FIXED POINT METHODS FOR THE COMPLEMENTARITY PROBLEM*1)

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Abstract

This paper is concerned with iterative procedures for the monotone complementarity problem. Our iterative methods consist of finding fixed points of appropriate continuous maps. In the case of the linear complementarity problem, it is shown that the problem is solvable if and only if the sequence of iterates is bounded in which case summability methods are used to find a solution of the problem. This procedure is then used to find a solution of the nonlinear complementarity problem satisfying certain regularity conditions for which the problem has a nonempty bounded solution set.

§ 1. Introduction

We are concerned in this paper with the complementarity problem, viz., that of finding a $z_0 \geqslant 0$ such that $F(z_0) \geqslant 0$ and such that $z_0^T F(z_0) = 0$. Here F is an operator from $\mathfrak{R}^n$ to $\mathfrak{R}^n$. In particular, we are concerned with the case when F is monotone, that is

$$(x-y)^T(F(x)-F(y)) \geqslant 0, \quad \forall x, y \in \mathfrak{R}^n.$$

The operator F is strongly monotone if there exists a positive real number λ such that

$$(x-y)^T(F(x)-F(y)) \geqslant \lambda \|x-y\|^2.$$

When F is an affine map, $F(x) = Mx + q$, we shall refer to the complementarity problem as the linear complementarity problem and write LCP(M, q) in this case. Otherwise we shall refer to it as the nonlinear complementarity problem and write NLCP(F). Clearly when F is affine and monotone, M is positive semidefinite.

In the case of LCP(M, q), when M is positive semidefinite, if the problem is feasible, that is there exists $x \geqslant 0$ such that $Mx + q \geqslant 0$, the problem is solvable [Eaves, 1971]. This is not the case for NLCP(F) ([Megiddo, 1977], [Garcia, 1977]). However, for $\varepsilon > 0$, if we consider the Tihonov regularization $F_\varepsilon := F + \varepsilon I$, then the corresponding problem NLCP(F_ε) has a unique solution since F_ε is strongly monotone [Karamardian, 1972]. When $\varepsilon \to 0$, x_ε converges to the least two-norm solution of NLCP(F), provided it is solvable [Brézis, 1973].

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$$x \mapsto (x-F(x))_+ := \max\{0,\ x-F(x)\}.$$

The principal aim of this paper is to consider iterative procedures to find such fixed points. We shall show that in the linear case the sequence of iterates is bounded if and only if LCP$(M,\ q)$ is solvable. When this is the case, we use summability methods to obtain a solution of the problem. Although feasibility of the monotone NLCP(F) does not imply its solvability, it is a theorem of Mangasarian and McLinden [1985] that when a regularity condition such as the distribute Slater constraint qualification is satisfied, then, in this case, the solution set is bounded. We show how the iterative procedure for the linear case may be adapted to find a solution in this special case.

We briefly describe the notation used in this paper. We use $\Re^n$ for the space of real ordered n-tuples. All vectors are column vectors and we use the Euclidean norm throughout. Given a vector x, we denote its i^{th} component by x_i. We say $x \geqslant 0$ if $x_i \geqslant 0 \forall i$. The nonnegative orthant is denoted by $\Re^n_+$.

We use superscripts to distinguish between vectors, e.g. x^1, x^2 etc. For x, $y \in \Re^n$, x^T indicates the transpose of x, $x^T y$ their inner product. Occasionally, the superscript T will be suppressed. All matrices are indicated by upper case letters A, B, C, etc. The i^{th} row of A is denoted by A_i while its j^{th} column is denoted by A_j. The transpose of A is denoted by A^T.

Given NLCP(F), we define the feasible set and solution set by $S(F)$ and $\bar{S}(F)$ respectively, that is,

$$S(F) = \{x \in \Re^n_+ :\ F(x) \in \Re^n_+\},$$

$$\bar{S}(F) = \{x \in S(F) :\ x^T F(x) = 0\}.$$

In the case of LCP$(M,\ q)$, we shall denote these sets by $S(M,\ q)$ and $\bar{S}(M,\ q)$ respectively. Finally the end of a proof is signified by ▮.

§ 2. Fixed Point Methods

We begin with the well known notion of a contraction mapping.

