ON THE STABILITY OF FINITE-DIFFERENCE SCHEMES OF HIGHER-ORDER APPROXIMATE ONE-WAY WAVE EQUATIONS*

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Abstract

The finite difference migration, proposed and developed by J. F. Claerbout[1], is now widely used in seismic data processing. The method has a limitation that the events are not dipping too much. Guanquan ZHANG derived a new version of higher-order approximation of one-way wave equation in the form of systems of lower-order equations[2]. For these systems he constructed some suitable difference schemes and developed a new algorithm of finite-difference migration for steep dips[3]. In this paper, we discuss the stability of these difference schemes by the method of energy estimation.

§1. Equations and Difference Schemes

For steep dip migration the following system of lower-order equations can be used[2]

$$\begin{cases} \dfrac{\partial p}{\partial z}=\sum\limits_{l=1}^{m/2}\dfrac{\partial q_l}{\partial t}, & (1.1a)\\ \dfrac{1}{c^2}\dfrac{\partial^2 q_l}{\partial t^2}=\alpha_{m,l}^2\dfrac{\partial^2 q_l}{\partial x^2}+\beta_{m,l}\dfrac{\partial^2 p}{\partial x^2}, \quad l=1, 2, \cdots, m/2. & (1.1b)\end{cases}$$

The initial and boundary conditions are

$$\begin{cases} p|_{z=0}=\varphi(x, t), & |x|<X,\ 0<t\leqslant T,\\ p=q_l=\dfrac{\partial q_l}{\partial t}=0, & |x|=X \ \text{or}\ t=0,\end{cases}$$

where m is an even integer, $l=1, 2, \cdots, \frac{m}{2}$,

$$\alpha_{m,l}=\cos(l\pi/(m+1)),$$

$$\beta_{m,l}=\prod_{j=1}^{m-1}(\alpha_{m,l}-\alpha_{m-1,j})\Big/\prod_{j\neq l}^{m}(\alpha_{m,l}-\alpha_{m,j}).$$

It can be easily verified[2] that

$$\beta_{m,l}>0, \quad \sum_{l=1}^{m/2}\beta_{m,l}=1/2. \tag{1.1c}$$

From (1.1a), (1.1b) and (1.1c) one obtains

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t\,\partial z}-\frac{1}{2}\frac{\partial^2 p}{\partial x^2}-\sum_{l=1}^{m/2}\alpha_{m,l}^2\frac{\partial^2 q_l}{\partial x^2}=0. \tag{1.1d}$$

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Obviously, problems (1.1d), (1.1b) and (1.1a), (1.1b) are equivalent. For $m=2$, (1.1a), (1.1b) can be simplified to

$$\begin{cases} \dfrac{\partial^2 p}{\partial z\,\partial t}=\dfrac{c^2}{2}\dfrac{\partial^2 p}{\partial x^2}+\dfrac{c^2}{4}\dfrac{\partial^2 q}{\partial x^2}, & (1.2a)\\ \dfrac{\partial p}{\partial z}=\dfrac{\partial q}{\partial t}. & (1.2b)\end{cases}$$

The approximations of $p(k\Delta x, j\Delta t, n\Delta z)$, $q(k\Delta x, (j-1/2)\Delta t, (n+1/2)\Delta z)$, $q(k\Delta x, j\Delta t, (n+1/2)\Delta z)$, $q_l(k\Delta x, (j-1/2)\Delta t, (n+1/2)\Delta z)$ are denoted respectively by

$$p^n_{k,j},\ q^{n+1/2}_{k,j-1/2},\ q^{n+1/2}_{k,j},\ q^{n+1/2}_{lk,j-1/2}.$$

Δ^+_x, Δ^-_x, and δ^2 are defined by

$$\Delta^+_x p^n_{k,j}=p^n_{k+1,j}-p^n_{k,j},\quad \Delta^-_x p^n_{k,j}=p^n_{k,j}-p^n_{k-1,j},\quad \delta^2 p^n_{k,j}=\Delta^+_x\Delta^-_x p^n_{k,j}.$$

Δ^+_t, Δ^-_t, Δ^+_z, Δ^-_z are similarly defined. If we let Δ denote Δ^+, we have the identities

$$\begin{aligned}\Delta(u_jv_j)&=u_{j+1}\Delta v_j+(\Delta u_j)v_j=u_j\Delta v_j+(\Delta u_j)v_{j+1}\\ &=\frac{1}{2}[(u_j+u_{j+1})\Delta v_j+(v_j+v_{j+1})\Delta u_j],\end{aligned}\tag{1.3}$$

$$(u_j+u_{j+1})\Delta u_j=\Delta(u_j^2).\tag{1.4}$$

We can approximate (1.2) by the difference scheme

$$\text{I:}\begin{cases}\Delta^+_t\Delta^+_z(1+\alpha\delta^2)p^n_{k,j}/\Delta t\Delta z=r^n_k\Delta^+_x\Delta^-_x(p^n_{k,j}+p^{n+1}_{k,j}+p^n_{k,j+1}+p^{n+1}_{k,j+1})/\Delta x^2\\ \quad +r^n_k\Delta^+_x\Delta^-_x(q^{n+1/2}_{k,j}+q^{n+1/2}_{k,j+1})/\Delta x^2, & (1.5a)\\ \Delta^+_t q^{n+1/2}_{k,j}/\Delta t=\Delta^+_z(p^n_{k,j}+p^n_{k,j+1})/2\Delta z, & (1.5b)\end{cases}$$

with the initial and boundary conditions

$$\begin{cases}p^0_{k,j}=\varphi(k\Delta x, j\Delta t),\\ p^n_{k,j}=q^{n+1/2}_{k,j}=0, & |k|=K,\\ p^n_{k,j}=q^{n+1/2}_{k,j}=0, & j=0.\end{cases}\tag{1.5c}$$

The interval $[-X, X]$ is divided into $2K$ equal parts, $\Delta x=X/K$.

We can also use the following scheme

$$\text{II:}\begin{cases}\Delta^+_t\Delta^+_z(1+\alpha\delta^2)p^n_{k,j}/\Delta t\Delta z-r^n_k\Delta^+_x\Delta^-_x(p^n_{k,j}+p^{n+1}_{k,j}+p^n_{k,j+1}\\ \quad +p^{n+1}_{k,j+1})/\Delta x^2-2r^n_k\Delta^+_x\Delta^-_x q^{n+1/2}_{k,j+1/2}/\Delta x^2=0, & (1.6a)\\ \Delta^+_t q^{n+1/2}_{k,j-1/2}/\Delta t=\Delta^+_z p^n_{k,j}/\Delta z, & (1.6b)\end{cases}$$

with the initial and boundary conditions

$$\begin{cases}p^0_{k,j}=\varphi(k\Delta x, j\Delta t),\\ p^n_{k,j}=q^{n+1/2}_{k,j-1/2}=0, & |k|=K,\\ p^n_{k,j}=q^{n+1/2}_{k,j+1/2}=0, & j=0.\end{cases}\tag{1.6c}$$

(1.1) can also be approximated by the difference scheme

$$\text{III:}\begin{cases}\Delta^+_t\Delta^+_z(1+\alpha\delta^2)p^n_{k,j}/\Delta t\Delta z=r^n_k\Delta^+_x\Delta^-_x(p^n_{k,j}+p^{n+1}_{k,j}+p^n_{k,j+1}+p^{n+1}_{k,j+1})/\Delta x^2\\ \quad +\sum\limits_{l=1}^{m/2}\alpha^n_{lk}\Delta^+_x\Delta^-_x q^{n+1/2}_{lk,j+1/2}/\Delta x^2, & (1.7a)\\ \Delta^+_t\Delta^-_t q^{n+1/2}_{lk,j+1/2}/\Delta t^2=\alpha^n_{lk}\Delta^+_x\Delta^-_x q^{n+1/2}_{lk,j+1/2}/\Delta x^2\\ \quad +\beta^n_{lk}\Delta^+_x\Delta^-_x(p^n_{k,j}+p^{n+1}_{k,j}+p^n_{k,j+1}+p^{n+1}_{k,j+1})/\Delta x^2, & (1.7b)\end{cases}$$

with the initial and boundary conditions

$$\begin{cases} p^0_{k,j}=\varphi(k\Delta x,\ j\Delta t), \\ p^n_{k,j}=q^{n+1/2}_{lk,j+1/2}=0, \quad j=0,1, \\ p^n_{k,j}=q^{n+1/2}_{lk,j+1/2}=0, \quad |k|=K, \end{cases} \tag{1.7c}$$

where
$$r^n_k=\frac{c^2\left(k\Delta x,\ \left(n+\frac{1}{2}\right)\Delta z\right)}{8}, \quad \alpha^n_{lk}=c^2\left(k\,\Delta x,\ \left(n+\frac{1}{2}\right)\Delta z\right)\alpha^2_{m,l},$$

$$\beta^n_{lk}=c^2\left(k\Delta x,\ \left(n+\frac{1}{2}\right)\Delta z\right)\beta_{m,l}/4.$$

From (1.1c) one has

$$\sum_{l=1}^{m/2}\beta^n_{lk}=r^n_k. \tag{1.7d}$$

The following lemma is very useful in our stability proof.

