Vol. 6 No. 2 JOURNAL OF COPMUTATIONAL MATHEMATICS April 1988

A NEW QUADRILATERAL ELEMENT APPROXIMATION TO THE STATIONARY STOKES PROBLEM*1)

WANG LIE-HENG (王烈衡)
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

The nearly linear triangular element approximation to the stationary Stokes problem has already been proposed[1,7]. This paper discusses new rectangular and quadrilateral element approximations to the same problem, and the first order error is proved for both elements.

§ 1. Introduction

In [7] and [1], the nearly linear triangular element approximation to the stationary Stokes problem has been proposed. The quadrilateral finite elements are attractive for discretization of a domain of arbitrary shapes, and some quadrilateral element approximations to the Stokes problem have been studied (c.f. [2], [5]). In the present paper, some new quadrilateral elements approximations for both rectangles and general quadrilaterals are discussed, and the first order error is obtained for the Stokes problem, in the same way as in [7].

Let us consider approximation to the Stokes problem,

$$
\text{(ST)} \quad \begin{cases} \text{find } (\boldsymbol{u}, p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega), \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \ \forall \boldsymbol{v} \in (H_0^1(\Omega))^2, & (1.1) \\ \qquad b(\boldsymbol{u}, q) = 0, \qquad \forall q \in L_0^2(\Omega), & (1.2) \end{cases}
$$

where Ω is a bounded convex polygon in plane with a boundary $\partial\Omega$, and

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu(\text{grad } \boldsymbol{u}, \text{grad } \boldsymbol{v}), \tag{1.3}$$

$$b(\boldsymbol{v}, q) = -(q, \text{div } \boldsymbol{v}), \tag{1.4}$$

$(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$, and in what follows, $H_0^1(\Omega)$, $H^m(\Omega)$ denote usual Sobolev spaces with the norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$, and

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx\, dy, \tag{1.5}$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega): \int_\Omega q dx\, dy = 0\}. \tag{1.6}$$

In equations (1.1) and (1.2), $\boldsymbol{u} = (u_1, u_2)^T$ denotes the velocity of fluid, p denotes the pressure and $\nu = \text{const.} > 0$ denotes the viscosity.

Let $X_h \subset (H_0^1(\Omega))^2$ and $M_h \subset L_0^2(\Omega)$ be two finite element spaces. Then the discrete analogy of the problem (ST) is the following

* Received April 19, 1986.
1) The Project Supported by National Natural Science Foundation of China.

$$(\mathrm{ST}_h)\qquad\begin{cases}\text{find}(\boldsymbol{u}_h,\ p_h)\in X_h\times M_h,\ \text{such that}\\ a(\boldsymbol{u}_h,\ \boldsymbol{v}_h)-(p_h,\ \operatorname{div}\boldsymbol{v}_h)=\langle\boldsymbol{f},\ \boldsymbol{v}_h\rangle,\quad\forall\boldsymbol{v}_h\in X_h, & (1.7)\\ \qquad(q_h,\ \operatorname{div}\boldsymbol{u}_h)=0,\qquad\forall q_h\in M_h. & (1.8)\end{cases}$$

It is known ([3], [6]) that if the discrete inf-sup condition holds, i.e., there exists $\beta^*=\text{const.}>0$, such that

$$\sup_{0\neq v_h\in X_h}\frac{(q_h,\ \operatorname{div}\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}}\geqslant\beta^*\|q_h\|_{0,\Omega},\quad\forall q_h\in M_h,\tag{1.9}$$

then the unique solution $(\boldsymbol{u}_h,\ p_h)\in X_h\times M_h$ of the problem (ST_h) satisfies the following error estimate

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\leqslant c\{\inf_{\boldsymbol{v}_h\in X_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega}+\inf_{q_h\in M_h}\|p-q_h\|_{0,\Omega}\}.\tag{1.10}$$

And it is also known[6] that if for each $q_h\in M_h$, there exists a function $\boldsymbol{v}_h\in X_h$, such that

$$(\operatorname{div}\boldsymbol{v}_h-q_h,\ s_h)=0,\quad\forall s_h\in M_h,\tag{1.11}$$

and

$$|\boldsymbol{v}_h|_{1,\Omega}\leqslant c\|q_h\|_{0,\Omega},\tag{1.12}$$

then the inf-sup condition (1.9) holds. Here and later c and c_i denote generic constants independent of h.

Furthermore, condition (1.9) can be verified by constructing an operator r_h: $(H_0^1(\Omega))^2\to X_h$ such that[6]

$$(\operatorname{div}\boldsymbol{v}-\operatorname{div}r_h\boldsymbol{v},\ s_h)=0,\quad\forall s_h\in M_h,\ \boldsymbol{v}\in(H_0^1(\Omega))^2,\tag{1.13}$$

$$|r_h\boldsymbol{v}|_{1,\Omega}\leqslant c|\boldsymbol{v}|_{1,\Omega},\quad\forall\boldsymbol{v}\in(H_0^1(\Omega))^2.\tag{1.14}$$