2.1. Definition. *Let P: $D \subseteq \Re^n \to \Re^n$. We say P is Lipschitzian with modulus $L > 0$ if*

$$\|P(x) - P(y)\| \leqslant L\|x-y\|, \quad \forall x,\ y \in D. \tag{2.1}$$

When $L \leqslant 1$ ($L < 1$) we say P is non-expansive (contractive).

The following theorem is classical; see e.g., [Ortega and Rheinboldt, 1970, page 120].

2.2. Theorem. (*Banach's contraction mapping principle*). *Let P: $D \subseteq \Re^n \to \Re^n$, D_0 a closed subset of D such that $PD_0 = \{P(x):\ x \in D_0\} \subseteq D_0$. If P is a contraction mapping on D_0 with modulus L, then P has a unique fixed point $\bar{x}$ in D_0. Further, for any point x^0 in D_0, the sequence $\{x^k\}$, where $x^{k+1} = P(x^k)$, converges to $\bar{x}$ with the following linear rate:*

$$\frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} \leqslant L. \tag{2.2}$$

The content of the following proposition is well known. We state it in the following form for later use and furnish a proof for the sake of completeness.

2.3. Proposition. Let F: $D\subseteq\Re^n\to\Re^n$ be monotone and Lipschitzian with modulus L. Suppose that $\varepsilon>0$, $\alpha>0$ and $\varepsilon\alpha\leqslant1$. Then the projection map $\mathbb{P}$ defined by

$$\mathbb{P}(x)=\{x-\alpha(F(x)+\varepsilon x)\}_+,\quad x\in D \tag{2.3}$$

is also Lipschitzian with modulus $k(\alpha)=\sqrt{(1-\alpha\varepsilon)^2+(\alpha L)^2}$. If $\alpha<2\varepsilon/\sqrt{\varepsilon^2+L^2}$, then $\mathbb{P}$ is a contraction and k attains its minimum value

$$k_{\min}(\alpha)=L/\sqrt{L^2+\varepsilon^2}\ \text{ for } \alpha=\varepsilon/L^2+\varepsilon^2.$$

Proof. We have

$$\begin{aligned}\|\mathbb{P}(x)-\mathbb{P}(y)\|^2&=\|\{x-\alpha(F(x)+\varepsilon x)\}_+-\{y-\alpha(F(y)+\varepsilon y)\}_+\|^2\\ &\leqslant\|\{x-\alpha(F(x)+\varepsilon x)\}-\{y-\alpha(F(y)+\varepsilon y)\}\|^2\end{aligned}$$

since projection on $\Re^n_+$ is non-expansive. Hence,

$$\begin{aligned}\|\mathbb{P}(x)-\mathbb{P}(y)\|^2&\leqslant\|(x-y)(1-\varepsilon\alpha)-\alpha(F(x)-F(y))\|^2\\ &=\|x-y\|^2(1-\alpha\varepsilon)^2+\alpha^2\|F(x)-F(y)\|^2\\ &\quad-2\alpha(1-\alpha\varepsilon)(x-y)(F(x)-F(y)).\end{aligned}$$

Since $\alpha\varepsilon\leqslant1$ and $\langle F(x)-F(y),\ x-y\rangle\geqslant0$ from the monotonicity of F,

$$\|\mathbb{P}(x)-\mathbb{P}(y)\|^2\leqslant\|x-y\|^2\{(1-\alpha\varepsilon)^2+(\alpha L)^2\}.$$

The other claims about $k(\alpha)$ are easy to verify. ▌

2.4. Theorem. *Let F: $\Re^n\to\Re^n$ be monotone and Lipschitzian with modulus L. Let $\{\varepsilon_n\}$ be a sequence of positive reals, $\varepsilon_n\downarrow0$. For $n=1, 2, \cdots$ let*

$$\mathbb{P}_n(x)=\{x-\alpha_n(F(x)+\varepsilon_n x)\}_+$$

and for $m=1, 2, \cdots$ and $x\in\Re^n$ let

$$\mathbb{P}_n^m(x)=\underbrace{\mathbb{P}_n\circ\cdots\circ\mathbb{P}_n}_{m\ \text{times}}(x)=x(n,\ m).$$