Lemma*. *If there exists a constant $K_0>0$ such that*

$$\frac{\Delta^+_t}{\Delta t}Q^{n,j}+\frac{\Delta^+_z}{\Delta z}P^{n,j}\leqslant K_0(P^{n,j}+P^{n+1,j}+Q^{n,j}+Q^{n,j+1}), \tag{1.8}$$

then for sufficiently small $l=\max\{\Delta z,\ \Delta t\}$,

$$\sum_{j=0}^{J=1}P^{N,j}\Delta t+\sum_{n=1}^{N-1}Q^{n,J}\Delta z\leqslant C_0(K_0)\exp[2K_0(N\Delta z+J\Delta t)]\left(\sum_{j=0}^{J-1}P^{0,j}\Delta t+\sum_{n=0}^{N-1}Q^{n,0}\Delta z\right),$$

where $C_0(K_0)$ is a constant.

Proof. From (1.8) one obtains

$$\left(\frac{1-K_0\Delta z}{1+K_0\Delta z}P^{n+1,j}-P^{n,j}\right)\Big/\Delta z+\frac{1+K_0\Delta t}{1+K_0\Delta z}\left(\frac{1-K_0\Delta t}{1+K_0\Delta t}Q^{n,j+1}-Q^{n,j}\right)\Big/\Delta t\leqslant 0.$$

Put
$$R^{n,j}=\left(\frac{1-K_0\Delta z}{1+K_0\Delta z}\right)^n\left(\frac{1-K_0\Delta t}{1+K_0\Delta t}\right)^j<1,$$

$$\tilde{P}^{n,j}=R^{n,j}P^{n,j},\quad \tilde{Q}^{n,j}=R^{n,j}Q^{n,j}.$$

Multiplying both sides of the above inequality by $R^{n,j}\Delta t\Delta z$, and summing up with respect to n, j, one obtains

$$\sum_{j=0}^{J-1}\tilde{P}^{N,j}\Delta t+\sum_{n=0}^{N-1}\frac{1+K_0\Delta t}{1+K_0\Delta z}\tilde{Q}^{n,J}\Delta z$$
$$\leqslant\sum_{j=0}^{J-1}\tilde{P}^{0,j}\Delta t+\sum_{n=0}^{N-1}\frac{1+K_0\Delta t}{1+K_0\Delta z}\tilde{Q}^{n,0}\Delta z.$$

When l is sufficiently small, one obtains

$$\left(\frac{1-K_0\Delta z}{1+K_0\Delta z}\right)^n\sim\exp(-2K_0n\Delta z),$$
$$\left(\frac{1-K_0\Delta t}{1-K_0\Delta t}\right)^j\sim\exp(-2K_0j\Delta t),$$
$$\frac{1}{2}\leqslant\frac{1+K_0\Delta t}{1+K_0\Delta z}\leqslant 2,$$

so that there exists $C_0(K_0)$ such that

$$\sum_{j=0}^{J-1}P^{N,j}\Delta t+\sum_{n=0}^{N-1}Q^{n,J}\Delta z$$
$$\leqslant C_0(K_0)\exp[2K_0(N\Delta z+J\Delta t)]\left[\sum_{j=0}^{J-1}P^{0,j}\Delta t+\sum_{n=0}^{N-1}Q^{n,0}\Delta z\right].$$

§ 2. Stability of Scheme I

In this section we will discuss the stability of (1.5).

From (1.3), (1.4) and (1.5c) one obtains

$$\frac{1}{\Delta z\Delta t^2}\sum_{k=-K}^{K-1}\Delta_t^+(p_{k,j}^n+p_{k,j}^{n+1})\Delta_z^+\Delta_t^+p_{k,j}^n\Delta x=\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\right)^2\Delta x, \tag{2.1}$$

$$\begin{aligned}&-\frac{\alpha}{\Delta z\Delta t^2}\sum_{k=-K}^{K-1}\Delta_t^+(p_{k,j}^n+p_{k,j}^{n+1})\Delta_z^+\Delta_t^+\delta^2p_{k,j}^n\,\Delta x=-\frac{\alpha}{\Delta z\,\Delta t^2}\sum_{k=-K}^{K-1}\Delta_t^+\Delta_x^+(p_{k,j}^n\\&\quad+p_{k,j}^{n+1})\Delta_z^+\Delta_t^+\Delta_x^+p_{k,j}^n\Delta x=-\alpha\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}\left(\Delta_x^+\frac{\Delta_t^+}{\Delta t}p_{k,j}^n\right)^2\Delta x,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\frac{-1}{\Delta t\Delta x^2}\sum_{k=-K}^{K-1}\Delta_t^+(p_{k,j}^n+p_{k,j}^{n+1})r_k^n\delta^2(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})\Delta x\\
&=\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\frac{\Delta_t^+(p_{k+1,j}^n+p_{k+1,j}^{n+1})}{\Delta t}\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\Delta x\\
&\quad+\sum_{k=-K}^{K-1}r_k^n\frac{\Delta_t^+\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta t\Delta x}\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\Delta x\\
&=\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\right)^2\Delta x\\
&\quad+\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\frac{\Delta_t^+(p_{k+1,j}^n+p_{k+1,j}^{n+1})}{\Delta t}\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta t}\Delta x.\end{aligned} \tag{2.3}$$

Using (1.5b) and the same deduction as used above, one obtains

$$\begin{aligned}&\frac{-1}{\Delta t\Delta x^2}\sum_{k=-K}^{K-1}r_k^n\Delta_t^+(p_{k,j}^n+p_{k,j}^{n+1})\delta^2(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})\Delta x\\
&=\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\frac{\Delta_t^+(p_{k+1,j}^n+p_{k+1,j}^{n+1})}{\Delta t}\frac{\Delta_x^+(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})}{\Delta x}\Delta x\\
&\quad+2\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}r_k^n\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\frac{\Delta_x^+q_{k,j}^{n+1/2}}{\Delta x}\Delta x\\
&\quad-\sum_{k=-K}^{K-1}r_k^n\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\frac{\Delta_z^+\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{2\Delta z\Delta x}\Delta x\\
&=-\frac{\Delta_z^+}{2\Delta z}\sum_{k=-K}^{K-1}r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\right)^2\Delta x\\
&\quad+\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\frac{\Delta_t^+(p_{k+1,j}^n+p_{k+1,j}^{n+1})}{\Delta t}\frac{\Delta_x^+(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})}{\Delta x}\Delta x\\
&\quad+\sum_{k=-K}^{K-1}\frac{\Delta_z^+r_k}{2\Delta z}\left(\frac{\Delta_x^+(p_{k,j}^{n+1}+p_{k,j+1}^{n+1})}{\Delta x}\right)^2\Delta x\\
&\quad+2\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}r_k^n\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\frac{\Delta_x^+q_{k,j}^{n+1/2}}{\Delta x}\Delta x.\end{aligned} \tag{2.4}$$

Multiplying both sides of (1.5a) by $\frac{1}{\Delta t}\Delta_t^+(p_{k,j}^n+p_{k,j}^{n+1})$ and summing up with respect to k, one obtains

$$\begin{aligned}&\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\right)^2\right]\Delta x+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\right)^2\Delta x\\
&-\frac{\Delta_z^+}{2\Delta z}\sum_{k=-K}^{K-1}r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\right)^2\Delta x+\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\frac{\Delta_t^+(p_{k+1,j}^n+p_{k+1,j}^{n+1})}{\Delta t}\end{aligned}$$

$$\times \frac{\Delta_x^+(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})}{\Delta x}\Delta x+\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r_k^n}{\Delta x}\,\frac{\Delta_t^+(p_{k+1,j}^n+p_{k+1,j}^{n+1})}{\Delta t}$$

$$\times\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\Delta x+2\,\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1} r_k^n\,\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\,\frac{\Delta_x^+ q_{k,j}^{n+1/2}}{\Delta x}$$

$$+\sum_{k=-K}^{K-1}\frac{\Delta_z^+ r_k^n}{2\Delta z}\left(\frac{\Delta_x^+(p_{k,j}^{n+1}+p_{k,j+1}^{n+1})}{\Delta x}\right)^2\Delta x=0. \tag{2.5}$$

