant Ce^{-(1.009796220\cdots-\varepsilon)(\alpha N)^{\frac{1}{3}}}.$$

These results are summarized in the following theorem:

Theorem 4.3.1. *Let the solution $u(x, y)$ take the form $r^\alpha g(\theta)$ with $g(\theta)$ analytic. Suppose $\Omega=[0, 1]\times[0, 1]$ is the domain of the problem. Then the h–p version which utilizes the geometric mesh of the first kind* (Fig. 3.3.1) *with ratio σ and the number of levels, m, and which adopts the bilinear mapping* (Section 3.1) *to construct C^0 hierarchical elements, gives an exponential rate of convergence*

$$E(N)\leqslant C(\alpha, g, \varepsilon)e^{-(\gamma-\varepsilon)\sqrt[3]{\alpha N}}, \tag{4.3.4}$$

where $E(N)$ is the error in energy norm for the finite element approximation with N as the number of degrees of freedom, $\varepsilon>0$. The constant $\gamma>0$ depends on the choice of the mesh and degrees of elements:

1° *If varied degrees are allowed, the degrees should be arranged in the following way: the elements in the same level i have the same degree p_i (the radial direction) and*

q_i *(the crossing direction) which are linearly decreasing down to the origin. The best choice then is*

$$\sigma=0.1715\cdots,$$
$$p_i=s_p\alpha\cdot(m-i),\quad s_p=2,$$
$$q_i=s_q\alpha\cdot(m-i),\quad s_q=1.1532\cdots.$$

In this case

$$\gamma=1.5272\cdots.$$

2° *If the degrees are arranged uniformly on all levels, but $p\neq q$ is allowed, then the best choice is*

$$\sigma=0.1715\cdots,$$
$$p=s_p\alpha\cdot m,\quad s_p=2,$$
$$q=s_q\alpha\cdot m,\quad s_q=1.1532\cdots.$$

In this case

$$\gamma=1.0589\cdots.$$

3° *If the degrees are arranged uniformly on all levels with $p=q$, then the best choice is*

$$\sigma=0.4142\cdots,$$
$$p=s\alpha\cdot m,\quad s=0.5766\cdots.$$

In this case

$$\gamma=1.0097\cdots.$$

4.4. The h-p Version using the Geometric Mesh of the Second Kind

Recall that the geometric mesh of the second kind is given by Fig. 3.3.2 on the sectorial domain $0\leqslant r\leqslant 1$, $0\leqslant\theta\leqslant\pi/2$. We will apply the corresponding notations as used in the last section.

To be specific, we let $u(r,\theta)=r^\alpha\cos(\alpha\theta)\,(\alpha>0)$. The singularities of the solutions on cornered domains usually have this form. On the standard element we have now (see (3.3.16)):

$$U(\xi,\eta)=\left[\frac{(1-\sigma)\sigma^{i-1}}{2}\right]^\alpha\left[\xi+\frac{1+\sigma}{1-\sigma}\right]^\alpha G(\eta)\qquad(4.4.1)$$

with
$$G(\eta)=\cos\left[\frac{\pi}{8}(\eta+1)\right].$$

Lemma 4.4.1. *For $G(\eta)$ defined in (4.4.1) we have*

$$\|G'-G'_p\|_{L_2(D)}\leqslant\frac{\sqrt{4p+2}}{(2p+1)!!}\left[\frac{\pi}{8}\right]^{p+1}.\qquad(4.4.2)$$

Proof. By (1.3.13) we have

$$\|G'-G'_p\|^2_{L_2(D)}\leqslant\frac{1}{(2p)!}\int_{-1}^{1}(1-\eta^2)^p\left[\left[\cos\frac{\pi}{8}(\eta+1)\right]^{(p+1)}\right]^2d\eta$$
$$\leqslant\frac{1}{(2p)!}\left[\frac{\pi}{8}\right]^{2p+2}\frac{2\cdot(2p)!!}{(2p+1)!!}\leqslant\frac{2(2p+1)}{[(2p+1)!!]^2}\left[\frac{\pi}{8}\right]^{2p+2}.$$