Suppose further that

$$\alpha_n=\frac{\varepsilon_n}{\varepsilon_n^2+L^2},\ k_n=\frac{L}{\sqrt{L^2+\varepsilon_n^2}},\ \delta_n=\varepsilon_n(1-k_n).$$

For $n=1, 2, \cdots$, let $\bar{x}^n$ be defined by

$$\bar{x}^n=x(n,\ m),\ \textit{where}\ \|x(n,\ m+1)-x(n,\ m)\|<\delta_n.$$

Then the sequence $\{\|\bar{x}^n\|\}$ is bounded if and only if NLCP(F) *is solvable and in this case, $\bar{x}_n\to\bar{x}$, the least two-norm solution of* NLCP(F).

Proof. From Proposition 2.3, $\mathbb{P}_n$ is a contraction with modulus $k_n<1$. By the contraction mapping principle, given any x^0,

$$\lim_{j\to\infty}\mathbb{P}_n^j\to z^n,\quad \mathbb{P}_n(z^n)=z^n. \tag{2.4}$$

Note that z^n solves NLCP$(F+\varepsilon_nI)$ uniquely. Since, by definition,

$$\mathbb{P}_n(x(n,\ m))=x(n,\ m+1),$$

we have

$$\delta_n > \|x(n, m+1) - x(n, m)\| \geq \|x(n, m) - z^n\| - \|x(n, m+1) - z^n\|$$

and

$$\|x(n, m+1) - z^n\| = \|\mathbb{P}_n(x(n, m) - \mathbb{P}_n(z^n)\| \leq k_n \cdot \|x(n, m) - z^n\|.$$

It follows that

$$\delta_n > \|x(n, m+1) - x(n, m)\| \geq (1-k_n)\|x(n, m) - z^n\|$$

and that

$$\|\bar{x}^n - z^n\| < \varepsilon_n. \tag{2.5}$$

The conclusions about $\{\bar{x}^n\}$ follow from [Brézis, 1973] (see also [Subramanian, 1985]). ∎

We remark that the last theorem is a two-step process in the sense that for a given ε_n, the contraction $\mathbb{P}_n$ is iterated m times until $x(n, m)$ is close enough to the solution z^n of NLCP$(F+\varepsilon_n I)$. One then takes a smaller ε_n and the process repeats. Our aim now is to prove convergence for an algorithm which combines both steps into a single step. We shall need the following notions from the theory of summability.

2.5. Definition. *An infinite matrix $A=(A_{ij})$, i, $j=1, 2, \cdots$, is said to be convergence preserving if for any sequence $\{x_n\}$, the sequence $\{y_n\}$ defined by*

$$y_n = \sum_{j=1}^{\infty} A_{nj} x_j \tag{2.6}$$

is well defined and $\lim x_n = \lim y_n$. *We call $\{y_n\}$ the A-transform of $\{x_n\}$ and write $y_n = A(\{x_n\})$.*

The following theorem is classical. Its proof may be found for instance in [Peyerimhoff, 1969].

2.6. Theorem. (*O. Toeptlitz*). *An infinite matrix $A=(A_{ij})$, i, $j=1, 2, \cdots$, is convergence preserving if and only if*

(1) $\sum_{j=1}^{\infty}|A_{ij}| = \sigma_i$ *exists,*

$\{\sigma_i\}$ *is bounded,*

(2) $\lim_i\left(\sum_{j=1}^{\infty} A_{ij}\right)=1,$ (2.7)

(3) $\lim_i A_{ij}=0.$ (2.8)

We are now ready to prove the principal theorem of this paper.

2.7. Theorem. *Let M be a positive semidefinite matrix. Assume that the sequences $\{\bar{\alpha}_n\}$ and $\{\bar{\varepsilon}_n\}$ of positive reals are such that*

$$\begin{cases} \sum_{n=1}^{\infty} \bar{\alpha}_n \text{ diverges}, \\ \sum_{n=1}^{\infty} \bar{\alpha}_n^2 \text{ converges}, \\ \sum_{n=1}^{\infty} \bar{\alpha}_n \bar{\varepsilon}_n \text{ converges, and} \\ \bar{\varepsilon}_n \leq 1, \ \bar{\rho}_n = \frac{\bar{\alpha}_n}{\bar{\varepsilon}_n} \downarrow 0. \end{cases} \tag{2.9}$$