In the case of $M_h\subset L_0^2(\Omega)\cap H^1(\Omega)$, $X_h\subset(H_0^1(\Omega))^2$, condition (1.13) becomes

$$\begin{aligned}0&=\int_\Omega s_h\cdot\operatorname{div}(\boldsymbol{v}-r_h\boldsymbol{v})dx\,dy=\sum_{K\in\mathscr{T}_h}\int_K\operatorname{div}(\boldsymbol{v}-r_h\boldsymbol{v})\cdot s_h dxdy\\&=\sum_{K\in\mathscr{T}_h}\int_{\partial K}(\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot\boldsymbol{\nu}_K\cdot s_h d\sigma-\sum_{K\in\mathscr{T}_h}\int_K(\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot\operatorname{grad}s_h dxdy,\end{aligned}\tag{1.15}$$

where $r_K\boldsymbol{v}=r_h\boldsymbol{v}|_K$, and $\boldsymbol{\nu}_K$ denotes the unit outward normol vector on the boundary ∂K of K. Since $r_h\boldsymbol{v}\in X_h\subset(H_0^1(\Omega))$, $s_h\in M_h\subset H^1(\Omega)$, then $r_h\boldsymbol{v}\in(C(\bar\Omega))^2$, $s_h\in C^0(\bar\Omega)$. And since $\boldsymbol{v}\in(H_0^1(\Omega))^2$, it can be easily seen that $\boldsymbol{v}$ is continuous across the element boundary ∂K of K. Summarizing the above argument, we can verify that the first term on the right-hand side of (1.15) vanishes. Thus, if $M_h\subset L_0^2(\Omega)\cap H^1(\Omega)$, $X_h\subset(H_0^1(\Omega))^2$, then condition (1.13) is equivalent to the following condition

$$\sum_{K\in\mathscr{T}_h}\int_K(\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot\operatorname{grad}s_h\,dx\,dy=0,\ \forall s_h\in M_h,\ \boldsymbol{v}\in(H_0^1(\Omega))^2.\tag{1.16}$$

§ 2. Basic Notations

Let $\mathscr{T}_h$ (with a parameter $h>0$) be a subdivision of a convex polygon Ω in plane, and for each convex quadrilateral element $K\in\mathscr{T}_h$, let $h_K=\operatorname{diam}(K)$, h'_K be the smallest length of the edges of K, and θ_i^K be the angles associated with the vertices $P_i(1\leqslant i\leqslant 4)$ of K (Fig. 1). We assume that $\mathscr{T}_h$ satisfies the following regularity condition ([4], p. 247): There exist positive constants σ and γ, such that $\forall K\in\mathscr{T}_h$,

$$\frac{h_K}{h'_K}\leqslant\sigma \quad\text{and}\quad \max_{1\leqslant i\leqslant 4}|\cos\theta_i^K|\leqslant\gamma<1. \tag{2.1}$$

Let $\hat{K}=[-1, 1]\times[-1, 1]$ be the reference square with the vertices $\hat{p}_i$ $(1\leqslant i\leqslant 4)$ (Fig. 1). Then a mapping F_K: $\hat{K}\to K$, i.e., $(x, y)=F_K(\xi, \eta)$, is defind by

$$\begin{cases} x=\frac{1}{4}\{(1+\xi)(1+\eta)x_1+(1-\xi)(1+\eta)x_2 \\ \qquad +(1-\xi)(1-\eta)x_3+(1+\xi)(1-\eta)x_4\}, \\ y=\frac{1}{4}\{(1+\xi)(1+\eta)y_1+(1-\xi)(1+\eta)y_2 \\ \qquad +(1-\xi)(1-\eta)y_3+(1+\xi)(1-\eta)y_4\}, \end{cases} \tag{2.2}$$

where $p_i=(x_i, y_i)$ $(1\leqslant i\leqslant 4)$, and $F_K(\hat{p}_i)=p_i$.