Remark 4.4.1. By Stirling's formula we have as $p\to\infty$

$$\frac{\sqrt{4p+2}}{(2p+1)!!}\left[\frac{\pi}{8}\right]^{p+1}\sim\frac{\pi}{4\sqrt{2p}}\left[\frac{\beta}{p}\right]^{p} \tag{4.4.3}$$

with

$$\beta=\frac{\pi e}{16}=0.5337\cdots.$$

Thus we have a simpler estimates

$$\|G'-G'_p\|_{L_2(D)}\leqslant C\left[\frac{\beta}{p}\right]^{p}. \tag{4.4.4}$$

A little different treatment is that there are two sectors Ω_0^1 and Ω_0^2 where linear interpolation will be used to construct the approximate function. The corresponding errors are denoted by E_{01} and E_{02} resp. Thus the total error is

$$E(N)^2=\sum_{i=0}^{m}\sum_{j=1}^{2}E_{i,j}^2. \tag{4.4.5}$$

On Ω_0^i we still have

$$E_{0i}\leqslant C\sigma^{\alpha m},\quad i=1,2. \tag{4.4.6}$$

From (4.4.1), (3.3.17), (4.4.4), Lemmas 4.2.1 and 4.2.3 it follows that

$$E_{i,j}(p_{ij},q_{ij})\leqslant C\left[r^{p_{ij}}+\left[\frac{\beta}{q_{ij}}\right]^{q_{ij}}\right]\sigma^{\alpha(i-1)} \tag{4.4.7}$$

with $r=\dfrac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}$, $\beta=0.5337\cdots$.

From (4.4.5), (4.4.6) and (4.4.7) we can also let the degrees on the same level be equal:

$$p_{i1}=p_{i2}\equiv p_i,\quad q_{i1}=q_{i2}\equiv q_i.$$

If we choose varied degrees on different levels, then the best choice is

$$r^{p_i}=\left[\frac{\beta}{q_i}\right]^{q_i}=\sigma^{\alpha(m-i+1)}. \tag{4.4.8}$$

Thus

$$p_i=\frac{\alpha\ln\sigma}{\ln r}(m-i+1).$$

Write $q_i=\beta y$ and $x=\dfrac{\alpha}{\beta}\ln\sigma^{-1}\cdot(m-i+1)$; then (4.4.8) gives $e^x=y^y$. Thus

$$y=\frac{x}{\ln x}\left[1+\frac{\ln(\ln y)}{\ln y}\right]=\frac{x}{\ln x}(1+o(1))$$

as $x\to\infty$. Therefore

$$q_i=\alpha\ln\sigma^{-1}\frac{m-i+1}{\ln(m-i+1)}(1+o(1))$$

as $m\to\infty$. The number of degrees of freedom is

$$N\sim\frac{2\alpha^2(\ln\sigma)^2}{-\ln r}\frac{m^3}{3\ln m}.$$

Therefore

$$m^3 \sim \frac{-\ln r}{2\alpha^2(\ln\sigma)^2}\,\frac{3N\ln N}{3},$$

$$E(N) \leqslant C\sqrt{m}\,\sigma^{\alpha m} \leqslant C(N\ln N)^{\frac{1}{6}} e^{-\left[\frac{1}{6}\ln r\cdot\ln\sigma\right]^{\frac{1}{3}}[3\alpha N\cdot\ln N]^{\frac{1}{3}}}.$$

Again for $\sigma=(\sqrt{2}-1)^2=0.1715\cdots$ we will achieve the best rate of convergence. In this case

$$p_i = 2\alpha\cdot(m-i+1),$$

$$q_i \approx 1.7627\alpha\,\frac{m-i+1}{\ln(m-i+1)},$$

$$E(N) \leqslant Ce^{-(0.6873817184\cdots-\varepsilon)(3\alpha N\cdot\ln N)^{\frac{1}{3}}}$$

$$\leqslant Ce^{-(0.9192635095\cdots-\varepsilon)(\alpha N\cdot\ln N)^{\frac{1}{3}}}.$$

If we choose the same p and q on different levels (p may differs from q), then we will have

$$E(N)^2 \leqslant C\left[\sigma^{2\alpha m} + r^{2p} + \left[\frac{\beta}{q}\right]^{2q}\right].$$

Let

$$r^p = \left[\frac{\beta}{q}\right]^q = \sigma^{\alpha m}.$$

Then

$$p = \frac{\alpha\ln\sigma}{\ln r}\,m$$

and

$$q = \alpha\ln\sigma^{-1}\,\frac{m}{\ln m}(1+o(1)), \quad m\to\infty.$$

The number of degrees of freedom is

$$N \sim \frac{2\alpha^2(\ln\sigma)^2}{-\ln r}\,\frac{m^3}{\ln m}.$$

Therefore

$$m^3 \sim \frac{-\ln r}{2\alpha^2(\ln\sigma)^2}\,\frac{N\ln N}{3},$$

$$E(N) \leqslant C\sigma^{\alpha m} \leqslant Ce^{-\left[\frac{1}{6}\ln r\cdot\ln\sigma\right]^{\frac{1}{3}}[\alpha N\cdot\ln N]^{\frac{1}{3}}}.$$