Multiplying both sides of (1.5a) by $\frac{\Delta_z^+}{\Delta z}(p_{k,j}^n+p_{k,j+1}^n)$, substituting (1.5b) into it, and summing up with respect to k, one obtains a similar equality

$$\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_z^+ p_{k,j}^n}{\Delta z}\right)^2-\alpha\left(\Delta_x^+\,\frac{\Delta_z^+ p_{k,j}^n}{\Delta z}\right)^2\right]\Delta x+\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1} r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\right)^2\Delta x$$

$$+2\,\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1} r_k^n\left(\frac{\Delta_x^+ q_{k,j}^{n+1/2}}{\Delta x}\right)^2\Delta x-\sum_{k=-K}^{K-1}\frac{\Delta_z^+ r_k^n}{\Delta z}\left(\frac{\Delta_x^+(p_{k,j}^{n+1}+p_{k,j+1}^{n+1})}{\Delta x}\right)^2\Delta x$$

$$+\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r_k^n}{\Delta x}\,\frac{\Delta_z^+(p_{k+1,j}^n+p_{k+1,j+1}^n)}{\Delta z}\,\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\Delta x$$

$$+\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r_k^n}{\Delta x}\,\frac{\Delta_z^+(p_{k+1,j}^n+p_{k+1,j+1}^n)}{\Delta z}\,\frac{\Delta_x^+(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})}{\Delta x}\Delta x=0. \tag{2.6}$$

Put

$$S_{11}^{n,j}=\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+ p_{k,j}^n}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\,\frac{\Delta_t^+ p_{k,j}^n}{\Delta t}\right)^2\right]\Delta x,$$

$$S_{21}^{n,j}=\frac{1}{2}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_z^+ p_{k,j}^n}{\Delta z}\right)^2-\alpha\left(\Delta_x^+\,\frac{\Delta_z^+ p_{k,j}^n}{\Delta x}\right)^2\right]\Delta x+\sum_{k=-K}^{K-1} r_k^n\left[\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\right.$$

$$\left.+\frac{\Delta_x^+ q_{k,j}^{n+1/2}}{\Delta x}\right]^2\Delta x.$$

Then

$$S_{11}^{n,j}\geqslant(1-4\alpha)\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+ p_{k,j}^n}{\Delta t}\right)^2\Delta x, \tag{2.7}$$

$$S_{21}^{n,j}\geqslant\frac{1-4\alpha}{2}\left[\sum_{k=-K}^{K-1}\left(\frac{\Delta_z^+ p_{k,j}^n}{\Delta z}\right)^2\Delta x+\sum_{k=-K}^{K-1} r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\right.\right.$$

$$\left.\left.+\frac{\Delta_x^+ q_{k,j}^{n+1/2}}{\Delta x}\right)^2\Delta x\right], \tag{2.8}$$

when $\alpha\in(0,\ 1/4)$.

Lemma 1. *Suppose $C(z, x)$ is a continuously differentiable function and there exist constants $C_0>0$, C_D, C_L such that*

$$C^2(z,\ x)\geqslant C_0,\quad \max\left|\frac{\partial C^2}{\partial x}\right|<C_L,\quad \max\left|\frac{\partial C^2}{\partial z}\right|<C_D.$$

Then for $\alpha\in(0,\ 1/4)$ and sufficiently small $l=\max\{\Delta z,\ \Delta x,\ \Delta t\}$,

$$\frac{\Delta_z^+ S_{11}^{n,j}}{\Delta z}+\frac{\Delta_t^+ S_{21}^{n,j}}{\Delta t}\leqslant K_1(S_{11}^{n,j}+S_{11}^{n+1,j}+S_{21}^{n,j}+S_{21}^{n,j+1}), \tag{2.9}$$

where K_1 is a constant.

Proof. From (2.5) + (2.6)/2 one obtains

$$\frac{\Delta_z^+}{\Delta z}\,S_{11}^{n,j}+\frac{\Delta_t^+}{\Delta t}\,S_{21}^{n,j}=-\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r_k^n}{\Delta x}\,\frac{\Delta_t^+(p_{k+1,j}^n+p_{k-1,j}^{n+1})}{\Delta t}\left[\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\right.$$

$$\left.+\frac{\Delta_x^+(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})}{\Delta x}\right]\Delta x-\frac{1}{2}\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r_k^n}{\Delta x}\,\frac{\Delta_z^+(p_{k+1,j}^n+p_{k+1,j+1}^n)}{\Delta z}$$

$$\times\left[\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}+\frac{\Delta_x^+(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})}{\Delta x}\right]\Delta x.$$

Since $r_k^n=\dfrac{C^2\left(k\Delta x,\ \left(n+\frac{1}{2}\right)\Delta z\right)}{8}$, for sufficiently small l one obtains

$$\left|\frac{\Delta_x^+ r_k^n}{\Delta x}\right|<\frac{C_L}{4},\qquad\left|\frac{\Delta_z^+ r_k^n}{\Delta z}\right|<\frac{C_D}{4}.$$

Using the Schwarz inequality one obtains

$$\frac{\Delta_z^+ S_{11}^{n,j}}{\Delta z}+\frac{\Delta_t^+ S_{21}^{n,j}}{\Delta t}\leqslant\frac{C_L}{8}\Bigg[\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+(p_{k+1,j}^n+p_{k+1,j}^{n+1})}{\Delta t}\right)^2\Delta x+\frac{3}{2}\sum_{k=-K}^{K-1}\Bigg(\frac{\Delta_x^+(q_{k,j}^{n+1/2}+q_{k,j+1}^{n+1/2})}{\Delta x}$$
$$+\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\Bigg)^2\Delta x+\frac{1}{2}\sum_{k=-K}^{K-1}\left(\frac{\Delta_z^+(p_{k+1,j}^n+p_{k+1,j+1}^n)}{\Delta z}\right)^2\Delta x\Bigg]$$
$$\leqslant\frac{C_L}{4}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\right)^2+\left(\frac{\Delta_t^+p_{k,j}^{n+1}}{\Delta t}\right)^2\right]\Delta x+\frac{3C_L}{C_0}\sum_{k=-K}^{K-1}r_k^n\Bigg[\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}+\frac{\Delta_x^+q_{k,j}^{n+1/2}}{\Delta x}\right)^2$$
$$+\left(\frac{\Delta_x^+(p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}+\frac{\Delta_x^+q_{k,j+1}^{n+1/2}}{\Delta x}\right)^2\Bigg]\Delta x+\frac{C_L}{8}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_z^+p_{k,j}^n}{\Delta z}\right)^2+\left(\frac{\Delta_z^+p_{k,j+1}^n}{\Delta z}\right)^2\right]\Delta x$$
$$\leqslant K_1(S_{11}^{n,j}+S_{11}^{n+1,j}+S_{21}^{n,j}+S_{21}^{n,j+1}),$$

where $K_1=(C_L/4(1-4\alpha))\max(24/C_0,\ 1)$.

Theorem 1. *Under the conditions of Lemma 1, difference scheme (1.5) is absolutely stable. That is there exists a constant l_0, such that for $\max(\Delta z,\ \Delta x,\ \Delta t)<l_0$, and all N, J satisfying $N\Delta z\leqslant D$, $J\Delta t\leqslant T$,*

$$\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}(p_{k,j}^N)^2\Delta x\Delta t\leqslant\bar{C}_1(K_1)\exp[2K_1(D+T)]\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\varphi_{k,j}^2\Delta x\Delta t,\tag{2.10}$$

where $\bar{C}_1(K_1)$ is a constant depending on the constant obtained in Lemma 1.

Proof. From Lemma*, (2.9) gives

$$\sum_{j=0}^{J-1}S_{11}^{N,j}\Delta t+\sum_{n=0}^{N-1}S_{21}^{n,J}\Delta z\leqslant C_1(K_1)\exp[2K_1(D+T)]\left[\sum_{j=0}^{J-1}S_{11}^{0,j}\Delta t+\sum_{n=0}^{N-1}S_{21}^{n,0}\Delta z\right].\tag{2.11}$$

From (1.5c),

$$S_{11}^{0,j}=\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+p_{k,j}^0}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\frac{\Delta_t^+p_{k,j}^0}{\Delta t}\right)^2\right]\Delta x\leqslant(1+4\alpha)\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+\varphi_{k,j}}{\Delta t}\right)^2\Delta x,$$
$$S_{21}^{n,0}=0.\tag{2.12}$$

From (2.8), (2.9),

$$\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+p_{k,j}^N}{\Delta t}\right)^2\Delta x\Delta t\leqslant\frac{1+4\alpha}{1-4\alpha}C_1(K_1)\exp[2K_1(D+T)]\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+\varphi_{k,j}}{\Delta t}\right)^2\Delta x\Delta t.$$

Put

$$\tilde{\tilde{p}}_{k,j}^n=\sum_{i=-\infty}^{j}p_{k,i}^n\Delta t,\quad\tilde{\tilde{q}}_{k,j}^{n+1/2}=\sum_{i=-\infty}^{j}q_{k,i}^{n+1/2}\Delta t,\quad\tilde{\tilde{\varphi}}_{k,j}^n=\sum_{i=-\infty}^{j}\varphi_{k,i}\Delta t,$$

where we define $p_{k,i}^n=q_{k,i}^{n+1/2}=0$ for $i\leqslant-1$.