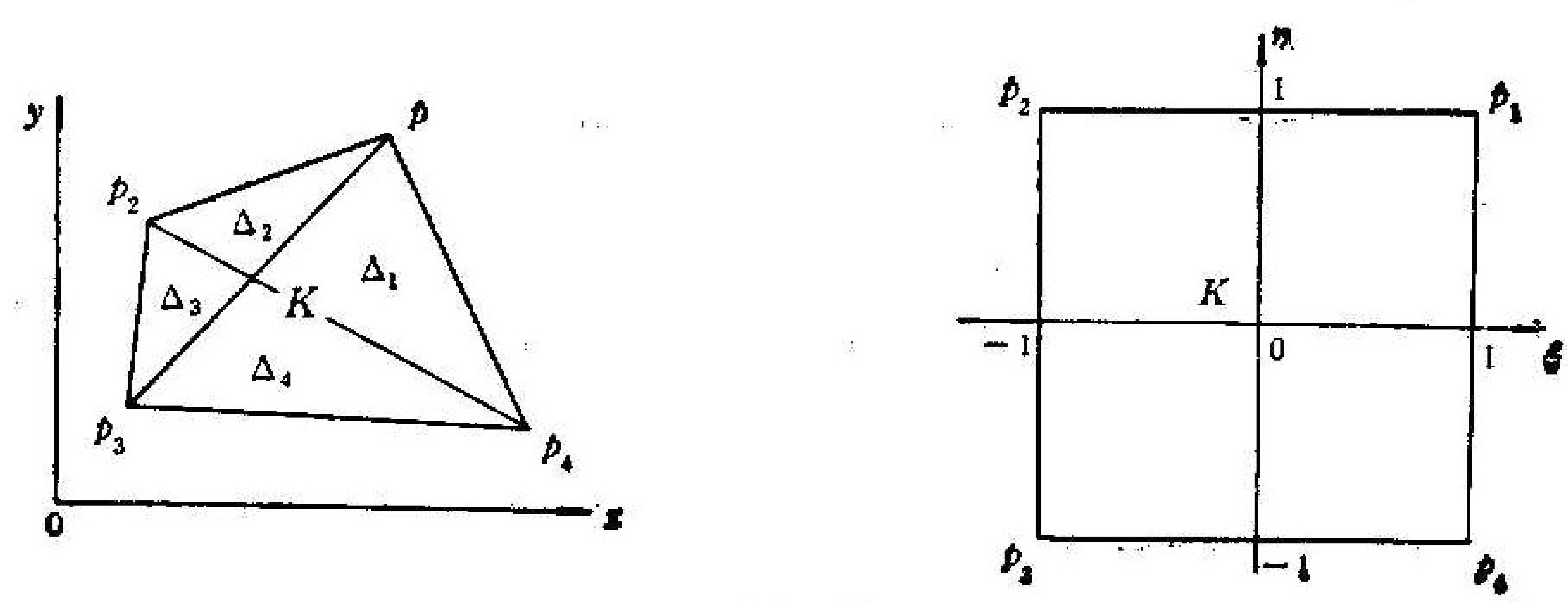

Fig. 1

Let the Jacobian of the mapping F_K be

$$J_K(\xi, \eta)=\begin{vmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{vmatrix}=\frac{1}{4}\{|K|+\xi(|\Delta_1|-|\Delta_3|)+\eta(|\Delta_2|-|\Delta_4|)\}, \tag{2.3}$$

where $|K|$ denotes the area of K, and $|\Delta_k|$ the area of the triangle Δ_k $(1\leqslant k\leqslant 4)$. Then under the regularity assumption (2.1), it can be easily seen that

$$c_1h_K^2\leqslant J_K\leqslant c_2h_K^2. \tag{2.4}$$

For each function $v(x, y)$ defined on K, let $(x, y)=F_K(\xi, \eta)$,

$$v(x, y)=v(x(\xi, \eta), y(\xi, \eta))=\hat{v}(\xi, \eta), \text{ or } \hat{v}=v\cdot F_K. \tag{2.5}$$

Under the assumption of the regularity (2.1) and convexity of element K, we have ([4], p. 247)

$$c_1|v|_{1,K}\leqslant|\hat{v}|_{1,\hat{K}}\leqslant c_2|v|_{1,K}, \quad \forall v\in H^1(K), \tag{2.6}$$

and

$$\begin{cases} |\hat{v}|_{2,\hat{K}}\leqslant c_2h_K|v|_{2,K}, \\ |v|_{2,K}\leqslant c_1(h_K^{-1}|\hat{v}|_{2,\hat{K}}+|\hat{v}|_{1,\hat{K}}). \end{cases} \tag{2.7}$$

For each $\hat{v}\in H^2(\hat{K})$, define the bilinear interpolation operator $\hat{Q}_1$ as follows:

$$\hat{Q}_1\hat{v}=\sum_{i=1}^4\hat{v}(\hat{p}_i)\hat{\varphi}_i(\xi, \eta), \tag{2.8}$$

where

$$\hat{\varphi}_1(\xi, \eta)=\frac{1}{4}(1+\xi)(1+\eta), \quad \hat{\varphi}_2(\xi, \eta)=\frac{1}{4}(1-\xi)(1+\eta),$$
$$\hat{\varphi}_3(\xi, \eta)=\frac{1}{4}(1-\xi)(1-\eta), \quad \hat{\varphi}_4(\xi, \eta)=\frac{1}{4}(1+\xi)(1-\eta). \tag{2.9}$$

And for each $v\in H^2(K)$, define an operator $Q_{1,K}$ as follows:

$$Q_{1,K}v=\sum_{i=1}^{4} v(p_i)\hat{\varphi}_i\circ F_K^{-1}(x, y). \tag{2.10}$$

Then

$$(Q_{1,K}\hat{v})=\hat{Q}_1\hat{v}. \tag{2.11}$$

By use of the estimates (2.6), (2.7) and the interpolation theory on the reference element $\hat{K}$, the following estimates hold: $\forall v\in H^2(K)$,

$$\begin{cases}\|v-Q_{1,K}v\|_{0,K}\leqslant ch_K|v|_{1,K},\\ |v-Q_{1,K}v|_{1,K}\leqslant c|v|_{1,K},\end{cases} \tag{2.12$_1$}$$

and

$$\begin{cases}\|v-Q_{1,K}v\|_{0,K}\leqslant ch_K^2\|v\|_{2,K},\\ |v-Q_{1,K}v|_{1,K}\leqslant ch_K\|v\|_{2,K}.\end{cases} \tag{2.12$_2$}$$

§ 3. Rectangular Element

Let Ω be a rectangle, $\mathscr{T}_h$ be a regular rectangular subdivision of Ω, and $K\in\mathscr{T}_h$ be a rectangular element (c.f. Fig. 2). Then the mapping F_K: $\hat{K}\to K$ is defined by

$$\begin{cases}x=x_0+a\xi,\\ y=y_0+b\eta,\end{cases} \tag{3.1}$$

and F_K^{-1}: $K\to\hat{K}$ defined by

$$\begin{cases}\xi=\dfrac{1}{a}(x-x_0),\\ \eta=\dfrac{1}{b}(y-y_0).\end{cases} \tag{3.2}$$