Suppose that k is the smallest positive integer satisfying

$$\sqrt{2\bar{\rho}_k}+2\bar{\rho}_k<L:=\frac{1}{1+\|M\|}. \tag{2.10}$$

Let $B=(B_{ij})$ be the infinite matrix whose n^{th} row B_n is defined by

$$B_n=\left(\frac{\bar{\alpha}_{1+k}}{S_n},\ \frac{\bar{\alpha}_{2+k}}{S_n},\ \cdots,\ \frac{\bar{\alpha}_{n+k}}{S_n},\ 0,\ \cdots\right),$$

where $S_n=\sum_{j=1}^{n}\bar{\alpha}_{j+k}$. Let $x^0=0$ and, having x^n, determine x^{n+1} from

$$x^{n+1}=\{(1-\bar{\alpha}_{n+k}\bar{\varepsilon}_{n+k})x^n-\bar{\alpha}_{n+k}(Mx^n+q)\}_+. \tag{2.11}$$

Let $\{y^n\}$ be the B-transform of $\{x^n\}$, that is

$$y^n=\frac{1}{S_n}\left(\sum_{j=1}^{n}\bar{\alpha}_j+kx^j\right). \tag{2.12}$$

Then $\bar{S}(M,\ q)\neq\emptyset\Leftrightarrow\{x^n\}$ is bounded. When this condition holds,

$$y^n\to y^*\in\bar{S}(M,\ q).$$

Proof. Assume that k satisfies (2.10). For notational convenience, we shall write

$$\alpha_n=\bar{\alpha}_{n+k},\quad \varepsilon_n=\bar{\varepsilon}_{n+k}\quad\text{and}\quad \rho_n=\bar{\rho}_{n+k}.$$

Obviously, the sequences $\{\alpha_n\}$, $\{\varepsilon_n\}$ and $\{\rho_n\}$ also satisfy conditions (2.9). We shall write

$$Fx=Mx+q,\quad F_nx=Fx+\varepsilon_nx.$$

Thus we can write (2.11) in the form

$$x^{n+1}=\{(1-\alpha_n\varepsilon_n)x^n-\alpha_nFx^n\}_+. \tag{2.13}$$

We first assume that $\{x^n\}$ is bounded and show that in this case $y^n\to y^*\in\bar{S}(M,$ $q)$ so that $\bar{S}(M,\ q)\neq\emptyset$.

Assume then that $\{x^n\}$ is bounded. Clearly, $\exists K_1>0$ and $K_2>0$ such that

$$\|x^n\|\leqslant K_1,$$

$$\begin{aligned}\|F_nx^n\|&=\|Mx^n+q+\varepsilon_nx^n\|\leqslant(1+\|M\|)\cdot\|x^n\|+\|q\|\\&\leqslant(1+\|M\|)\cdot K_1+\|q\|:=K_2.\end{aligned}$$

Let $x\in\Re^n_+$ be arbitrary but fixed. Then from (2.13) we have

$$\begin{aligned}\|x^{n+1}-x\|^2&=\|(x^n-\alpha_n(F_nx^n))_+-x\|^2\leqslant\|(x^n-x)-\alpha_n(Fx^n+\varepsilon_nx^n)\|^2\\&\leqslant\|(x^n-x)\|^2-2\alpha_n(Fx^n)(x^n-x)-2\alpha_n\varepsilon_nx^n(x^n-x)+\alpha_n^2K_2^2.\end{aligned} \tag{2.14}$$

Since M is positive semidefinite, we also have

$$(Fx^n)(x^n-x)\geqslant(Fx)(x^n-x).$$

Let

$$K_3=\sup_n\|x^n\|\cdot\|x^n-x\|.$$

From (2.14) we now get

$$2\alpha_n(Fx)(x^n-x)\leqslant\|x^n-x\|^2-\|x^{n+1}-x\|^2+2\alpha_n\varepsilon_nK_3+\alpha_n^2K_2^2.$$

Summing this from 1 to k we obtain

$$2(Fx)\sum_{n=1}^{k}\alpha_n(x^n-x)\le\|x^1-x\|^2-\|x^{k+1}-x\|^2+2K_3\sum_{n=1}^{k}\alpha_n\varepsilon_n+K_2^2\sum_{n=1}^{k}\alpha_n^2.$$

