HOW TO RECOVER THE CONVERGENT RATE FOR RICHARDSON EXTRAPOLATION ON BOUNDED DOMAINS*

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Abstract

We are interested in solving elliptic problems on bounded convex domains by higher order methods using the Richardson extrapolation. The theoretical basis for the application of the Richardson extrapolation is the asymptotic error expansion with a remainder of higher order. Such an expansion has been derived by the method of finite difference, where, in the neighborhood of the boundary one must reject the elementary difference analogs and adopt complex ones. This plight can be changed if we turn to the method of finite elements, where no additional boundary approximation is needed but an easy triangulation is chosen, i.e. the higher order boundary approximation is replaced by a chosen triangulation. Specifically, a global error expansion with a remainder of fourth order can be derived by the linear finite element discretization over a chosen triangulation, which is obtained by decomposing the domain first and then subdividing each subdomain almost uniformly. A fourth order method can thus be constructed by the simplest linear finite element approximation over the chosen triangulation using the Richardson extrapolation.

§ 1. Problem and Result

The Richardson extrapolation to the limit is a common way of increasing the accuracy of low order finite difference schemes applied to ordinary differential equations[23]. For elliptic equations, for example, the two-dimensional model problem

$$-\Delta u=f \quad \text{in } \Omega,\ u=0 \quad \text{on } \partial\Omega \tag{1}$$

on a curved domain Ω, the elementary difference analogs do not, near the boundary, allow us to expand the approximation error in powers of the mesh size h. Therefore, near the boundary we must reject the elementary difference analogs and adopt complex ones which usually lead to a large number of nonzero coefficients in the equations near the boundary. In so doing we shall succeed in obtaining an expression for the approximation error[7, 18, 19, 25]

$$u^h(z)-u(z)=h^2e(z)+O(h^4) \tag{2}$$

at nodal points z.

What will happen to the method of finite elements? Can the additional higher order boundary approximation be avoided by choosing a proper triangulation?

Let us recall the L_2-error estimate for linear finite element approximation u^h,

$$u^h-u=O(h^2) \text{ in } L_2\text{-norm}.$$

It is hopeless, in contrast to the usual imagination, to prove the further error

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expansion like

$$u^h - u = h^2 e + O(h^4) \quad \text{in } L_2\text{-norm}$$

since the combination of approximations on two triangulations (denoted by T^h and $T^{h/2}$, respectively) is still a piecewise linear function over $T^{h/2}$ which does not give an $O(h^4)$ approximation of u in L_2.

Let us turn to the pointwise estimate

$$u^h(z) - u(z) = O(h^2 |\log h|),$$

where the factor $\log h$ cannot be moved at the nodal points unless the surrounding meshes have strong symmetry[6,13] (forming a six-polygon at least). So, it seems that we cannot hope to find a united error expansion (2) for all nodal points of a general (regular) triangulation.

So, the problem is how to choose a triangulation such that there will exist a united error expansion (2) for all nodal points.

In [6], [14] and [15] a piecwise uniform triangulation, and in [17] a piecewise almost uniform triangulation are constructed in order to obtain the united error expansions like (2). These kinds of triangulation are easy to be constructed for the polygonal domains, for instance, by first choosing coarse triangles or quadrilaterals and then subdividing each triangle or quadrilateral almost uniformly.

An interior uniform triangulation has been used in [6, 16] for the curved domains. By an arbitrary arrangement of triangular meshes near the boundary we get only an interior error expansion with a reduced order $O(h^3|\log h|)$ for the remainder. It seems that unproper meshes near the boundary pollute the remainder even in the interior of Ω.

In [2], a transformed uniform triangulation was introduced in combining with a transformed bilinear element approximation. It is the purpose of this paper to describe how to recover usual linear elements from the transformed linear elements used in [2].

Define a triangulation T^h by first decomposing the domain and then subdividing each subdomain almost uniformly, for example, by the following possible process (see Fig. 1).

Suppose, for simplicity, that Ω is a star domain with respect to a point O. Firstly, choose a square Ω_0 with its center at O and divide $\Omega\backslash\Omega_0$ into four pieces $\{\Omega_i,\ 1\leqslant i\leqslant 4\}$ by four rays passing through O and the four vertices of Ω_0. Secondly, make n-equipartition along each edge of Ω_0 and draw $n-1$ rays through O and the $n-1$ equinodes. Linking the n-equinodes along each ray lying in Ω_i we obtain an almost uniform triangulation T_i^h over Ω_i. Finally, let T_0^h be a uniform triangulation over Ω_0. We obtain a piecewise almost uniform triangulation