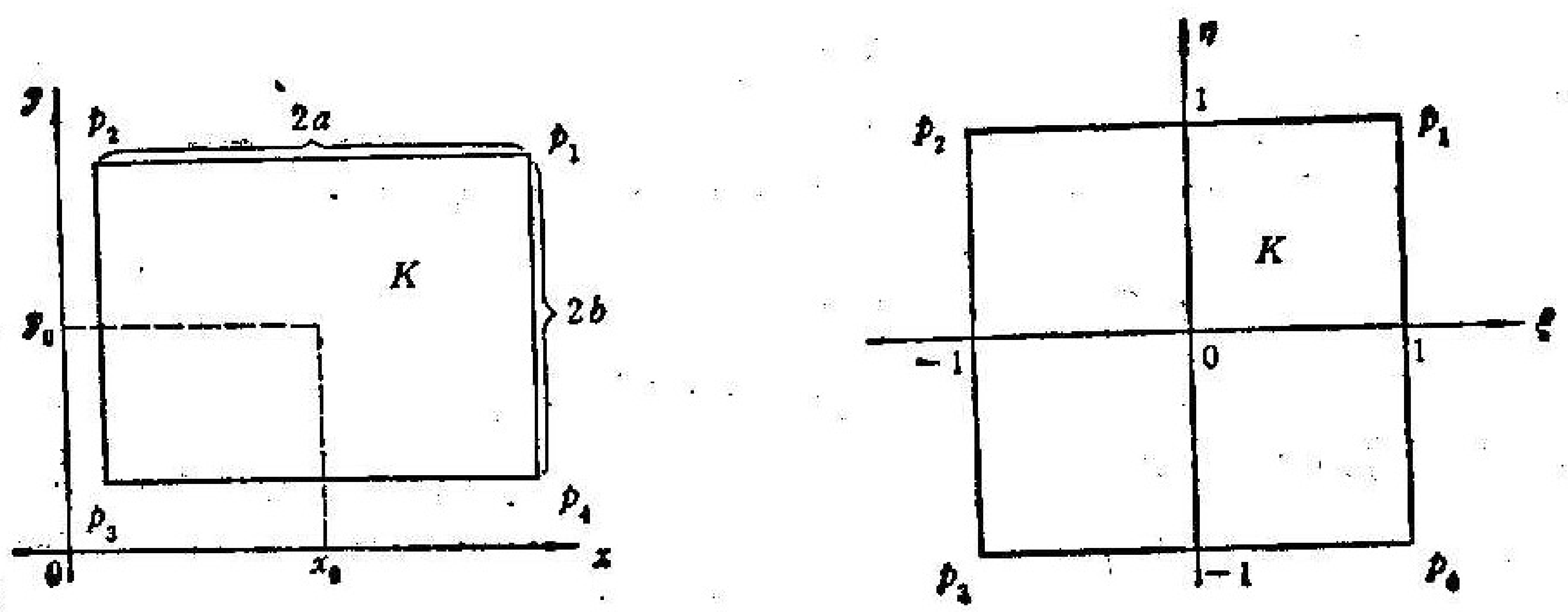

Fig. 2

And

$$\begin{cases} DF_K=\begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \\ DF_K^{-1}=\begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{pmatrix}=\begin{pmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{b} \end{pmatrix}, \\ J_K=\det\,(DF_K)=|K|/4. \end{cases} \tag{3.3}$$

We consider a new rectangular element approximation to the problem (ST). First, the space $M_h\subset L_0^2(\Omega)\cap H^1(\Omega)$ consists of the functions $q_h\in M_h$, such that $q_h|_K=q_K$, $\forall K\in\mathscr{T}_h$, and

$$\begin{aligned} &q_K=\sum_{i=1}^4 q_i\hat{\varphi}_i(\xi,\eta)=Q_{1,K}q_K, \\ &\int_\Omega q_h\,dx\,dy=\sum_{K\in\mathscr{T}_h}\Big(\sum_{i=1}^4 q_i\Big)|K|=0. \end{aligned} \tag{3.4}$$

Next, we consider the construction of the space $X_h\subset(H_0^1(\Omega))^2$. From (1.16), the following should be satisfied: $\forall K\in\mathscr{T}_h$,

$$\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot\text{grad}\,q_K\,dx\,dy=0,\quad \forall q_h\in M_h. \tag{3.5}$$

With the following equality taken into account:

$$\text{grad}\,q_K=\frac{1}{4}\begin{pmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{b} \end{pmatrix}\left\{\begin{pmatrix} q_1-q_2-q_3+q_4 \\ q_1+q_2-q_3-q_4 \end{pmatrix}+(q_1-q_2+q_3-q_4)\begin{pmatrix} \eta \\ \xi \end{pmatrix}\right\}, \tag{3.6}$$

(3.5) is equivalent to that $\forall K\in\mathscr{T}_h$,

$$\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T\begin{pmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{b} \end{pmatrix}dx\,dy=\boldsymbol{0}^T, \tag{3.7}$$

$$\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T\begin{pmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{b} \end{pmatrix}\begin{pmatrix} \eta \\ \xi \end{pmatrix}dx\,dy=0. \tag{3.8}$$