Divide this last inequality by S_k and let $k\to\infty$. From the assumed properties in (2.9) of the sequences $\{\alpha_n\}$, $\{\varepsilon_n\}$ and from the definition of $\{y^n\}$, we now have

$$\liminf_k\langle Fx,\ x-y^k\rangle\ge0.$$

Since y^n is a convex combination of $x^1, x^2, \cdots, x^n$, it follows that $\{x^n\}$ is bounded $\Rightarrow$ $\{y^n\}$ is bounded. Hence $\{y^n\}$ has a limit point y^* for which

$$\langle Fx,\ x-y^*\rangle\ge0.$$

Since $x\in\Re^n_+$ is arbitrary, y^* solves LCP(M, q). This completes our proof that

$$\{x^n\}\ \text{bounded}\Rightarrow\bar{S}(M,\ q)\ne\emptyset.$$

Next we prove $y^n\to y^*$.

Since $\bar{S}(M, q)\ne\emptyset$, choose $z\in\bar{S}(M, q)$ arbitrary but fixed. By [Subramanian, 1985 Theorem 2.4.2],

$$\langle Fx^n,\ x^n-z\rangle\ge0 \tag{2.15}$$

since $x^n\ge0$. From (2.13) and (2.15) we have

$$\begin{aligned}\|x^{n+1}-z\|^2&\le\|(x^n-z)-\alpha_n(Fx^n+\varepsilon_nx^n)\|^2\\&\le\|x^n-z\|^2-2\alpha_n(Fx^n)(x^n-z)-2\alpha_n\varepsilon_nx^n(x^n-z)+\alpha_n^2K_2^2\\&\le\|x^n-z\|^2+2\alpha_n\varepsilon_n|x^n(x^n-z)|+\alpha_n^2K_2^2.\end{aligned}$$

Define $\beta_n(z)$ by

$$\beta_n(z):=2\alpha_n\varepsilon_n\|x^n\|\,\|x^n-z\|+\alpha_n^2K_2^2 \tag{2.16}$$

and we now have

$$\|x^{n+1}-z\|^2\le\|x^n-z\|^2+\beta_n(z). \tag{2.17}$$

Let $\bar{S}$ denote $\bar{S}(M, q)$ and let

$$z^n=P_{\bar{S}}(x^n).$$

We are going to show that $\exists z^*$ such that

$$z^n\to z^*,\quad y^n\to z^*.$$

From (2.17) and the definition of z^n,

$$\|x^{n+1}-z^{n+1}\|^2\le\|x^{n+1}-z^n\|^2\le\|x^n-z^n\|^2+\beta_n(z).$$

Since $\sum_n\beta_n(z)$ converges, by [Cheng, 1981, Lemma 2.2.12], we can conclude that

$$\|x^n-z^n\| \tag{2.18}$$

converges. By the parallelogram law, for $m>0$,

$$\|z^{n+m}-z^n\|^2=2\|x^{n+m}-z^n\|^2+2\|x^{n+m}-z^{n+m}\|^2-4\left\|x^{n+m}-\frac{1}{2}(z^n+z^{n+m})\right\|^2.$$

Since $\bar{S}$ is convex, $(z^n+z^{n+m})/2\in\bar{S}$. Also, z^{n+m} is the closest point to x^{n+m} in $\bar{S}$. Hence,

$$\|z^{n+m}-z^n\|^2\le2\|x^{n+m}-z^n\|^2-2\|x^{n+m}-z^{n+m}\|^2. \tag{2.19}$$

Letting $z=z^n$ in (2.17) and noting that z^n is the closest point to x^n in $\bar{S}$, it follows that $\beta_n(z^n)\le\beta_n(z)$. Now let $z=z^n$ in (2.17) and use induction to get

$$\|x^{n+m}-z^n\|^2\leqslant\|x^n-z^n\|^2+\sum_{j=n}^{n+m}\beta_j(z), \quad m>0.$$

Substitute this in (2.19) and we have

$$\|z^{n+m}-z^n\|^2\leqslant 2\|x^n-z^n\|^2-2\|x^{n+m}-z^{n+m}\|^2+2\sum_{j=n}^{n+m}\beta_j(z). \tag{2.20}$$

From (2.18) and the fact $\sum_n \beta_n(z)$ converges, we have by letting n, $m\to\infty$ in (2.20) that

$$\|z^{n+m}-z^n\|\to 0$$

so that $\{z^n\}$ is Cauchy. Since $\bar{S}$ is closed, $\exists z^*\in\bar{S}$ such that $z^n\to z^*$. We shall now show that $y^n\to z^*$ as well.