Then $\tilde{\tilde{p}}_{k,j}^n$, $\tilde{\tilde{q}}_{k,j}^{n+1/2}$ also satisfy equation (1.5). So we have the estimate

$$\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+\tilde{\tilde{p}}_{k,j}^N}{\Delta t}\right)^2\Delta x\Delta t\leqslant\bar{C}_1(K_1)\exp[2K_1(D+T)]\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+\tilde{\tilde{\varphi}}_{k,j}}{\Delta t}\right)^2\Delta x\Delta t,$$

where
$$\bar{C}_1(K_1)=C_1(K_1)\frac{1+4\alpha}{1-4\alpha}.$$

Obviously
$$\frac{\Delta_t^+ \tilde{p}^N_{k,j}}{\Delta t}=p^N_{k,j}, \quad \frac{\Delta_t^+ \tilde{\varphi}_{k,j}}{\Delta t}=\varphi_{k,j},$$
from which (2.10) is obtained immediately.

Theorem 2. *Under the conditions of Lemma 1, there exists a constant l_0, such that for* max $(\Delta x, \Delta z, \Delta t)<l_0$, *and all N, J satisfying*
$$N\Delta z\leqslant D, \quad J\Delta t\leqslant T,$$
$$\sum_{n=0}^{N-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_z^+ p^n_{k,J}}{\Delta z}\right)^2 \Delta t\ \Delta z\leqslant \bar{C}_1'(K_1)\exp[2K_1(D+T)]\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+\varphi_{k,j}}{\Delta t}\right)^2\Delta x\Delta t,$$
where $\bar{C}_1'(K_1)$ is a constant.

Proof. From (2.11), (2.7), (2.8) and (2.12) one obtains immediately
$$\sum_{n=0}^{N-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_z^+ p^n_{k,J}}{\Delta z}\right)^2\Delta t\ \Delta z\leqslant\frac{2(1+4\alpha)}{1-4\alpha}C_1(K_1)\exp[2K_1(D+T)]$$
$$\times\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+\varphi_{k,j}}{\Delta t}\right)^2\Delta x\ \Delta t.$$

§ 3. Stability of Scheme II

In this section we will discuss the stability of (1.6). From (1.3), (1.4), (1.6b) and (1.6c) one obtains
$$\frac{-2}{\Delta t\Delta x^2}\sum_{k=-K}^{K-1} r^n_k\Delta_t^+(p^n_{k,j}+p^{n+1}_{k,j})\Delta_x^+\Delta_x^- q^{n+1/2}_{k,j+1/2}\,\Delta x$$
$$=2\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r^n_k}{\Delta x}\frac{\Delta_t^+(p^n_{k+1,j}+p^{n+1}_{k+1,j})}{\Delta t}\frac{\Delta_x^+ q^{n+1/2}_{k,j+1/2}}{\Delta x}\Delta x+2\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1} r^n_k\frac{\Delta_x^+}{\Delta x}(p^n_{k,j}+p^{n+1}_{k,j})$$
$$\times\frac{\Delta_x^+ q^{n+1/2}_{k,j-1/2}}{\Delta x}\Delta x-2\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1} r^n_k\left(\frac{\Delta_x^+ p^n_{k,j}}{\Delta x}\right)^2\Delta x+2\sum_{k=-K}^{K-1}\frac{\Delta_z^+ r^n_k}{\Delta z}\left(\frac{\Delta_x^+ p^{n+1}_{k,j}}{\Delta x}\right)^2\Delta x. \tag{3.1}$$

Multiplying both sides of (1.6a) by $\frac{\Delta_t^+}{\Delta t}(p^n_{k,j}+p^{n+1}_{k,j})$, summing up with respect to k and using (2.1)—(2.3), (3.1), one obtains
$$\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+ p^n_{k,j}}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\frac{\Delta_t^+ p^n_{k,j}}{\Delta t}\right)^2\right]\Delta x-2\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1} r^n_k\left(\frac{\Delta_x^+ p^n_{k,j}}{\Delta x}\right)^2\Delta x$$
$$+2\sum_{k=-K}^{K-1}\frac{\Delta_z^+ r^n_k}{\Delta z}\left(\frac{\Delta_x^+ p^{n+1}_{k,j}}{\Delta x}\right)^2\Delta x+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\left[r^n_k\left(\frac{\Delta_x^+(p^n_{k,j}+p^{n+1}_{k,j})}{\Delta x}\right)^2\right.$$
$$\left.+2r^n_k\frac{\Delta_x^+ q^{n+1/2}_{k,j-1/2}}{\Delta x}\frac{\Delta_x^+(p^n_{k,j}+p^{n+1}_{k,j})}{\Delta x}\right]\Delta x+\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r^n_k}{\Delta x}\frac{\Delta_t^+(p^n_{k+1,j}+p^{n+1}_{k+1,j})}{\Delta t}$$
$$\times\frac{\Delta_x^+(p^n_{k,j}+p^{n+1}_{k,j}+p^n_{k,j+1}+p^{n+1}_{k,j+1})}{\Delta x}\Delta x$$
$$+2\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r^n_k}{\Delta x}\frac{\Delta_x^+ q^{n+1/2}_{k,j+1/2}}{\Delta x}\frac{\Delta_t^+(p^n_{k+1,j}+p^{n+1}_{k+1,j})}{\Delta t}\Delta x=0. \tag{3.2}$$

Applying $(1+\alpha\delta^2)\Delta_t^+/\Delta t$ to both sides of (1.6b), and substituting (1.6a) into it, one obtains
$$\Delta_t^+\Delta_t^-(1+\alpha\delta^2)q^{n+1/2}_{k,j+1/2}/\Delta t^2=r^n_k\Delta_x^+\Delta_x^-(p^n_{k,j}+p^{n+1}_{k,j}+p^n_{k,j+1}+p^{n+1}_{k,j+1})/\Delta x^2$$
$$+2r^n_k\Delta_x^+\Delta_x^- q^{n+1/2}_{k,j+1/2}/\Delta x^2. \tag{3.3}$$

By a similar deduction for (2.2) and (2.1) one obtains

$$\frac{1}{\Delta t^3}\sum_{k=-K}^{K-1}\Delta_t^+(q_{k,j-1/2}^{n+1/2}+q_{k,j+1/2}^{n+1/2})\Delta_t^+\Delta_t^-(1+\alpha\delta^2)q_{k,j+1/2}^{n+1/2}\Delta x$$
$$=\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\ \frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\right)^2\right]\Delta x. \tag{3.4}$$

Using (1.3), (1.4), (1.6b) and (1.6c) one obtains

$$\frac{-1}{\Delta t\Delta x^2}\sum_{k=-K}^{K-1}\Delta_t^+(q_{k,j-1/2}^{n+1/2}+q_{k,j+1/2}^{n+1/2})r_k^n\Delta_x^+\Delta_x^-(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})\Delta x$$
$$=\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\right)^2\Delta x-\sum_{k=-K}^{K-1}\frac{\Delta_z^+r_k^n}{\Delta z}\left(\frac{\Delta_x^+(p_{k,j}^{n+1}+p_{k,j+1}^{n+1})}{\Delta x}\right)^2\Delta x$$
$$+\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\,\frac{\Delta_t^+(q_{k+1,j-1/2}^{n+1/2}+q_{k+1,j+1/2}^{n+1/2})}{\Delta t}\,\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\Delta x. \tag{3.5}$$

Since

$$\frac{\Delta_x^+q_{k,j+1/2}^{n+1/2}}{\Delta x}\,\frac{\Delta_t^+\Delta_x^+(q_{k,j-1/2}^{n+1/2}+q_{k,j+1/2}^{n+1/2})}{\Delta t\Delta x}$$
$$=\frac{\Delta_x^+(q_{k,j+1/2}^{n+1/2}+q_{k,j-1/2}^{n+1/2})}{\Delta x}\,\frac{\Delta_t^+\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta t\Delta x}$$
$$+\frac{\Delta_x^+q_{k,j+1/2}^{n+1/2}}{\Delta x}\,\frac{\Delta_t^+\Delta_x^+q_{k,j+1/2}^{n+1/2}}{\Delta t\Delta x}-\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\,\frac{\Delta_t^+\Delta_x q_{k,j-1/2}^{n+1/2}}{\Delta t\Delta x}$$
$$=\frac{\Delta_t^+}{\Delta t}\left(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right)^2+\frac{\Delta_t^+}{\Delta t}\left[\frac{\Delta_t^+\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\,\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right],$$

then

$$\frac{-2}{\Delta t\Delta x^2}\sum_{k=-K}^{K-1}r_k^n\Delta_t^+(q_{k,j-1/2}^{n+1/2}+q_{k,j+1/2}^{n+1/2})\Delta_x^+\Delta_x^-q_{k,j+1/2}^{n+1/2}\Delta x$$
$$=2\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\,\frac{\Delta_t^+(q_{k+1,j-1/2}^{n+1/2}+q_{k+1,j+1/2}^{n+1/2})}{\Delta t}\,\frac{\Delta_x^+q_{k,j+1/2}^{n+1/2}}{\Delta x}\Delta x+2\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}r_k^n$$
$$\times\left(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right)^2\Delta x+2\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}r_k^n\frac{\Delta_t^+\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\,\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Delta x. \tag{3.6}$$