Since $r_h\boldsymbol{v}\in X_h\subset(C^0(\overline{\Omega}))^2$, let

$$r_K\boldsymbol{v}=Q_{1,K}\boldsymbol{v}+\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} \alpha_1+\beta_1\xi+\gamma_1\eta \\ \alpha_2+\beta_2\xi+\gamma_2\eta \end{pmatrix}\hat{\varphi}_0, \tag{3.9}$$

where α_i, β_i and γ_i $(i=1, 2)$ are undetermined coefficients, and

$$\hat{\varphi}_0=\frac{1}{16}(1-\xi^2)(1-\eta^2). \tag{3.10}$$

The coefficients α_i, β_i and γ_i $(i=1, 2)$ should be chosen such that (3.7) and (3.8) hold, i.e.,

$$\int_K (\boldsymbol{v}-Q_{1,K}\boldsymbol{v})^T \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} dx\,dy = \int_K \begin{pmatrix} \alpha_1+\beta_1\xi+\gamma_1\eta \\ \alpha_2+\beta_2\xi+\gamma_2\eta \end{pmatrix}^T \hat{\varphi}_0\,dx\,dy$$

$$=\frac{|K|}{4}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}^T\cdot\int_{\hat{K}}\hat{\varphi}_0\,d\xi\,d\eta=\frac{|K|}{36}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}^T, \tag{3.11}$$

and

$$\int_K (\boldsymbol{v}-Q_{1,K}\boldsymbol{v})^T \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix} dx\,dy = \int_K \begin{pmatrix} \alpha_1+\beta_1\xi+\gamma_1\eta \\ \alpha_2+\beta_2\xi+\gamma_2\eta \end{pmatrix}^T \begin{pmatrix}\eta\\ \xi\end{pmatrix}\hat{\varphi}_0\,dx\,dy$$

$$=\frac{|K|}{4}\int_{\hat{K}}(\gamma_1\eta^2+\beta_2\xi^2)\hat{\varphi}_0\,d\xi\,d\eta=\frac{|K|}{180}(\gamma_1+\beta_2), \tag{3.12}$$

since the function $\hat{\varphi}_0(\xi,\eta)$ is an odd function with respect to both variables ξ and η. From (3.11) and (3.12), the coefficients α_i, β_i and $\gamma_i (i=1, 2)$ can be chosen in such a way that $\beta_1=\gamma_2=0$,

$$\beta=\gamma_1=\beta_2=\frac{90}{|K|}\int_K(\boldsymbol{v}-Q_{1,K}\boldsymbol{v})^T\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix}dx\,dy, \tag{3.13}$$

and

$$\boldsymbol{\alpha}=\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}^T=\frac{36}{|K|}\int_K(\boldsymbol{v}-Q_{1,K}\boldsymbol{v})^T\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}dx\,dy. \tag{3.14}$$

Thus, the space X_h consists of those vector functions $\boldsymbol{v}_h\in X_h$ that satisfy $\boldsymbol{v}_h|_K=\boldsymbol{v}_K$, $\forall K\in\mathscr{T}_h$,

$$\boldsymbol{v}_K=\sum_{i=1}\boldsymbol{v}_i\hat{\varphi}_i+\begin{pmatrix}a&0\\0&b\end{pmatrix}\boldsymbol{\alpha}_K\hat{\varphi}_0+\beta_K\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix}\hat{\varphi}_0, \tag{3.15}$$

and for $\forall\boldsymbol{v}\in(H^1(K)\cap C^0(\overline{K}))^2$,

$$r_K\boldsymbol{v}=Q_{1,K}\boldsymbol{v}+\begin{pmatrix}a&0\\0&b\end{pmatrix}\boldsymbol{\alpha}_K(\boldsymbol{v})\hat{\varphi}_0+\beta_K(\boldsymbol{v})\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix}\hat{\varphi}_0, \tag{3.16}$$

where $\boldsymbol{\alpha}_K(v)$ and $\beta_K(v)$ are defined by (3.14) and (3.13) respectively.

Remark 3.1. In the case of $\boldsymbol{v}\in(H_0^1(\Omega))^2$, $Q_{1,K}\boldsymbol{v}$ and $Q_{1,h}\boldsymbol{v}(Q_{1,h}\boldsymbol{v}|_K=Q_{1,K}\boldsymbol{v})$, which are not defined, should be modified as follows: Let

$$V_h=\{\boldsymbol{v}_h:\ \hat{\boldsymbol{v}}_h=\boldsymbol{v}_h\circ F_K\in(Q_1(K))^2,\quad\forall K\in\mathscr{T}_h\}\cap(H_0^1(\Omega))^2,$$
$$Q_1(\hat{K})=\text{span}\{\hat{\varphi}_i,\ 1\leqslant i\leqslant 4\}, \tag{3.17}$$

and for any given $\boldsymbol{v}\in(H_0^1(\Omega))^2$, let $\boldsymbol{w}_h\in V_h$ be the solution of the problem

$$(\text{grad}\,\boldsymbol{w}_h,\ \text{grad}\,\boldsymbol{v}_h)=(\text{grad}\,\boldsymbol{v},\ \text{grad}\,\boldsymbol{v}_h),\ \forall\boldsymbol{v}_h\in V_h, \tag{3.18}$$

from which, and by the general analysis of the finite element method, we have

$$|\boldsymbol{v}-\boldsymbol{w}_h|_{1,\Omega}\leqslant c|\boldsymbol{v}|_{1,\Omega},$$
$$\|\boldsymbol{v}-\boldsymbol{w}_h\|_{0,\Omega}\leqslant ch^{K+1}|\boldsymbol{v}|_{K+1,\Omega}, \tag{3.19}$$
$$\|\boldsymbol{v}-\boldsymbol{w}_h\|_{1,\Omega}\leqslant ch^{K}|\boldsymbol{v}|_{K+1,\Omega},\quad K=0,\ 1.$$

Then we take $w_h|_K$ instead of $Q_{1,K}v$ in (3.13), (3.14) and (3.16).