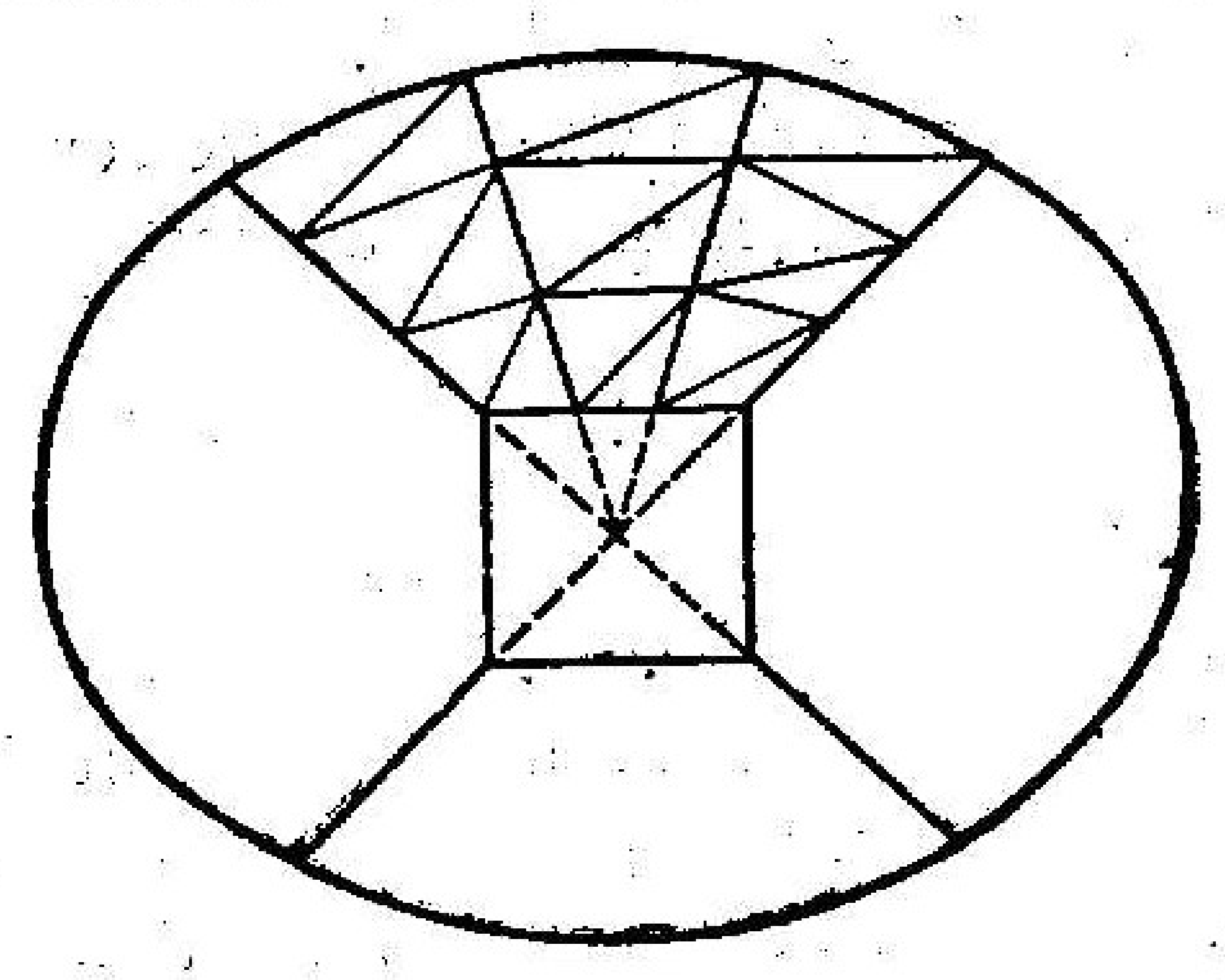

Fig. 1

$$T^h = \bigcup T_i^h \tag{3}$$

over $\Omega^h = \bigcup\{K\in T^h\}$.

The result is

Theorem 1. *Suppose the solution* $u\in W^{4,\infty}(\Omega)$. *Let* u^h *be the usual linear finite element approximation to* u *over a triangulation defined as in* (3). *Then the approximation error can be expanded in the form*

$$u^h(z)-u(z)=h^2e(z)+O(h^4|\log h|) \tag{4}$$

at nodal points z *uniformly bounded away from the vertices of* Ω_0. *In particular, a fourth order method can be obtained by the combination of linear finite element approximations over two triangulations* T^h *and* $T^{h/2}$:

$$\frac{1}{3}(4u^{h/2}-u^h)(z)=u(z)+O(h^4|\log h|) \tag{5}$$

at nodal points $z\in T^h$ *uniformly bounded away from the vertices of* Ω_0.