Multiplying both sides of (3.3) by $\frac{1}{\Delta t}\Delta_t^+(q_{k,j-1/2}^{n+1/2}+q_{k,j+1/2}^{n+1/2})$, and summing up with respect to k, one obtains

$$\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\ \frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\right)^2+2r_k^n\left(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right)^2\right.$$
$$\left.+2r_k^n\frac{\Delta_t^+\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\,\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right]\Delta x-\sum_{k=-K}^{K-1}\frac{\Delta_z^+r_k^n}{\Delta z}\left(\frac{\Delta_x^+(p_{k,j}^{n+1}+p_{k,j+1}^{n+1})}{\Delta x}\right)^2\Delta x$$
$$+\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\right)^2\Delta x+\sum_{k=-K}^{K-1}\frac{\Delta_x^+r_k^n}{\Delta x}\,\frac{\Delta_t^+(q_{k+1,j-1/2}^{n+1/2}+q_{k+1,j+1/2}^{n+1/2})}{\Delta t}$$
$$\times\left[2\ \frac{\Delta_x^+q_{k,j+1/2}^{n+1/2}}{\Delta x}+\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1}+p_{k,j+1}^n+p_{k,j+1}^{n+1})}{\Delta x}\right]\Delta x=0. \tag{3.7}$$

Put

$$S_{12}^{n,j}=2M\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\ \frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\right)^2+2r_k^n\left(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right)^2\right.$$
$$\left.+2r_k^n\frac{\Delta_t^+\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\,\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right]\Delta x+\sum_{k=-K}^{K-1}\left[r_k^n\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\right)^2\right.$$
$$\left.+2r_k^n\left(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\right)\left(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\right)\right]\Delta x, \tag{3.8}$$

$$S_{22}^{n,j}=\sum_{k=-K}^{k-1}\Big[\Big(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2-\alpha\Big(\Delta_x^+\ \frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2-2r_k^n\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2$$
$$+2Mr_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\Big)^2\Big]\Delta x, \tag{3.9}$$

where M is a constant.

Suppose

$$0<\alpha<1/4,\quad \max_{k,n} r_k^n\Delta t^2/\Delta x^2<(1-4\alpha)/2. \tag{3.10}$$

Then there exists σ, such that

$$\max_{k,n} r_k^n\Delta t^2/\Delta x^2=(1-4\alpha)/2(1+2\sigma)<(1-4\alpha)/2(1+\sigma). \tag{3.11}$$

Lemma 2. *Suppose $\alpha\in(0,\ 1/4)$, Δt and Δx satisfy (3.10). Then*

$$S_{12}^{n,j}\geqslant S_1(\sigma)\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_t^+q_{k,j-1/2}^n}{\Delta t}\Big)^2+r_k^n\Big(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Big)^2$$
$$+r_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\Big)^2\Big]\Delta x, \tag{3.12}$$

$$S_{22}^{n,j}\geqslant S_2(\sigma)\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2+r_k^n\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2$$
$$+r_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\Big)^2\Big]\Delta x, \tag{3.13}$$

for some sufficiently large M. Here $S_1(\sigma)$, $S_2(\sigma)$ are constants depending on σ, M.

Proof. From the definition of $S_{12}^{n,j}$, one obtains

$$S_{12}^{n,j}\geqslant\sum_{k=-K}^{K-1}\Big\{2M\Big[(1-4\alpha)\Big(\frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\Big)^2+2r_k^n\Big(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Big)^2-\frac{r_k^n}{\varepsilon'}\Big(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Big)^2$$
$$-r_k^n\varepsilon'\Big(\frac{\Delta_t^+\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Big)^2\Big]+r_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\Big)^2-\frac{r_k^n}{2}\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\Big)^2$$
$$-2r_k^n\Big(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Big)^2\Big\}\Delta x.$$

Choose $\varepsilon''\in(0,\ 1)$ such that $\dfrac{1}{(1-\varepsilon'')(1+2\sigma)}<1$ and put $\varepsilon'=\dfrac{1}{2(1-\varepsilon'')}$. Let M be sufficienty large so that

$$2M\varepsilon''>1\quad\text{and}\quad \sigma-\frac{1+\sigma}{2M}>0.$$

Then
$$S_{12}^{n,j}\geqslant\sum_{k=-K}^{K-1}\Big\{2M\Big[(1-4\alpha)-2r_k^n\ \frac{\Delta t^2}{\Delta x^2}\Big/(1-\varepsilon'')\Big]\Big(\frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\Big)^2+2(2M\varepsilon''-1)r_k^n$$
$$\times\Big(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Big)^2+\frac{r_k^n}{2}\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\Big)^2\Big\}\Delta x$$
$$\geqslant\sum_{k=-K}^{K-1}\Big\{2M(1-4\alpha)\Big(1-\frac{1}{(1-\varepsilon'')(1+2\sigma)}\Big)\Big(\frac{\Delta_t^+q_{k,j-1/2}^{n+1/2}}{\Delta t}\Big)^2$$
$$+2(2M\varepsilon''-1)r_k^n\Big(\frac{\Delta_x^+q_{k,j-1/2}^{n+1/2}}{\Delta x}\Big)^2+\frac{r_k^n}{2}\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j}^{n+1})}{\Delta x}\Big)^2\Big\}\Delta x.$$

It gives (3.12) with

$$S_1(\sigma)=\min\Big\{2M(1-4\alpha)\Big(1-\frac{1}{(1-\varepsilon'')(1+2\sigma)}\Big),\ \frac{1}{2},\ 2(2M\varepsilon''-1)\Big\}.$$

From the definition one has

$$S_{22}^{n,j}=\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2-\alpha\Big(\Delta_x^+\ \frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2-2r_k^n\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2+Mr_k^n$$
$$\times\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\Big)^2\Big]\Delta x+Mr_k^n\sum_{k=-K}^{K-1}\Big[4\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2+4\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)\Big(\frac{\Delta_t^+\Delta_x^+p_{k,j}^n}{\Delta x}\Big)$$
$$+\Big(\frac{\Delta_t^+\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2\Big]\Delta x\geqslant\sum_{k=-K}^{K-1}\Big[(1-4\alpha)\Big(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2-2r_k^n\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2$$
$$+Mr_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\Big)^2\Big]\Delta x+\sum_{k=-K}^{K-1}M\Big[4r_k^n\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2$$
$$-2r_k^n\varepsilon\Big(\Delta_t^+\ \frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2-\frac{2r_k^n}{\varepsilon}\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2+r_k^n\Big(\frac{\Delta_t^+\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2\Big]\Delta x. \tag{3.14}$$

Choosing $\varepsilon=\frac{1}{2}+\frac{1+\sigma}{4M}$, from (3.14) and (3.11) one obtains

$$S_{22}^{n,j}\geqslant\sum_{k=-K}^{K-1}\Big[(1-4\alpha)\Big(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2-\frac{1+\sigma}{2}r_k^n\Big(\frac{\Delta_t^+\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2+\Bigg(4M-2-\frac{2M}{\frac{1}{2}+\frac{1+\sigma}{4M}}\Bigg)r_k^n$$
$$\times\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2+Mr_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\Big)^2\Big]\Delta x$$
$$\geqslant\sum_{k=-K}^{K-1}\Big[(1-4\alpha)\Big(1-\frac{1+\sigma}{1+2\sigma}\Big)\Big(\frac{\Delta_t^+p_{k,j}^n}{\Delta t}\Big)^2+\frac{\sigma-\frac{1+\sigma}{2M}}{\frac{1}{2}+\frac{1+\sigma}{4M}}r_k^n\Big(\frac{\Delta_x^+p_{k,j}^n}{\Delta x}\Big)^2$$
$$+Mr_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\Big)^2\Big]\Delta x.$$

It gives (3.13) with

$$S_2(\sigma)=\min\Bigg[(1-4\alpha)\Big(1-\frac{1+\sigma}{1+2\sigma}\Big),\quad\frac{\sigma+\frac{1+\sigma}{2M}}{\frac{1}{2}+\frac{1+\sigma}{4M}},\ M\Bigg]>0.$$

Lemma 3. *Under the conditions of Lemma* 1 *and* (3.10), $S_{12}^{n,j}$, $S_{22}^{n,j}$ *satisfy*

$$\frac{\Delta_t^+}{\Delta t}S_{12}^{n,j}+\frac{\Delta_z^+}{\Delta z}S_{22}^{n,j}\leqslant K_2(S_{12}^{n,j}+S_{12}^{n,j+1}+S_{22}^{n,j}+S_{22}^{n+1,j}), \tag{3.15}$$

where K_2 is a constant depending on the constants σ, M, C_0, C_L, C_D.