Since $\{y^n\}$ is also bounded, let y^* be any of its limit points. Assume that the subsequence y^{n_k} converges to y^*. From our proof earlier, $y^*\in\bar{S}$. Observe that [Bazaraa and Shetty, 1979, Theorem 2.3.1]

$$z^j=P_{\bar{S}}(x^j)\Rightarrow\langle x^j-z^j,\ y^*-z^j\rangle\leqslant 0. \tag{2.21}$$

Multiply (2.21) by α_j^2 and sum from $j=1, 2, \cdots n_k$ to get

$$\left\langle\sum_{j=1}^{n_k}\alpha_j(x^j-z^j),\ \sum_{j=1}^{n_k}\alpha_j(y^*-z^j)\right\rangle\leqslant 0.$$

Divide the last inequality by $S_{n_k}^2$ to obtain

$$\left\langle y^{n_k}-\frac{1}{S_{n_k}}\sum_{j=1}^{n_k}\alpha_j z^j,\ y^*-\frac{1}{S_{n_k}}\sum_{j=1}^{n_k}\alpha_j z^j\right\rangle\leqslant 0. \tag{2.22}$$

Notice however that

$$\xi^{n_k}:=\frac{1}{S_{n_k}}\left(\sum_{j=1}^{n_k}\alpha_j z^j\right)$$

is simply a subsequence of the B–transform of $\{z^n\}$, that is of

$$\{\xi^n\}=B(\{z^n\}).$$

Since B satisfies all the conditions of Theorem 2.7, it is a convergence preserving matrix. However, $z^n\to z^*$ so that both ξ^n and ξ^{n_k} also converge to z^*. If we take limits as $k\to\infty$ in (2.22), we get

$$\langle y^*-z^*,\ y^*-z^*\rangle\leqslant 0$$

so that $y^*=z^*$. But y^* is any arbitrary limit point of $\{y^n\}$. Hence $y^n\to z^*$. This completes our proof that

$$\{x^n\}\ \text{bounded}\Rightarrow\bar{S}(M,q)\neq\emptyset \quad\text{and}\quad y^n\to z^*\in\bar{S}(M,q).$$

We now prove the converse, that is we shall assume that $\bar{S}(M, q)\neq\emptyset$ and show that $\{x^n\}$ is bounded.

Recall from (2.10) that k satisfies

$$\sqrt{2\bar{\rho}_k}+2\bar{\rho}_k<L=\frac{1}{1+\|M\|}.$$

Hence there exists $\sigma\,(0<\sigma<1/2)$ for which

$$\sqrt{2\bar{\rho}_k}+2\bar{\rho}_k<\frac{1}{(1+\sigma)(1+\|M\|)}:=L_\sigma.$$

The function $f(r)$,

$$f(r):=\frac{r}{r(1+\sigma)(1+\|M\|)+\|q\|}$$

is strictly increasing in $[0, \infty]$, $\lim_{r} f(r)=L_\sigma$. Thus $\exists \bar{r}>0$ such that for $r>\bar{r}$,

$$\sqrt{2\bar{\rho}_n}+2\bar{\rho}_n<f(\bar{r})<f(r).$$

Since $\bar{\rho}_n \downarrow 0$ and $\bar{\rho}_{n+k}=\rho_n$, we have for all $n>0$ and $r>\bar{r}$

$$\sqrt{2\rho_n}+2\rho_n<f(\bar{r})<f(r). \tag{2.23}$$

By assumption, $\bar{S}\neq\emptyset$. Let $z=P_{\bar{S}}(0)$, that is z is the least two-norm solution of LCP(M, q). Define

$$r=\max\left(\bar{r}, \frac{1}{\sigma}\|z\|\right)+1.$$

Our aim is to show that

$$\|x^n-z\|\leqslant r, \quad \forall n\geqslant 0,$$

that is $\{x^n\}$ is bounded and this would complete our proof.