Therefore, the Richardson extrapolation (5) to the h-version of the finite element method is worthy of recommendation. Since, in the h-method, successive refinement of the meshes has been used, why not use Richardson extrapolation?

We now compare the treatments of finite element and finite difference.

(i) On the uniform meshes the finite element approximation of elliptic problems with constant coefficients reduces to a special finite difference method since all equations for interior grid points are the same[4]. The essential difference is that the expansion (4) holds for the finite element analysis even at nodal points in Ω_i ($i\neq 0$, shown in Fig. 1) where the triangulation is not uniform. So, the finite element treatment admits more flexibility in the choice of the meshes which can be used fully to avoid the higher order boundary approximation needed in the finite difference treatment.

(ii) The finite element analysis needs much weeker smoothness assumption of u than the finite difference analysis, which is useful in dealing with extrapolation methods on reentrant domains[4].

(iii) In contrast to most of the proofs in the finite difference context no discrete maximum principle is needed. Therefore, it is possible to derive asymptotic expansions for second order problems which are not separable and also for second order elliptic systems[20, 21].

At the end of this paper we compare the extrapolation of linear elements and the superconvergence of quadratic elements.

§ 2. Preparations and Lemmas

The success of error expansion (4) is based on the special type of triangulation as shown in Fig. 1. In fact, such kind of triangulation comes from a uniform triangulation over a unit square in the following sense.

Following [2], we assume that Ω can be represented as a collection of transformed unit squares. To this end we make the following assumption on Ω: there is a finite number of subdomains Ω_i ($i=0, 1, 2, \cdots$) such that

(i) $\Omega_i\cap\Omega_j=\varnothing\ \forall\ i\neq j$;

(ii) $\Omega=\bigcup\Omega_i$;

(iii) there is an invertible transformation ψ_i: $\bar{\Omega}_i\to[0, 1]^2$ which, together with its inverse $\phi_i=\psi_i^{-1}$, is sufficiently smooth;

(iv) $\Omega_i=\phi_i((0,1)^2)$;

(v) $|\psi_i(x)-\psi_i(y)|=|\psi_j(x)-\psi_j(y)| \quad \forall\, x, y\in\bar{\Omega}_i\cap\bar{\Omega}_j$.

Let $\hat{T}^h$ be a uniform triangulation over $[0, 1]^2$ with node set $\hat{N}^h$, and let

$$N_i^h=\phi_i(\hat{N}^h).$$

Linking the nodes in N_i^h we obtain a triangulation T_i^h over Ω_i. By condition (v), the nodes along the common side $\bar{\Omega}_j\cap\bar{\Omega}_i$ coincide:

$$N_i^h\cap\bar{\Omega}_j\cap\bar{\Omega}_i=N_j^h\cap\bar{\Omega}_j\cap\bar{\Omega}_i.$$

Thus, $T^h=\bigcup T_i^h$ is a regular triangulation over Ω.

Fig. 1 shows a possible partitioning of a star domain Ω into subdomains Ω_i and the construction of triangulations T_i^h.

Then, the properties of T^h can be described through $\hat{T}^h$.

We use some local notations in the reference domain. For any fixed triangle $\hat{K}\in\hat{T}^h$, we introduce the notations:

$\hat{p}_d=$ vertex of $\hat{K}$, $\hat{s}_d=$ side of $\hat{K}$ opposite to $\hat{p}_d$,

$\hat{h}_d=$ length of $\hat{s}_d$, $h=\max \hat{h}_d$,

$\hat{n}_d=$ outer normal unit vector along $\hat{s}_d$,

$\hat{t}_d=$ tangent unit vector along $\hat{s}_d$,

$\hat{q}_d=$ midpoint of s_d, $\hat{q}=$ center of $\hat{K}$.

Corresponding to $\hat{K}$, let $K\in T_i^h$ be a triangular element with vertices

$$p_d=\phi_i(\hat{p}_d)\in\Omega_i,\quad 1\leqslant d\leqslant 3.$$

We use a for its area and s_d, h_d, n_d, t_d, q_d, q for its side, length, outer normal vector, tangent vector, midpoint, center, respectively.