The proof of this lemma can be derived by adding (3.2) to (3.7) multiplied by $2M$ Here M is the constant obtained in Lemma 2. The deduction is similar to that in Lemma 1.

Theorem 3. *Suppose the conditions of Lemma* 1 *are satisfied and*

$$\max_{k,n}\ r_k^n\,\Delta t^2/\Delta x^2<(1-4\alpha)/2.$$

Then
$$\sum_{n=0}^{N-1}\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_t^+q_{k,J-1/2}^{n+1/2}}{\Delta t}\Big)^2+r_k^n\Big(\frac{\Delta_x^+q_{k,J-1/2}^{n+1/2}}{\Delta x}\Big)^2+r_k^n\Big(\frac{\Delta_x^+(p_{k,J}^n+p_{k,J}^{n-1})}{\Delta x}\Big)^2\Big]\Delta x\Delta z$$
$$+\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_t^+p_{k,j}^N}{\Delta t}\Big)^2+r_k^n\Big(\frac{\Delta_x^+(p_{k,j}^n+p_{k,j+1}^n)}{\Delta x}\Big)^2\Big]\Delta x\,\Delta t\leqslant\overline{C}_2(K_2)$$
$$\times\exp[2K_2(D+T)]\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_t^+\varphi_{k,j}}{\Delta t}\Big)^2+r_k^0\Big(\frac{\Delta_x^+(\varphi_{k,j}+\varphi_{k,j+1})}{\Delta x}\Big)^2\Big]\Delta x\Delta t \tag{3.16}$$

for sufficiently small Δt, Δz, Δx and all N, J satisfying $N\Delta z\leqslant D$, $J\Delta t<T$, where $\overline{C}_2(K_2)$ is a constant.

The proof is similar to that of Theorem 2.

Define

$$\|p^n\|_{1,(2)}=\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left[(p^n_{k,j})^2+\left(\frac{\Delta_t^+p^n_{k,j}}{\Delta t}\right)^2+r^n_k\left(\frac{\Delta_x^+(p^n_{k,j}+p^n_{k,j+1})}{\Delta x}\right)^2\right]\Delta x\Delta t.$$

One obtains

Corollary. Under the conditions of Lemma 3, difference scheme (1.6) is stable in the norm $\|\cdot\|_{1,(2)}$, that is, there exists l_0 such that

$$\|p^N\|_{1,(2)}\leqslant C_2^*(K_2)\|p^0\|_{1,(2)}$$

for Δt, Δx, Δz satisfying $\max(\Delta z,\Delta x,\Delta t)\leqslant l_0$ and (3.10), N, J satisfying $N\Delta z\leqslant D$, $J\Delta t\leqslant T$, where $C_2^*(K_2)$ is a constant.

Proof. From (1.6c) one obtains

$$(p^n_{k,j})^2=\left(\sum_{i=0}^{j-1}\Delta_t^+p^n_{k,i}\right)^2\leqslant T\sum_{i=0}^{j-1}\left(\frac{\Delta_t^+p^n_{k,i}}{\Delta t}\right)^2\Delta t,$$

so that

$$\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}(p^n_{k,j})^2\Delta x\Delta t\leqslant T^2\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\left(\frac{\Delta_t^+p^n_{k,j}}{\Delta t}\right)^2\Delta x\Delta t.$$

By (3.16), the above inequality gives

$$\|p^n\|_{1,(2)}\leqslant\{T^2+\bar{C}_2(K_2)\exp[2K_2(D+T)]\}\|p^0\|_{1,(2)}.$$

The corollary is proved with $C_2^*(K_2)=T^2+\bar{C}_2(K_2)\exp[2K_2(D+T)]$.

§ 4. Stability of Scheme III

In this section we will discuss the stability of (1.7). Directly from (1.7) and the initial and boundary conditions one has

$$\sum_{l=1}^{m/2}\Delta_t^+q^{n+1/2}_{lk,j-1/2}/\Delta t=\Delta_z^+(1+\alpha\delta)^2p^n_{k,j}/\Delta z. \tag{4.1}$$

To obtain this equation, summing up (1.7b) with respect to l and using (1.1′), one obtains

$$\Delta_t^+\left[\sum_{l=1}^{m/2}\Delta_t^+q^{n+1/2}_{lk,j-1/2}/\Delta t-\Delta_z^+(1+\alpha\delta^2)p^n_{k,j}/\Delta z\right]=0.$$

Because of the initial condition, it gives (4.1).

Put $\quad\tilde{q}^{n+1/2}_{lk,j-1/2}=q^{n+1/2}_{lk,j-1/2}+q^{n+1/2}_{lk,j+1/2},\quad \tilde{p}^n_{k,j}=p^n_{k,j}+p^n_{k,j+1}.$

Then $\tilde{q}^{n+1/2}_{lk,j-1/2}$, $\tilde{p}^n_{k,j}$ satisfy (1.7a), (1.7b), (4.1) and the relevant initial and boundary conditions.

Applying $\Delta_x^+(\tilde{p}^n_{k,j}+\tilde{p}^{n+1}_{k,j})\Delta_x^+/\Delta x^2$ to both sides of (4.1) for $\tilde{p}$ and $\tilde{q}$ and summing up with respect to k, one obtains

$$\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_t^+\Delta_x^+\tilde{q}^{n+1/2}_{lk,j-1/2}}{\Delta t\Delta x}\frac{\Delta_x^+(\tilde{p}^n_{k,j}+\tilde{p}^{n+1}_{k,j})}{\Delta x}\Delta x=\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_x^+\tilde{p}_{k,j}}{\Delta x}\right)^2-\alpha\left(\Delta_z^+\frac{\Delta_x^+\tilde{p}^n_{k,j}}{\Delta x}\right)^2\right]\Delta x. \tag{4.2}$$

Multiplying both sides of (1.7b) by $\dfrac{1}{\beta^n_{lk}}\dfrac{\Delta_t^+(q^{n+1/2}_{lk,j-1/2}+q^{n+1/2}_{lk,j+1/2})}{\Delta t}$ and summing up with respect to l, k one obtains

$$\sum_{l=1}^{m/2}\sum_{k=-K}^{K-1}\frac{\Delta_t^+\Delta_t^-q^{n+1/2}_{lk,j+1/2}}{\beta^n_{l,k}\Delta t^2}\frac{\Delta_t^+(q^{n+1/2}_{lk,j-1/2}+q^{n+1/2}_{lk,j+1/2})}{\Delta t}\Delta x$$

$$-\sum_{l=1}^{m/2}\sum_{k=-K}^{K-1}\frac{\alpha^n_{lk}}{\beta^n_{lk}}\frac{\Delta_t^+(q^{n+1/2}_{lk,j-1/2}+q^{n+1/2}_{lk,j+1/2})}{\Delta t}\frac{\Delta_x^+\Delta_x^-q^{n+1/2}_{lk,j+1/2}}{\Delta x^2}\Delta x$$

$$=-\sum_{l=1}^{m/2}\sum_{k=-K}^{K-1}\frac{\Delta_x^+(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})}{\Delta x}\frac{\Delta_t^+\Delta_x^+\widetilde{q}_{lk,j-1/2}^{n+1/2}}{\Delta t\Delta x}\Delta x.$$

Substituting (4.2) into the above equality and using the relations similar to (3.6) one obtains

$$\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{1}{\beta_{lk}^n}\left(\frac{\Delta_t^+q_{lk,j-1/2}^{n+1/2}}{\Delta t}\right)^2\Delta x+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_x^+}{\Delta x}\left(\frac{\alpha_{lk}^n}{\beta_{lk}^n}\right)\frac{\Delta_x^+q_{lk,j+1/2}^{n+1/2}}{\Delta x}$$
$$\times\frac{\Delta_t^+(q_{lk+1,j-1/2}^{n+1/2}+q_{lk+1,j-1/2}^{n+1/2})}{\Delta t}\Delta x+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\alpha_{lk}^n}{\beta_{lk}^n}\left(\frac{\Delta_x^+q_{lk,j-1/2}^{n+1/2}}{\Delta x}\right)^2\Delta x$$
$$+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\alpha_{lk}^n}{\beta_{lk}^n}\left[\frac{\Delta_t^+\Delta_x^+q_{lk,j-1/2}^{n+1/2}}{\Delta x}\frac{\Delta_x^+q_{lk,j-1/2}^{n+1/2}}{\Delta x}\right]\Delta x$$
$$+\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_x^+\widetilde{p}_{k,j}^n}{\Delta x}\right)^2-\alpha\left(\Delta_x^+\frac{\Delta_x^+\widetilde{p}_{k,j}^n}{\Delta x}\right)^2\right]\Delta x=0. \qquad (4.3)$$