From the above analyses, we have

Lemma 3.1. *Condition* (1.13) *is satisfied.*

Proof. The lemma can be deduced directly from (1.16), (3.5)—(3.8), (3.13), (3.14), (3.16), and Remark 3.1.

Lemma 3.2.

$$|r_h v|_{1,\Omega} \leqslant c|v|_{1,\Omega}, \quad \forall v \in (H_0^1(\Omega)), \tag{3.20}$$

$$\|v - r_h v\|_{1,\Omega} \leqslant ch|v|_{2,\Omega}, \quad \forall v \in (H_0^1(\Omega) \cap H^2(\Omega))^2. \tag{3.21}$$

Proof. From (3.13), (3.14) and Remark 3.1,

$$\|\alpha_K(v)\| \leqslant ch_K^{-2}\|v - Q_{1,K}v\|_{0,K},$$

$$|\beta_K(v)| \leqslant ch_K^{-2}\|v - Q_{1,K}v\|_{0,K},$$

from which and (3.16), (3.19),

$$|r_h v|_{1,\Omega} = \Big(\sum_{K \in \mathscr{T}_h} |r_K v|_{1,K}^2\Big)^{1/2} \leqslant |w_h|_{1,\Omega} + ch^{-1}\|v - w_h\|_{0,\Omega} \leqslant c|v|_{1,\Omega},$$

and

$$\|v - r_h v\|_{1,\Omega} \leqslant \|v - w_h\|_{1,\Omega} + ch^{-1}\|v - w_h\|_{0,\Omega} \leqslant ch|v|_{2,\Omega}.$$

Now we define an operator ρ_h: $H^1(\Omega) \cap L_0^2(\Omega) \to M_h$, as follows. For any given $q \in H^1(\Omega) \cap L_0^2(\Omega)$, $\rho_h q \in M_h$ is the solution of the problem

$$(\text{grad}\ \rho_h q, \text{grad}\ s_h) = (\text{grad}\ q, \text{grad}\ s_h), \ \forall s_h \in M_h, \tag{3.22}$$

which has a unique solution, since $|\cdot|_{1,\Omega}$ is norm equivalent to $\|\cdot\|_{1,\Omega}$ in the space $H^1(\Omega) \cap L_0^2(\Omega)$. From (3.22) and the general analysis of the finite element method,

$$\|q - \rho_h q\|_{0,\Omega} \leqslant ch|q|_{1,\Omega}, \quad \forall q \in L_0^2(\Omega) \cap H^1(\Omega). \tag{3.23}$$

Summarizing Lemmas 3.1—3.2 and (3.23) we have

Theorem 3.1. *Let Ω be a rectangular domain, $\mathscr{T}_h$ be a regular subdivision of Ω, $K \in \mathscr{T}_h$ be a rectangular element, $\hat{K} = [-1, 1] \times [-1, 1]$ be the reference square, and the mapping F_K: $\hat{K} \to K$ be defined by* (3.1). *Assume that the solution $(u, p) \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^1(\Omega) \cap L_0^2(\Omega))$ of the problem* (ST), *and the finite element spaces $X_h \subset (H_0^1(\Omega))^2$, $M_h \subset L_0^2(\Omega) \cap H^1(\Omega)$ are defined by* (3.15) *and* (3.4) *respectively. Then the following error estimate holds:*

$$\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leqslant ch(|u|_{2,\Omega} + |p|_{1,\Omega}). \tag{3.24}$$

§ 4. Quadrilateral Element

In this section, a new quadrilateral element method for the Stokes problem (ST) is considered by use of the notation in section 2.

As in section 3, the space $M_h \subset L_0^2(\Omega) \cap H^1(\Omega)$ consists of the functions $q_h \in M_h$, such that $q_h|_K = q_K$, $\forall K \in \mathscr{T}_h$, and

$$q_K = \sum_{i=1}^{4} q_i \hat{\varphi}_i(\xi, \eta), \quad (\xi, \eta) = F_K^{-1}(x, y), \tag{4.1}$$

and

$$\int_\Omega q_h\, dx\, dy = \sum_{K \in \mathscr{T}_h} \Big(\sum_{i=1}^{4} q_i\Big) K = 0. \tag{4.2}$$

And the space $X_h \subset (H_0^1(\Omega))^2$ consists of the vector functions $v_h \in X_h$, such that

$v_h|_K = v_K,\ \forall K \in \mathscr{T}_h,$

$$v_K = \sum_{i=1}^{4} v_i\hat{\varphi}_i + (DF_K)^T a_K \hat{\varphi}_0 + \beta_K (CF_K)^T \binom{\eta}{\xi} \hat{\varphi}_0, \tag{4.3}$$

$$(\xi, \eta) = F_K^{-1}(x, y),$$

where F_K is defined by (2.2),

$$DF_K = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}, \quad DF_K^{-1} = (DF_K)^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix}, \tag{4.4}$$

$$\hat{\varphi}_0(\xi, \eta) = \frac{1}{16}(1-\xi^2)(1-\eta^2) \tag{4.5}$$

and $J_K = \det(DF_K)$ is denoted in (2.3).