Note that all $\hat{K}\in\hat{T}^h$ will coincide under translation and reflection. Let us choose the reference vectors $\hat{\tau}_d$ such that either

$$\hat{t}_d=\hat{\tau}_d \text{ or } \hat{t}_d=-\hat{\tau}_d,\quad 1\leqslant d\leqslant 3.$$

For $K=\Delta p_1p_2p_3$ corresponding to $\hat{K}=\Delta\hat{p}_1\hat{p}_2\hat{p}_3$ we define

$$\delta(K)=\hat{t}_d\cdot\hat{\tau}_d$$

and we have, for any two adjacent triangles K and K'

$$\delta(K)=-\delta(K'). \tag{6}$$

The difference of the lengths, the normal vectors and the areas between K and K' are of higher order:

$$h_d=h_d'+O(h^2),\ t_d=-t_d'+O(h),\ a=a'+O(h^3).$$

The normal vector can be expressed as a combination of tangent vectors along two sides[16].

Lemma 1. *On the triangle K, there hold*

(i) $n_1=\alpha t_1+\beta t_2$ *with* $\alpha=\frac{h_1h_2}{2a}t_1\cdot t_2$, $\beta=-\frac{h_1h_2}{2a}$,

(ii) $t_1\cdot n_2=\frac{2a}{h_1h_2}$, $t_1\cdot n_3=-\frac{2a}{h_1h_3}$.

For a function v defined on Ω_i, let

$$\hat{v}(\hat{x})=v(\phi(\hat{x}))$$

and we will use the simple notation

$$v(\hat{x})=\hat{v}(\hat{x}).$$

So, a function defined on Ω_i can be regarded as a function defined on $[0, 1]^2$ and the inverse is also true. Thus, it is needless to make a distinction, for a function, between the definitions on $[0, 1]^2$ and on Ω_i.

The lengths $\hat{h}_d$ or vectors $\hat{t}_d$ are united for all $\hat{K}\in\hat{T}^h$. Correspondently, h_d or t_d are almost united for all $K\in T^h$.

Lemma 2. *There exist smooth functions a_{de}, b_{de} and a_e defined on $[0, 1]^2$ and $[0, 1]^2\times[0, 1]^2$ independent of h such that, for all $K\in T_i^h$,*

(i) $h_d=ha_{d1}(\hat{q}_d)+h^3a_{d2}(\hat{q}_d)+O(h^4)$,

(ii) $t_d=\delta(b_{d1}(\hat{q}_d)+h^2b_{d2}(\hat{q}_d))+O(h^3)$,

(iii) $\alpha=\alpha_1(\hat{q}_1, \hat{q}_2)+h^2\alpha_2(\hat{q}_1, \hat{q}_2)+O(h^3)$.

Proof. Let $K=\Delta p_1p_2p_3$ with $p_d=\phi(\hat{p}_d)$. By the Taylar expansion at midpoint $\hat{q}_1$, there hold

$$\phi(\hat{p}_3)-\phi(\hat{q}_1)=D\phi(\hat{q}_1)(\hat{p}_3-\hat{q}_1)+\frac{1}{2}D^2\phi(\hat{q}_1)(\hat{p}_3-\hat{q}_1)^2+\frac{1}{6}D^3\phi(\hat{q}_1)(\hat{p}_3-\hat{q}_1)^3+O(h^4),$$

$$\phi(\hat{p}_2)-\phi(\hat{q}_1)=D\phi(\hat{q}_1)(\hat{p}_2-\hat{q}_1)+\frac{1}{2}D^2\phi(\hat{q}_1)(\hat{p}_2-\hat{q}_1)^2+\frac{1}{6}D^3\phi(\hat{q}_1)(\hat{p}_2-\hat{q}_1)^3+O(h^4).$$

By subtraction we have

$$p_3-p_2=\hat{h}_1\hat{\partial}_1\phi(\hat{q}_1)\hat{t}_1+\frac{1}{24}\hat{h}_1^3\hat{\partial}_1^3\phi(\hat{q}_1)\hat{t}_1^3+O(h^4)$$

where $\hat{h}_1=\frac{h}{\sqrt{2}}$, and hence

$$h_1=|p_3-p_2|=ha_{11}(\hat{q}_1)+h^3a_{12}(\hat{q}_1)+O(h^4).$$

Note that

$$t_1=\frac{p_3-p_2}{h_1},\quad \hat{\partial}_1\phi=\delta\frac{\partial\phi}{\partial\hat{t}_1}.$$

(ii) follows by an analogous argument.

Note that

$$\alpha=-\frac{t_1\cdot t_2}{|t_1\times t_2|}.$$

(iii) follows from (ii).

We now define a geometric point set X consisting of the points with the same geometric position for all $K\in T_i^h$. X may be, for example,

$$\{p_1\},\ \{q_1\},\ \{\phi(\hat{q}_1)\},\ \{g\},\ \{\phi(\hat{q})\},\ \cdots.$$

Then, we can transfer one geometric point to another.