From (1.3) one obtains

$$\frac{-1}{\Delta t\Delta x^2}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\alpha_{lk}^n\Delta_t^+(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})\delta^2\widetilde{q}_{lk,j+1/2}^{n+1/2}\Delta x=\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_x^+\alpha_{lk}^n}{\Delta x}$$
$$\times\frac{\Delta_x^+(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})}{\Delta x}\frac{\Delta_t^+\widetilde{q}_{lk+1,j+1/2}^{n+1/2}}{\Delta tt}\Delta x+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_x^+\alpha_{lk}^n}{\Delta x}\frac{\Delta_t^+(\widetilde{p}_{k+1,j}^n+\widetilde{p}_{k+1,j}^{n+1})}{\Delta t}$$
$$\times\frac{\Delta_x^+\widetilde{q}_{lk,j+1/2}^{n+1/2}}{\Delta x}\Delta x+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\alpha_{lk}^n\frac{\Delta_x^+(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})}{\Delta x}\frac{\Delta_x^+\widetilde{q}_{lk,j+1/2}^{n+1/2}}{\Delta x}\Delta x$$
$$+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\alpha_{lk}^n\frac{\Delta_x^+\Delta_x^-(\widetilde{p}_{k,j+1}^n+\widetilde{p}_{k,j+1}^{n+1})}{\Delta x^2}\frac{\Delta_t^+\widetilde{q}_{lk,j+1/2}^{n+1/2}}{\Delta t}\Delta x. \qquad (4.4)$$

From (1.7b) and the relation similar to (3.6) one obtains

$$\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\alpha_{lk}^n\frac{\Delta_x^+\Delta_x^-(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})}{\Delta x^2}\frac{\Delta_t^+\widetilde{q}_{lk,j-1/2}^{n+1/2}}{\Delta t}\Delta x$$
$$=\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\left[\frac{\alpha_{lk}^n}{\beta_{lk}^n}\frac{\Delta_+^t\Delta_t^-q_{lk,j+1/2}^{n+1/2}}{\Delta t^2}-\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\frac{\Delta_x^+\Delta_x^-q_{lk,j+1/2}^{n+1/2}}{\Delta x^2}\right]\frac{\Delta_t^+\widetilde{q}_{lk,j-1/2}^{n+1/2}}{\Delta t}\Delta x$$
$$=\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\alpha_{lk}^n}{\beta_{lk}^n}\left(\frac{\Delta_t^+q_{lk,j-1/2}^{n+1/2}}{\Delta t}\right)^2\Delta x+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_x^+}{\Delta x}\left(\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\right)\frac{\Delta_t^+\widetilde{q}_{lk+1,j-1/2}^{n+1/2}}{\Delta t}$$
$$\times\frac{\Delta_x^+q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Delta x+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\left(\frac{\Delta_x^+q_{lk,j-1/2}^{n+1/2}}{\Delta x}\right)^2\Delta x$$
$$+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\frac{\Delta_t^+\Delta_x^+q_{lk,j-1/2}^{n+1/2}}{\Delta x}\frac{\Delta_x^+q_{lk,j-1/2}^{n+1/2}}{\Delta x}\Delta x. \qquad (4.5)$$

By (4.4), (4.5), using $\widetilde{p}_{k,j}^n$, $\widetilde{q}_{lk,j+1/2}^{n+1/2}$ instead of $p_{k,j}^n$, $q_{lk,j+1/2}^{n+1/2}$ in (1.7a) multiplied by $\frac{1}{\Delta t}(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})$ and summing up with respect to k one obtains

$$\frac{\Delta_z^+}{\Delta z}\sum_{k=-K}^{K-1}\left[\left(\frac{\Delta_t^+\widetilde{p}_{k,j}^n}{\Delta t}\right)^2-\alpha\left(\Delta_x^+\frac{\Delta_t^+\widetilde{p}_{k,j}^n}{\Delta t}\right)^2\right]\Delta x+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}r_k^n\left(\frac{\Delta_x^+(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})}{\Delta x}\right)^2\Delta x$$
$$+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\alpha_{lk}^n\frac{\Delta_x^+(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})}{\Delta x}\frac{\Delta_x^+\widetilde{q}_{lk,j+1/2}^{n+1/2}}{\Delta x}\Delta x+\frac{\Delta_t^+}{\Delta t}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\left[\frac{\alpha_{lk}^n}{\beta_{lk}^n}\right.$$
$$\left.\times\left(\frac{\Delta_t^+q_{lk,j+1/2}^{n+1/2}}{\Delta t}\right)^2+\left(\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\right)\left(\frac{\Delta_x^+q_{lk,j+1/2}^{n+1/2}}{\Delta x}\right)^2+\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\frac{\Delta_t^+\Delta_x^+q_{lk,j+1/2}^{n+1/2}}{\Delta x}\frac{\Delta_x^+q_{lk,j+1/2}^{n+1/2}}{\Delta x}\right]\Delta x$$
$$+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_x^+\alpha_{lk}^n}{\Delta x}\frac{\Delta_t^+(\widetilde{p}_{k+1,j}^n+\widetilde{p}_{k+1,j}^{n+1})}{\Delta t}\frac{\Delta_x^+\widetilde{q}_{lk,j+1/2}^{n+1/2}}{\Delta x}\Delta x+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_x^+\alpha_{lk}^n}{\Delta x}$$
$$\times\frac{\Delta_x^+(\widetilde{p}_{k,j}^n+\widetilde{p}_{k,j}^{n+1})}{\Delta x}\frac{\Delta_t^+\widetilde{q}_{lk+1,j+1/2}^{n+1/2}}{\Delta t}\Delta x+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\Delta_x^+}{\Delta x}\left(\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\right)\frac{\Delta_t^+\widetilde{q}_{lk+1,j+1/2}^{n+1/2}}{\Delta t}$$

$$\times \frac{\Delta_x^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}\Delta x+\sum_{k=-K}^{K-1}\frac{\Delta_x^+ r_k^n}{\Delta x}\frac{\Delta_t^+(\tilde p_{k+1,j}^n+\tilde p_{k+1,j}^{n+1})}{\Delta t}$$

$$\times \frac{\Delta_x^+(\tilde p_{k,j}^n+\tilde p_{k,j}^{n+1}+\tilde p_{k,j+1}^n+\tilde p_{k,j+1}^{n+1})}{\Delta x}\Delta x=0. \tag{4.6}$$

Put

$$S_{13}^{n,j}=\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{M'}{\beta_{lk}^n}\Big\{\Big[\Big(\frac{\Delta_t^+ q_{lk,j+3/2}^{n+1/2}}{\Delta t}\Big)^2+\Big(\frac{\Delta_t^+ q_{lk,j+1/2}^{n+1/2}}{\Delta t}\Big)^2\Big]+\alpha_{lk}^n\Big[\Big(\frac{\Delta_x^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}\Big)^2$$
$$+\Big(\frac{\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Big)^2+\frac{\Delta_t^+\Delta_x^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}\frac{\Delta_x^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}$$
$$+\frac{\Delta_t^+\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\frac{\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Big]\Big\}\Delta x+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\Big\{\frac{\alpha_{lk}^n}{\beta_{lk}^n}\Big(\frac{\Delta_t^+ q_{lk,j+1/2}^{n+1/2}}{\Delta t}\Big)^2$$
$$+\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\Big(\frac{\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Big)^2+\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\frac{\Delta_t^+\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\frac{\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}$$
$$+\alpha_{lk}^n\frac{\Delta_x^+(\tilde p_{k,j}^n+\tilde p_{k,j}^{n+1})}{\Delta x}\frac{\Delta_x^+\tilde q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Big\}\Delta x+\sum_{k=-K}^{K-1}r_k^n\Big(\frac{\Delta_x^+(\tilde p_{k,j}^n+\tilde p_{k,j}^{n+1})}{\Delta x}\Big)^2\Delta x,$$

$$S_{23}^{n,j}=M'\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_x^+\tilde p_{k,j}^n}{\Delta x}\Big)^2-\alpha\Big(\frac{\Delta_x^+\Delta_x^+\tilde p_{k,j}^n}{\Delta x}\Big)^2+\Big(\frac{\Delta_x^+\tilde p_{k,j+1}^n}{\Delta x}\Big)^2$$
$$-\alpha\Big(\frac{\Delta_x^+\Delta_x^+\tilde p_{k,j+1}^n}{\Delta x}\Big)^2\Big]\Delta x+\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_t^+\tilde p_{k,j}^n}{\Delta t}\Big)^2-\alpha\Big(\Delta_x^+\frac{\Delta_t^+\tilde p_{k,j}^n}{\Delta t}\Big)^2\Big]\Delta x. \tag{4.7}$$

Suppose $\max\limits_{l,k,n}(\alpha_{lk}^n\Delta t^2/\Delta x^2)<1$. Then there exists σ' such that

$$\max_{l,k,n}(\alpha_{lk}^n\Delta t^2/\Delta x^2)\leqslant 1-\sigma'. \tag{4.8}$$