Next we define an operator r_h: $(H_0^1(\Omega))^2 \to X_h$ as follows. For any given $v \in (H_0^1(\Omega))^2$, $w_h \in V_h$ is defined by (3.18) associated with the general quadrilateral subdivision of Ω, $r_h v|_K = r_K v$, $\forall K \in \mathscr{T}_h$,

$$r_K v = w_K + (DF_K)^T a_K(v) \hat{\varphi}_0 + \beta_K(v) (DF_K)^T \binom{\eta}{\xi} \hat{\varphi}_0, \tag{4.6}$$

and

$$\beta_K(v) = D^{-1}\Big\{\frac{|K|}{36}\int_K (v - w_K)^T DF_K^{-1} \binom{\eta}{\xi} dx\,dy$$

$$- \int_K (v - w_K)^T DF_K^{-1}\, dx\,dy \cdot \int_K \binom{\eta}{\xi} \hat{\varphi}_0\, dx\,dy\Big\}, \tag{4.7}$$

$$a_K^T(v) = \frac{36}{|K|}\Big\{\int_K (v - w_K)^T \cdot DF_K^{-1} dxdy - \beta_K(v) \cdot \int_K (\eta, \xi) \hat{\varphi}_0\, dx\,dy\Big\}, \tag{4.8}$$

where $w_K = w_h|_K$, and

$$D = \frac{|K|^2}{36 \times 90} - \Big(\int_K \xi\hat{\varphi}_0\, dx\,dy\Big)^2 - \Big(\int_K \eta\hat{\varphi}_0\, dx\,dy\Big)^2. \tag{4.9}$$

Lemma 4.1.

$$D \geqslant \Big(\frac{|K|}{60}\Big)^2. \tag{4.10}$$

Proof. By use of (2.3) and considering

$$\int_K \xi\hat{\varphi}_0\, dx\,dy = \frac{1}{180}(|\Delta_1| - |\Delta_3|), \tag{4.11}$$

$$\int_K \eta\,\hat{\varphi}_0\, dx\,dy = \frac{1}{180}(|\Delta_2| - |\Delta_4|), \tag{4.12}$$

we have

$$D = \frac{1}{360 \times 9}\{|K|^2 - \frac{1}{10}[(|\Delta_1| - |\Delta_3|)^2 + (|\Delta_2| - |\Delta_4|)^2]\} \geqslant \frac{|K|^2}{3600}.$$

From (4.10), it can be seen that the mapping F_K (2.2) is invertible and $a_K(v)$, $\beta_K(v)$ are definite.

Lemma 4.2. $\forall v \in (H_0^1(\Omega))^2$, $K \in \mathscr{T}_h$,

$$\int_K (v - r_K v)^T DF_K^{-1}\, dx\,dy = 0^T, \tag{4.13}$$

$$\int_K (v - r_K v)^T DF_K^{-1} \begin{pmatrix} \eta \\ \xi \end{pmatrix} dxdy = 0. \tag{4.14}$$

Proof. First, it can be easily seen that

$$\int_K \hat{\varphi}_0\, dx\, dy = |K|/36, \tag{4.15}$$

$$\int_K (\xi^2+\eta^2)\hat{\varphi}_0\, dx\, dy = |K|/90. \tag{4.16}$$

Then from (4.7)—(4.9) and (4.6),

$$\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T DF_K^{-1} dx\, dy = \int_K (\boldsymbol{v}-\boldsymbol{w}_K)^T DF_K^{-1} dx\, dy - \boldsymbol{\alpha}_K^T(v)\cdot\int_K \hat{\varphi}_0\, dx\, dy$$
$$-\beta_K(\boldsymbol{v})\int_K (\eta,\ \xi)\hat{\varphi}_0 dxdy = \int_K (\boldsymbol{v}-\boldsymbol{w}_K)^T DF_K^{-1}\, dxdy$$
$$-\left\{\int_K (\boldsymbol{v}-\boldsymbol{w}_K)^T DF_K^{-1}\, dx\, dy - \beta_K(\boldsymbol{v})\cdot\int_K (\eta,\ \xi)\hat{\varphi}_0\, dx\, dy\right\}$$
$$-\beta_K(\boldsymbol{v})\int_K (\eta,\ \xi)\hat{\varphi}_0\, dx\, dy = \boldsymbol{0}^T.$$

By (4.11), (4.12),

$$\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T DF_K^{-1}\begin{pmatrix}\eta\\ \xi\end{pmatrix} dx\, dy$$
$$=\int_K (\boldsymbol{v}-\boldsymbol{w}_K)^T DF_K^{-1}\begin{pmatrix}\eta\\ \xi\end{pmatrix} dx\, dy - \boldsymbol{\alpha}_K^T(\boldsymbol{v})\int_K \begin{pmatrix}\eta\\ \xi\end{pmatrix}\hat{\varphi}_0\, dx\, dy - \beta_K(\boldsymbol{v})\int_K (\xi^2+\eta^2)\hat{\varphi}_0\, dx\, dy$$
$$=\int_K (\boldsymbol{v}-\boldsymbol{w}_K)^T DF_K^{-1}\begin{pmatrix}\eta\\ \xi\end{pmatrix} dx\, dy - \frac{1}{5|K|}\int_K (\boldsymbol{v}-\boldsymbol{w}_K)^T DF_K^{-1} dx\, dy \begin{pmatrix}|\Delta_2|-|\Delta_4|\\ |\Delta_1|-|\Delta_3|\end{pmatrix}$$
$$-\frac{36}{|K|}\cdot D\cdot\beta_K(\boldsymbol{v}) = 0.$$

The last equation can be deduced from (4.7), (4.11) and (4.12).