Lemma 3. *Let f be a smooth function defined on Ω_i^2 and $\{X_1, X_2, X_3\}$ three geometric point sets. Then there exist functions $f_d(0\leqslant d\leqslant 3)$ defined on Ω_i independent of h such that, for all $K\in T_i^h$,*

$$f(x, y)=\sum_0^3(\delta h)^d f_d(z)+O(h^4),$$

where $x\in X_1$, $y\in X_2$ *and* $z\in X_3$ *in a triangle* K.

Proof. For clarity, let us take the concrete example

$$X_1=X_3=\{\phi(\hat{q}_1)\},\quad X_2=\{\phi(\hat{q}_2)\}.$$

Thus, by the Taylar expension

$$f(\hat{q}_1,\ \hat{q}_2)=f(\hat{q}_1,\ \hat{q}_1)+D_2f(\hat{q}_1,\ \hat{q}_1)(\hat{q}_2-\hat{q}_1)$$
$$+\frac{1}{2}D_2^2f(\hat{q}_1,\ \hat{q}_1)(\hat{q}_2-\hat{q}_1)^2+\frac{1}{6}D_2^3f(\hat{q}_1,\ \hat{q}_1)(\hat{q}_2-\hat{q}_1)^3+O(h^4).$$

Since

$$\hat{q}_2-\hat{q}_1=-\frac{1}{2}(\hat{p}_2-\hat{p}_1)=-\frac{1}{2}\hat{h}_3\hat{t}_3=-\frac{1}{2}\delta\hat{h}_3\hat{\tau}_3,$$

we have

$$f(\hat{q}_1,\ \hat{q}_2)=f(\hat{q}_1,\ \hat{q}_1)-\frac{\delta}{2}\hat{h}_3D_2f(\hat{q}_1,\ \hat{q}_1)\hat{\tau}_3$$
$$+\frac{1}{16}\hat{h}_3^2D_2^2f(\hat{q}_1,\ \hat{q}_1)\hat{\tau}_3^2-\frac{\delta}{48}\hat{h}_3^3D_2^3f(\hat{q}_1,\ \hat{q}_1)\hat{\tau}_3^3+O(h^4)$$

and hence

$$f(\hat{q}_1,\ \hat{q}_2)=\sum_0^3(\delta h)^d f_d(\hat{q}_1)+O(h^4).$$

The line integral can be expanded in an area integral with a remainder of higher order line integral.

Lemma 4. *For* $v\in C^3(K)$, *there hold*

(i) $\displaystyle\int_{s_1} v\,ds=\frac{h_1}{a}\int_K v\,dx-\frac{h_2}{6}\int_{s_1}\partial_2 v\,ds+O(h^3)$,

(ii) $\displaystyle\int_{s_1} v\,ds=h_1v(q_1)+\frac{1}{24}h_1^2\int_{s_1}\partial_1^2 v\,ds+O(h^4)$.

The proof is based on the Bramble Lemma. We refer to [16] for detail.

We will use some other notations:

$$\Omega_i^h=\bigcup_{K\in T_i^h}K,\ \partial^1\Omega_i^h=\partial\Omega_i^h\backslash\partial\Omega^h,\ \Gamma_d=\bigcup_{s_d\subset\partial^1\Omega_i^h}s_d,\qquad V=\bigcup_{i<j<k}(\bar{\Omega}_i\cap\bar{\Omega}_j\cap\bar{\Omega}_k).$$

§ 3. Proof of Theorem 1

Let i^hu, g_z^h and $\tilde{g}_z$ be the interpolant of u, the discrete Green function and the regularized Green function over T^h, respectively, as defined in [6]. Then, by the definition,

$$(u^h-i^hu)(z)=\int_{\Omega^h}\nabla(u-i^hu)\nabla g_z^h\,dx=\sum_i\sum_{K\in T_i^h}\int_K\nabla(u-i^hu)\nabla g_z^h\,dx$$
$$=\sum_i\sum_{K\in T_i^h}\int_{\partial K}(u-i^hu)\frac{\partial}{\partial n}g_z^h\,ds$$
$$=\sum_i\sum_{d=1}^3\sum_{K\in T_i^h}\frac{\partial}{\partial n_d}g_z^h\int_{s_d}(u-i^hu)\,ds.$$

Inserting the Euler–Maclaurin formula

$$\int_{s_d}(u-i^h u)\,ds=-\frac{1}{12}h_d^2\int_{s_d}\partial_d^2 u\,ds+h_d^4\int_{s_d}c(s)\partial_d^4 u\,ds$$

into the last term we obtain

$$(u^h-i^h u)(z)=\sum_i\sum_d\sum_{K\in T_i^h}\left(-\frac{1}{12}h_d^2\int_{s_d}\partial_d^2 u\frac{\partial}{\partial n_d}g_z^h\,ds\right)$$

$$+\sum_i\sum_d\sum_{K\in T_i^h}h_d^4\int_{s_d}c(s)\partial_d^4 u\frac{\partial}{\partial n_d}g_z^h\,ds.$$