If we take the best choice of σ, i.e. $\sigma=(\sqrt{2}-1)^2=0.1715\cdots$,

$$p = 2\alpha\cdot m,$$

$$q \approx 1.7627\alpha\,\frac{m}{\ln m},$$

$$E(N) \leqslant Ce^{-0.687381784\cdots(\alpha N\cdot\ln N)^{\frac{1}{3}}}.$$

At last, if we let the degrees be uniform on all elements and $q=p$, then we will have

$$E(N)^2 \leqslant C\left[\sigma^{2\alpha m}+r^{2p}+\left[\frac{\beta}{p}\right]^{2p}\right].$$

Since $\left[\frac{\beta}{p}\right]^p$ diminishes with higher order than r^p does, it can be neglected. In this case the best choice is

$$r^p=\sigma^{\alpha m}.$$

Thus

$$p=\frac{\alpha \ln \sigma}{\ln r}\, m.$$

The number of degrees of freedom is

$$N \sim \frac{2\alpha^2(\ln \sigma)^2}{(\ln r)^2}\, m^3.$$

Therefore

$$E(N) \leqslant C\sigma^{\alpha m} \leqslant Ce^{-\left[\frac{1}{2}|\ln \sigma|\cdot(\ln r)^2\right]^{\frac{1}{3}}(\alpha N)^{\frac{1}{3}}}.$$

Since function $|\ln \sigma|\cdot(\ln r)^2 (0<\sigma<1)$ has a unique maximum at

$$\sigma=0.5717172496\cdots,$$

it yields

$$p=s\alpha\cdot p$$

with

$$s=0.2832108839\cdots$$

and

$$E(N) \leqslant Ce^{-1.028998009(\alpha N)^{\frac{1}{3}}}.$$

These results are summarized in the following theorem:

Theorem 4.4.1. *If the solution $u(x, y)$ is of the form $r^\alpha \cos(\alpha\theta)$ and if $\Omega=[0\leqslant r\leqslant 1]\times\left[0\leqslant\theta\leqslant\frac{\pi}{2}\right]$ is the domain of the problem, then the h–p version, which utilizes the geometric mesh of the second kind* (Fig. 3.3.2) *with ratio σ and the number of levels, m, and adopts the polar mapping* (Section 3.2) *to construct C^0-hierarchical elements, gives an exponential rate of convergence.*

Let $E(N)$ be the error in energy norm for the finite element approximation with N as the number of degrees of freedom.

1° *If varied degrees are allowed, the degrees should be arranged in the following way: the elements in the same level i have the same degree p_i (the radial direction) and q_i (the crossing direction) which are linearly decreasing to the origin. The best choice is*

$$\sigma=0.1715\cdots,$$

$$p_i=s_p\alpha\cdot(m-i+1), \quad s_p=2,$$

$$q_i=s_q\alpha\frac{m-i+1}{\ln(m-n+1)}, \quad s_q=1.7627\cdots.$$

In this case we have

$$E(N) \leqslant Ce^{-0.9192(\alpha N \cdot \ln N)^{\frac{1}{3}}}. \tag{4.4.9}$$

2° *If the degrees are arranged uniformly on all levels, but it is allowed that $p \neq q$, then the best choice is*

$$\sigma = 0.1715\cdots,$$
$$p = s_p \alpha \cdot m, \quad s_p = 2,$$
$$q = s_q \alpha \frac{m}{\ln m}, \quad s_q = 1.7627\cdots.$$

In this case

$$E(N) \leqslant Ce^{-0.6373(\alpha N \cdot \ln N)^{\frac{1}{3}}}. \tag{4.4.10}$$

3° *If the degrees are arranged uniformly on all levels with $p = q$, then the best choice is*

$$\sigma = 0.5717\cdots,$$
$$p = s_p \alpha \cdot m, \quad s_p = 0.2832\cdots.$$

In this case

$$E(N) \leqslant Ce^{-1.0289(\alpha N)^{\frac{1}{3}}}. \tag{4.4.11}$$

Remark 4.4.2. As seen above, when choosing variable degrees one can achieve a rate of convergence higher than $\exp(-\gamma\sqrt[3]{\alpha N})$. In fact if $u(r, \theta) = r^{\alpha}$, then the rate of convergence can be $\exp(-\gamma\sqrt{\alpha N})$ since degree q could be bounded. In all the cases the best choice of σ is independent of α, the strength of the singularity.

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