Lemma 4.3. *Condition* (1.15) *holds.*

Proof. $\forall q_h \in M_h,\ \boldsymbol{v}\in (H_0^1(\Omega))^2$,

$$\sum_{K\in\mathcal{J}_h}\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot \operatorname{grad} q_h\, dx\, dy$$
$$=\frac{1}{4}\sum_{K\in\mathcal{J}_h}\left\{\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T DF_K^{-1} dx\, dy \begin{pmatrix} q_1-q_2-q_3+q_4\\ q_1+q_2-q_3-q_4\end{pmatrix}\right\}$$
$$+\frac{1}{4}\sum_{K\in\mathcal{J}_h}\left\{\int_K (\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot DF_K^{-1}\cdot\begin{pmatrix}\eta\\ \xi\end{pmatrix} dx\, dy\cdot (q_1-q_2+q_3-q_4) = 0,\right.$$

in which the last equality is a direct result from Lemma 4.2. This is condition (1.15) The proof is thus completed.

Lemma 4.4.

$$|r_h\boldsymbol{v}|_{1,\Omega}\leqslant c|\boldsymbol{v}|_{1,\Omega},\quad \forall \boldsymbol{v}\in (H_0^1(\Omega))^2, \tag{4.17}$$

$$\|\boldsymbol{v}-r_h\boldsymbol{v}\|_{1,\Omega}\leqslant ch\|\boldsymbol{v}\|_{2,\Omega},\quad \forall \boldsymbol{v}\in (H_0^1(\Omega)\cap H^2(\Omega))^2. \tag{4.18}$$

Proof. From (4.7), (4.8), (4.10) and taking into account the inequalities (c.f. [4], p. 247)

$$\left.\begin{aligned}&\sup_{(x,y)\in K}\|DF_K^{-1}\| = |F_K^{-1}|_{1,\infty,k}\leqslant ch_K^{-1},\\ &\sup_{(x,y)\in K}\|DF_K\| = |F_K|_{1,\infty,k}\leqslant ch_K,\end{aligned}\right. \tag{4.19}$$

we have

$$|\beta_K(\boldsymbol{v})|\leqslant ch_K^{-2}\|\boldsymbol{v}-\boldsymbol{w}_h\|_{0,K}, \tag{4.20}$$

$$\|\boldsymbol{a}_K(\boldsymbol{v})\|\leqslant ch_K^{-2}\|\boldsymbol{v}-\boldsymbol{w}_h\|_{0,K}. \tag{4.21}$$

Then, in the same way as in the proof of Lemma 3.2, from (4.6), the lemma can be proved.

Finally, the operator ρ_h: $H^1(\Omega)\cap L_0^2(\Omega)\to M_h$ is defined as in (3.22) associated with the general quadrilateral subdivision of Ω, and we have

Lemma 4.5.

$$\|q-\rho_h q\|_{0,\Omega}\leqslant ch|q|_{1,\Omega},\quad \forall q\in H^1(\Omega)\cap L_0^2(\Omega).. \tag{4.22}$$

Summarizing Lemmas 4.3, 4.4, 4.5 we have

Theorem 4.1. *Assume that the solution of the problem* (ST), $(\boldsymbol{u}, p)\in (H^2(\Omega)\cap H_0^1(\Omega))^2\times(H^1(\Omega)\cap L_0^2(\Omega))$, *the subdivision* $\mathcal{T}_h$ *is regular, and the spaces* $X_h\subset (H_0^1(\Omega))^2$, $M_h\subset L_0^2(\Omega)\cap H^1(\Omega)$ *are defined by* (4.3) *and* (4.1)—(4.2) *respectively. Then the following error estimate holds:*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\leqslant ch(\|\boldsymbol{u}\|_{2,\Omega}+|p|_{1,\Omega}). \tag{4.23}$$

References

[1] Arnold D. N.; Brezzi F.; Fortin M.: A stable finite element for the Stokes equations, *Calcolo*, **21:4** (1984), 337—344.

[2] Boland J. M.; Nicolaides R. A.: Stable and semistable low order finite elements for viscous flows, *SIAM J. Numer. Anal.*, **22**: 3 (1985), 474—492.

[3] Brezzi F.: On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers, *RAIRO Numer. Anal.*, **8**: 2 (1974), 129—151.

[4] Ciarlet P. G.: The Finite Element Method for Elliptic Problems, North-Holland, 1978.

[5] Fortin M.: Old and new finite elements for incompressible flows, *Inter. J. for Numer. Methods in fluids*, **1**: 3 (1981), 347—364.

[6] Girault V.; Reviart P. A: Finite element approximation of the Navier-Stokes equations, Lecture Notes in Mathematics 749, Springer-Verlag, 1979.

[7] Wang Lie-heng: An analysis of the mixed finite element approximation for the Stokes problem, *Mathematica Numerica Sinica*, **9**: 1 (1987), 70—81. (in Chinese)