For the remainder, replace g_z^h by $\tilde{g}_z$ and note from $\tilde{g}_z\in H^2(\Omega)$ that

$$\sum_{K\in T_i^h}h_d^4\int_{s_d}c(s)\,\partial_d^4 u\frac{\partial}{\partial n_d}\tilde{g}_z\,ds=\sum_{s_d\subset\partial\Omega_i^h}h_d^4\int_{s_d}c(s)\partial_d^4 u\frac{\partial}{\partial n_d}\tilde{g}_z\,ds \tag{7}$$

and from a trace theorem and the estimates for g_z^h and $\tilde{g}_z$ that

$$\sum_{K\in T_i^h}\int_{s_d}|\nabla(g_z^h-\tilde{g}_z)|\,ds\leqslant ch^{-1}\int_{\Omega_i}|\nabla(g_z^h-\tilde{g}_z)|\,ds+c\int_{\Omega_i}|\nabla^2\tilde{g}_z|\,ds\leqslant c|\log h|. \tag{8}$$

We obtain

$$(u^h-i^h u)(z)=\sum_i\sum_d\sum_{K\in T_i^h}\left(-\frac{1}{12}h_d^2\int_{s_d}\partial_d^2 u\frac{\partial}{\partial n_d}g_z^h\,ds\right)+O(h^4|\log h|).$$

Hence, it remains to derive asymptotic expansions for terms like $(d=1)$

$$I^h=\sum_{K\in T_i^h}h_1^2\frac{\partial}{\partial n_1}g_z^h\int_{s_1}\partial_1^2 u\,ds.$$

Break up the integral in I^h, by Lemma 4, as follows:

$$\int_{s_1}\partial_1^2 u\,ds=\partial_1^2 u(q_1)h_1+\frac{1}{24}h_1^2\int_{s_1}\partial_1^4 u\,ds+O(h^4).$$

For the second term in the right hand side there holds, by using the argument in (7), (8),

$$\left|\sum_{K\in T_i^h}h_1^4\frac{\partial}{\partial n_1}g_z^h\int_{s_1}\partial_1^4 u\,ds\right|\leqslant ch^4|\log h|.$$

Hence, we obtain

$$I^h=\sum_{K\in T_i^h}h_1^3\frac{\partial}{\partial n_1}g_z^h\partial_1^2 u(q_1)+O(h^4|\log h|).$$

Reduce the normal vector, by Lemma 1, to tangent vectors along two sides,

$$I^h=\sum_{K\in T_i^h}\alpha h_1^3\partial_1 g_z^h\partial_1^2 u(q_1)+\sum_{K\in T_i^h}\beta h_1^3\partial_2 g_z^h\partial_1^2 u(q_1)+O(h^4|\log h|).$$

We want to derive asymptotic expansions for terms

$$L^h=\sum_{K\in T_i^h}\alpha h_1^3\partial_1 g_z^h\partial_1^2 u(q_1),$$

$$M^h=\sum_{K\in T_i^h}\beta h_1^3\partial_2 g_z^h\partial_1^2 u(q_1).$$

Let us consider L^h first. Using Lemmas 2 and 3 we can expand the function

$$\alpha h_1^2\partial_1^2 u(q_1)=\alpha h_1^2\nabla^2 u(q_1)t_1^2$$

by the united functions $W_i(\hat{q}_1)$:

$$ah_1^2\partial_1^2 u(q_1)=h^2\sum_{d=0}^{2}(\delta h)^d W_d(\hat{q}_1)+O(h^5).$$

Then

$$L^h=h^2\sum_{d=0}^{2}\sum_{K\in T_1^h} h_1(\delta h)^d\,\partial_1 g_z^h W_d(\hat{q}_1)+O(h^4).$$

We split L^h as follows

$$L^h=h^2\sum_{d=0}^{2}L_d^h+O(h^4),\quad L_0^h=\sum_{K\in T_1^h}\partial_1 g_z^h W_0(\hat{q}_1)h_1,\ \cdots.$$

Note that $\partial_1=-\partial_1'$ for two adjacent triangles K and K' with a common side s_1. The sum in L_0^h over interior sides s_1 is cancelled. Hence, L_0^h reduces to

$$L_0^h=\sum_{s_1\subset\partial^1\Omega_1^h}\partial_1 g_z^h W_0(\hat{q}_1)h_1.$$