We use induction. For $n=0$, $\|x^0-z\|=\|z\|<\sigma r<r$.

Suppose now that $\|x^n-z\|\leqslant r$. Let $\mu_n=\|x^n-z\|$. From (2.13),

$$\begin{aligned}\mu_{n+1}^2&=\|x^{n+1}-z\|^2\\ &\leqslant\|x^n-z\|^2-2\alpha_n\varepsilon_n x^n(x^n-z)-2\alpha_n(Fx^n)(x^n-z)+\alpha_n^2\|Fx^n+\varepsilon_n x^n\|^2.\end{aligned} \tag{2.24}$$

Since $z\in\bar{S}$, $(Fx^n)(x^n-z)\geqslant 0$. Also if $\mu_{n+1}\leqslant\mu_n$, we are done. So assume $\mu_{n+1}>\mu_n$. From (2.24) we thus get

$$2\alpha_n\varepsilon_n x^n(x^n-z)<\alpha_n^2\|Fx^n+\varepsilon_n x^n\|^2,$$

that is

$$x^n(x^n-z)<\frac{\rho_n}{2}\|Fx^n+\varepsilon_n x^n\|^2. \tag{2.25}$$

Since

$$\|x^n\|\leqslant\|x^n-z\|+\|z\|\leqslant r+\sigma r=(1+\sigma)r,$$

we have

$$\begin{aligned}\|Fx^n+\varepsilon_n x^n\|&\leqslant\|Mx^n+q\|+\varepsilon_n\|x^n\|\leqslant\|M\|\,\|x^n\|+\|q\|+\|x^n\|\\ &\leqslant(1+\sigma)r\cdot(1+\|M\|)+\|q\|=\frac{r}{f(r)}=:\xi \quad \text{(say)}.\end{aligned} \tag{2.26}$$

From (2.25) we now get

$$x^n(x^n-z)<\frac{\rho_n}{2}\xi^2,$$

$$(x^n-z)(x^n-z)<\frac{\rho_n}{2}\xi^2-z(x^n-z)\leqslant\frac{\rho_n}{2}\xi^2+\|z\|\cdot\|x^n-z\|.$$

Rewriting this last inequality,

$$\mu_n^2<\frac{\rho_n}{2}\xi^2+r\sigma\mu_n,$$

whence

$$2\mu_n^2 - 2r\sigma\mu_n - \rho_n\xi^2 < 0.$$

Since $\mu_n \geqslant 0$, we must have

$$\mu_n < \frac{2r\sigma + \sqrt{4r^2\sigma^2 + 8\xi^2\rho_n}}{4} \leqslant \frac{r\sigma}{2} + \left(\frac{2r\sigma + 2\xi\sqrt{2\rho_n}}{4}\right)$$

and finally

$$\mu_n < r\sigma + \frac{\xi}{2}\sqrt{2\rho_n}. \tag{2.27}$$

Again from the definition of x^{n+1} in (2.13),

$$\mu_{n+1} = \|x^{n+1} - z\| \leqslant \|x^n - z - \alpha_n(F_n x^n)\| \leqslant \mu_n + \alpha_n \|F_n x^n\| \leqslant \mu_n + \rho_n \xi, \tag{2.28}$$

where we have used (2.26) and the fact $\alpha_n < \rho_n$. If we use our estimate of μ_n from (2.27) in (2.28) we get

$$\mu_{n+1} < r\sigma + \frac{\xi}{2}\sqrt{2\rho_n} + \rho_n \xi.$$

Substituting for ξ from (2.26) and using (2.23) we finally get

$$\mu_{n+1} < r\sigma + \frac{(\sqrt{2\rho_n} + 2\rho_n)}{2} \cdot \frac{r}{f(r)} < r\sigma + \frac{f(r)}{2} \cdot \frac{r}{f(r)}$$

$$= r\sigma + \frac{r}{2} < r$$

since $\sigma < 1/2$. Hence $\mu_{n+1} < r$. This completes our induction and also the proof of the theorem. ▌

2.8. Remark. Our proof showing that $\{y^n\}$ converges by considering $z^n = P_{\bar{S}}(x^n)$ is patterned after [Baillon, 1975], who uses this technique to construct fixed points of non-expansive maps. Notice also Baillon's use of the Cesàro matrix C where $C_{ij} = 1/i$ for $j \leqslant i$, while $C_{ij} = 0$ for $j > i$.