Lemma4. *Suppose $\alpha\in(0, 1/4)$ and (4.7) is true. Then for sufficiently large M' there exist S_1', S_2' such that*

$$S_{13}^{n,j}\geqslant S_1'\sum_{k=-K}^{K-1}\Big\{\sum_{l=1}^{m/2}\Big[\Big(\frac{\Delta_t^+ q_{lk,j+3/2}^{n+1/2}}{\Delta t}\Big)^2+\Big(\frac{\Delta_t^+ q_{lk,j+1/2}^{n+1/2}}{\Delta t}\Big)^2\Big]\frac{1}{\beta_{lk}^n}+\frac{\alpha_{lk}^n}{\beta_{lk}^n}\Big[\Big(\frac{\Delta_x^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}\Big)^2$$
$$+\Big(\frac{\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Big)^2\Big]+r_k^n\Big(\frac{\Delta_x^+(\tilde p_{k,j}^n+\tilde p_{k,j}^{n+1})}{\Delta x}\Big)^2\Big\}\Delta x, \tag{4.9}$$

$$S_{23}^{n,j}\geqslant S_2'\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_x^+\tilde p_{k,j}^n}{\Delta x}\Big)^2+\Big(\frac{\Delta_x^+\tilde p_{k,j+1}^n}{\Delta x}\Big)^2+\Big(\frac{\Delta_t^+\tilde p_{k,j}^n}{\Delta t}\Big)^2\Big]\Delta x, \tag{4.10}$$

where S_1', S_2' are constants depending on M', σ'.

Proof. Since $\alpha\in(0, 1/4)$, (4.10) is obviously true.

Let $\varepsilon\in(2(1-\sigma'), 2)$, $\varepsilon'\in(0,2)$. From (4.8) and inequality $ab\geqslant-\frac{a^2+b^2}{2}$ one obtains

$$S_{13}^{n,j}\geqslant\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{M'}{\beta_{lk}^n}\Big\{\Big(1-\frac{2\alpha_{lk}^n\Delta t^2}{\Delta x^2\varepsilon}\Big)\Big[\Big(\frac{\Delta_t^+ q_{lk,j+3/2}^{n+1/2}}{\Delta t}\Big)^2+\Big(\frac{\Delta_t^+ q_{lk,j+1/2}^{n+1/2}}{\Delta t}\Big)^2\Big]$$
$$+\alpha_{lk}^n(1-\varepsilon/2)\Big[\Big(\frac{\Delta_x^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}\Big)^2+\Big(\frac{\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Big)^2\Big]\Big\}\Delta x+\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{\alpha_{lk}^n}{\beta_{lk}^n}$$
$$\times(1-\alpha_{lk}^n\Delta t^2/\Delta x^2)\Big(\frac{\Delta_t^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}\Big)^2\Delta x+\sum_{k=-K}^{K-1}r_k^n(1-\varepsilon'/2)\Big(\frac{\Delta_x^+(\tilde p_{k,j}^n+\tilde p_{k,j}^{n+1})}{\Delta x}\Big)^2\Delta x$$
$$-\frac{1}{2\varepsilon'}\sum_{k=-K}^{K-1}\sum_{l=1}^{m/2}\frac{(\alpha_{lk}^n)^2}{\beta_{lk}^n}\Big[\Big(\frac{\Delta_x^+ q_{lk,j+3/2}^{n+1/2}}{\Delta x}\Big)^2+\Big(\frac{\Delta_x^+ q_{lk,j+1/2}^{n+1/2}}{\Delta x}\Big)^2\Big]\Delta x.$$

By choosing $M'>(\max\limits_{l,k,n}\alpha_{lk}^n)/\varepsilon'(1-\varepsilon/2)$, it gives (4.9) for some S_1' depending on M' and σ'.

Lemma 5. *Suppose $C(x,z)$ satisfies all conditions of Lemma 1 and assume (4.7) to be valid. Then*

$$\frac{\Delta_z^+}{\Delta z}S_{23}^{n,j}+\frac{\Delta_t^+}{\Delta t}S_{13}^{n,j}\leqslant K_3(S_{23}^{n,j}+S_{23}^{n+1,j}+S_{13}^{n,j+1}+S_{13}^{n,j}), \tag{4.11}$$

where K_3 is a constant depending on C_0, C_L, C_D, M'.

Multiplying the sum of (4.3) for j and $(j+1)$ by M and adding it to (4.6), one can obtain an equality. Then using the deduction similar to that of Lemma 1 one may easily prove this lemma.

Theorem 4. *Suppose the conditions of Lemma 5 are satisfied and $\max\limits_{l,k,n}(\alpha_{lk}^n\Delta t^2/\Delta x^2)<1$. Then the solution of (1.7) satisfies*

$$\sum_{n=0}^{N-1}\sum_{k=-K}^{K-1}\Big[\sum_{l=1}^{n'/2}\Big\{\frac{1}{\beta_{lk}^n}\Big[\Big(\frac{\Delta_t^+q_{lk,J+3/2}^{n+1/2}}{\Delta t}\Big)^2+\Big(\frac{\Delta_t^+q_{lk,J+1/2}^{n+1/2}}{\Delta t}\Big)^2\Big]+\frac{\alpha_{lk}^n}{\beta_{lk}^n}\Big[\Big(\frac{\Delta_x^+q_{lk,J+3/2}^{n+1/2}}{\Delta x}\Big)^2$$
$$+\Big(\frac{\Delta_x^+q_{lk,J+1/2}^{n+1/2}}{\Delta x}\Big)^2\Big]+r_k^n\Big(\frac{\Delta_x^+(\tilde p_{k,J}^n+\tilde p_{k,J}^{n+1})}{\Delta x}\Big)^2\Big]\Big\}\Delta x\,\Delta z+\sum_{j=-1}^{J-1}\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_x^+\tilde p_{k,j}^N}{\Delta x}\Big)^2$$
$$+\Big(\frac{\Delta_x^+\tilde p_{k,j+1}^N}{\Delta x}\Big)^2+\Big(\frac{\Delta_t^+\tilde p_{k,j}^N}{\Delta t}\Big)^2\Big]\Delta x\Delta t\leqslant\bar C_3(K_3)\exp[2K_3(D+T)]$$
$$\times\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\Big[\Big(\frac{\Delta_x^+\tilde\varphi_{k,j}}{\Delta x}\Big)^2+\Big(\frac{\Delta_x^+\tilde\varphi_{k,j+1}}{\Delta x}\Big)^2+\Big(\frac{\Delta^+\tilde\varphi_{k,j}}{\Delta t}\Big)^2\Big]\Delta x\Delta t, \tag{4.12}$$

for sufficiently small l_0 and all N, J satisfying $N\Delta z\leqslant D$, $J\Delta t\leqslant T$. Here $\tilde\varphi_{k,j}=\varphi_{k,j}+\varphi_{k,j+1}$, $\bar C_3(K_3)$ is a constant depending on M_0, K_3.

The proof is similar to that of Theorem 1.

Define

$$\|p^n\|_{1,(3)}=\sum_{j=-1}^{J-1}\sum_{k=-K}^{K-1}\Big[(p_{k,j}^n)^2+\Big(\frac{\Delta_x^+\tilde p_{k,j}^n}{\Delta x}\Big)^2+\Big(\frac{\Delta_x^+\tilde p_{k,j+1}^n}{\Delta x}\Big)^2+\Big(\frac{\Delta_t^+\tilde p_{k,j}^n}{\Delta t}\Big)^2\Big]\Delta x\Delta t.$$

Corollary. Under the conditions of Lemma 5, difference scheme (1.7) is stable in the norm $\|\cdot\|_{1,(3)}$. That is, there exists l_0 such that

$$\|p^N\|_{1,(3)}\leqslant C_3^*(K_3)\|p^0\|_{1,(3)},$$

for Δx, Δt, Δz satisfying $\max(\Delta x,\Delta t,\Delta z)\leqslant l_0$ and (4.7), N, J satisfying $N\Delta z\leqslant D$, $J\Delta t\leqslant T$, where $C_3^*(K_3)$ is a constant.

Proof. From (1.7c) one obtains

$$(p_{k,j}^n)^2=\Big(\sum_{s=0}^{[\frac{j+1}{2}]}\Delta_t^+\tilde p_{k,j-2(s+1)}^n\Big)^2\leqslant T\sum_{s=0}^{[\frac{j+1}{2}]}\Big(\frac{\Delta_t^+\tilde p_{k,j-2(s+1)}^n}{\Delta t}\Big)^2\Delta t,$$

so that
$$\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}(p_{k,j}^n)^2\Delta x\Delta t\leqslant T^2\sum_{j=0}^{J-1}\sum_{k=-K}^{K-1}\Big(\frac{\Delta_t^+\tilde p_{k,j}^n}{\Delta t}\Big)^2\Delta x\Delta t.$$

From (4.12) and the above inequality one can derive the result.

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