In view of

$$W_0(\hat{q}_1)=W_0(\phi(\hat{q}_1)),\quad |\phi(\hat{q}_1)-q_1|\leqslant ch^2, \tag{9}$$

L_0^h turns back to an integral form:

$$\begin{aligned}L_0^h&=\sum_{s_1\subset\partial^1\Omega_1^h}\partial_1 g_z^h W_0(q_1)h_1+O(h^2|\log h|)\\&=\sum_{s_1\subset\partial^1\Omega_1^h}\partial_1 g_z^h\int_{s_1}W_0\,ds+O(h^2|\log h|)\\&=\int_{\Gamma_1}\partial_1 g_z^h W_0\,ds+O(h^2|\log h|)\\&=-\int_{\Gamma_1}g_z^h\,\partial_1 W_0\,ds+(g_z^h W_0)\big|_b^c+O(h^2|\log h|),\end{aligned}$$

where b and c denote the endpoints of Γ_1. Set

$$L_0=-\int_{\Gamma_1}g_z\,\partial_1 W_0\,ds+(g_z W_0)\big|_b^c.$$

We obtain, for dist $(z, V)\leqslant s>0$, by Lemmas A 3 and A 4 in[6]

$$L_0^h(z)=L_0(z)+O(h^2|\log h|).$$

For L_1^h we have, observing (9),

$$L_1^h=\sum_{K\in T_1^h}\delta h\,\partial_1 g_z^h W_1(\hat{q}_1)h_1=h\sum_{K\in T_1^h}\delta\int_{s_1}\partial_1 g_z^h W_1 ds+O(h^2|\log h|). \tag{10}$$

Note that

$$\delta=-\delta',\quad \partial_1=-\partial_1'$$

for two adjacent triangles K and K' with a common side s_1. The sum in L_1^h over interior sides s_1 cannot be cancelled. We have to reduce, by Lemma 4, the line integral in L_1^h to an area integral:

$$\begin{aligned}\int_{s_1}\partial_1 g_z^h W_1\,ds&=\frac{h_1}{a}\int_K\partial_1 g_z^h W_1\,dx-\frac{h_2}{6}\int_{s_1}\partial_1 g_z^h\,\partial_2 W_1\,ds+O(h^3)\\&=-\frac{h_1}{a}\int_K g_z^h\,\partial_1 W_1\,dx+\frac{h_1}{a}\int_{\partial K}g_z^h W_1 t_1\cdot n\,ds\\&\quad-\frac{h_2}{6}\int_{s_1}\partial_1 g_z^h\,\partial_2 W_1\,ds+O(h^3).\end{aligned} \tag{11}$$

For line integral $\int_{\partial K}$ in (11) we have, in view of $t_1\cdot n_1=0$ and Lemma 1,

$$\frac{h_1}{a}\int_{s_3} g_z^h W_1 t_1\cdot n_3\,ds=-\frac{2}{h_3}\int_{s_3} g_z^h W_1\,ds,$$
$$\frac{h_1}{a}\int_{s_2} g_z^h W_1 t_1\cdot n_2\,ds=\frac{2}{h_2}\int_{s_2} g_z^h W_1\,ds.$$

For the first formula we have, observing $\delta=-\delta'$ and s_3-interior sides,

$$\sum_{K\in T_1^h}\delta\frac{2}{h_3}\int_{s_3} g_z^h W_1\,ds=0. \tag{12}$$

For the second formula we have, observing that the lengths h_2 along Γ_2 are the same,

$$\sum_{K\in T_1^h}\delta\frac{2}{h_2}\int_{s_2} g_z^h W_1\,ds=\frac{2\delta}{h_2}\int_{\Gamma_2} g_z^h W_1\,ds. \tag{13}$$

For line integral $\int_{s_1}$ in (11) we use the following estimate: for two adjacent triangles K and K' with a common side s_1,

$$\left|h_2\int_{s_1}\partial_1 g_z^h\,\partial_2 W_1\,ds-h_2'\int_{s_1'}\partial_1' g_z^h\,\partial_2' W_1\,ds\right|\le ch\int_{K\cup K'}|g_z^h|\,dx.$$

So, the sum over interior sides s_1 is almost cancelled, and hence

$$\sum_{K\in T_1^h}\delta h_2\int_{s_1}\partial_1 g_z^h\,\partial_2 W_1\,ds=\sum_{s_1\subset\partial'\Omega_1}\delta h_2\int_{s_1}\partial_1 g_z^h\,\partial_2 W_1\,ds+O(h|\log h|).$$

Using a trace theorem

$$\int_{\Gamma_1}|\nabla g_z^h|\,ds\le c\int_{\Omega}|\nabla g_z^h|\,dx\le c|\log h|$$

we conclude that

$$\left|\sum_{K\in T_1^h}\delta h_2\int_{s_2}\partial_1 g_z^h\,\partial_2 W_1\,ds\right|\le ch|\log h|. \tag{14}$$