§ 3. Application to NLCP(F)

We shall now show that the proof of 2.8 can be used to construct a solution of NLCP(F) when F is monotone and satisfies some regularity conditions such as the distributed Slater constraint qualification [Mangasarian and McLinden, 1985].

3.1. Definition. *Let $F: D \subseteq \Re^n \to \Re^n$. We say that F satisfies the distributed Slater constraint qualification* (DSCQ) *if there exist p points $z^1, z^2, \cdots, z^p \in D$ and nonnegative weights $\lambda_1, \lambda_2, \cdots, \lambda_p$ ($\sum_j \lambda_j = 1$) such that $\hat{z} = \sum_j \lambda_j z^j \geqslant 0$ and $\hat{w} = \sum_j \lambda_j w^j > 0$ where $w^j = F(z^j)$.*

Mangasarian and McLinden have proved the following theorem.

3.2. Theorem. *Let $F: D \subseteq \Re^n \to \Re^n$, $\Re^n_+ \subset D$ and suppose that F is monotone and continuous on D. Assume that F satisfies* (DSCQ). *Let*

$$\gamma > \max\left(1, \; -\hat{z}\hat{w} + \sum_{j=1}^{p} \lambda_j z^j w^j\right),$$

$$C = \{z \in \Re^n_+ : \hat{w}z \leqslant \hat{w}\hat{z} + \gamma\},$$

where λ_j, z^j, w^j, $\hat{z}$ and $\hat{w}$ are as in (DSCQ). *Then* NLCP(F) *is solvable and has a* *solution* z^* *such that* $\hat{w}z^* < \hat{w}\hat{z} + \gamma$.

We shall now show that the technique used in the proof of 2.9 can be used to construct a solution of NLCP(F) guaranteed by Theorem 3.2.

3.3. Theorem. *Assume that* F *satisfies the hypotheses of Theorem* 3.2 *and let* C *be the compact convex set as defined in that theorem. Let* $x^0=0$ *and given* x^n *find* x^{n+1}

$$x^{n+1} = P_C\left\{x^n - \frac{F(x^n)}{n}\right\}.$$

Let B *be the Césaro matrix with*

$$B_n = \left(\frac{1}{S_n}, \frac{1}{2S_n}, \cdots, \frac{1}{nS_n}, 0, 0, \cdots\right), \quad S_n = \sum_{j=1}^{n} \frac{1}{j}$$

and let $\{y^n\} = B(\{x^n\})$. *Then* y^n *converges to a solution of* NLCP(F).

Proof. We shall give only a brief outline. Since $\{x^n\}$ and hence $\{y^n\}$ are both bounded, $\{y^n\}$ has a limit point y^*. One uses the monotonicity of F to show that

$$\langle F(y^*), x - y^*\rangle \geq 0, \quad \forall x \in C.$$

Hence y^* is a fixed point of the map $x \mapsto P_C(x - F(x))$. However, Mangasarian and McLinden show that any such fixed point satisfies $\hat{w}y^* < \hat{w}\hat{z} + \gamma$. Hence y^* solves NLCP(F). One can now show that $y^n \to y^*$ by considering the projection z^n of x^n on $\bar{S}(F)$. ∎

3.4. Remarks. 1. It is easy to see that Theorem 2.8 may be extended to the nonlinear case if F is Lipschitzian. In this case $\|M\|$ is replaced by the Lipschitz constant of F in (2.10).

2. Unfortunately, from a computational point of view the fixed point methods in general, and those considered in this paper in particular, are not viable methods. They are extremely slow and particularly so in the vicinity of a solution point since the step sizes taken in such a vicinity are extremely small. Their slowness is in part also due to the fact that they do not utilize special features of the matrix M in the case of LCP(M, q). Their real utility is perhaps in generating good starting points for fast Newton-type algorithms. However, the SOR methods are much faster than the fixed point methods even for generation of starting points.

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