For area integral $\int_K$ in (11) we use the following expansion:

$$\int_K g_z^h\,\partial_1 W_1\,dx=g_z^h(q)\,\partial_1 W_1(q)\,a+O(h^2)\int_K|\nabla g_z^h|\,dx.$$

Again, we expand the un-united function $\frac{h_1}{a}\partial_1 W_1(q)$, using Lemmas 2 and 3, by united functions $v_i(q)$:

$$-\partial_1 W_1(q)\frac{h_1}{a}=\sum_{i=0}^{1}(\delta h)^{i-1}v_i(q)+O(h).$$

Hence

$$-\frac{h_1}{a}\int_K g_z^h\,\partial_1 W_1\,dx=\frac{a}{\delta h}\,v_0(q)g_z^h(q)+av_1(q)g_z^h(q)+O(h^3)+O(h)\int_K|\nabla g_z^h|\,dx.$$

Thus

$$\sum_{K\in T_1^h}\left(-\frac{\delta h h_1}{a}\int_K g_z^h\,\partial_1 W_1\,dx\right)=\sum_{K\in T_1^h}a(v_0 g_z^h)(q)+\sum_{K\in T_1^h}\delta h a(v_1 g_z^h)(q)+O(h^2).$$

Again in view of $\delta=-\delta'$ and $a-a'=O(h^3)$, there holds

$$\sum_{K\in T_1^h} \delta h a(v_1 g_z^h)(q) = O(h^2|\log h|).$$

Then, the area integral in (11) can have a united expansion:

$$\sum_{K\in T_1^h}\left(-\frac{\delta h h_1}{a}\int_K g_z^h \partial_1 W_1\, dx\right) = \int_{\Omega_1} v_0 g_z^h\, dx + O(h^2|\log h|). \quad (15)$$

Combining (15), (12), (13) with (14), we obtain from (10), (11) that

$$L_1^h = \int_{\Omega_1} v_0 g_z^h\, dx + \frac{2\delta h}{h_2}\int_{\Gamma_1} g_z^h W_1\, ds + O(h^2|\log h|).$$

Set

$$L_1 = \int_{\Omega_1} v_0 g_z\, dx + \frac{2\delta h}{h_2}\int_{\Gamma_1} g_z W_1\, ds$$

where $\frac{h}{h_2}$ is a constant since the lengths h_2 along Γ_2 are a constant. We obtain, for dist $(z, V)\geqslant \varepsilon>0$, by Lemmas 4 and A4 in [6],

$$L_1^h(z) = L_1(z) + O(h^2|\log h|).$$

For L_2^h we have, by the same treatment for L_0^h,

$$L_2^h = h^2 \sum_{K\in T_1^h} \partial_1 g_z^h W_2(\hat{q}_1) h_1 = O(h^2|\log h|).$$

As a result of the above discussion, the following expansion

$$L^h(z) = h^2(L_0(z) + L_1(z)) + O(h^4|\log h|) \quad (16)$$

holds true for dist $(z, V)\geqslant \varepsilon > 0$.

A similar expansion like (16) can be obtained for M^h by observing

$$\beta h_1^3 \partial_1^2 u(q_1) = -h_2\frac{h_1}{h_2 t_1\cdot t_2}\alpha h_1^2 \nabla^2 u(q_1) t_1^2 = h_2 h^2 \sum_{d=0}^{2}(\delta h)^d r_d(\hat{q}_2) + O(h^6).$$

This completes the proof of Theorem 1.

§ 4. Comparison with Quadratic Elements

An error expansion for quadratic element approximation to Poisson equation (1) can be derived in the same way as in [16] for the eigenvalue problem. Below, we shall consider uniform triangulation T^h, generated by a set of three-direction vectors.

Let S_0^h be the piecewise quadratic element space over T^h and $i^h u\in S_0^h$ the interpolant of u. Consider the integral

$$I(v) = \int_\Omega \nabla(u - i^h u)\nabla v\, dx \text{ for } v\in S_0^h.$$

We split I as follows:

$$I = \sum_0^3 I_d, \quad I_0 = -\sum_{K\in T^h}\int_K (u - i^h u)\Delta v\, dx,$$

$$I_d = \sum_{K\in T^h}\int_{s_d}(u - i^h u)\frac{\partial v}{\partial n_d}\, ds, \quad 1\leqslant d\leqslant 3.$$

We consider first the line integral I_d, say I_1. By Lemma 1

$$I_1 = \sum_K \alpha\int_{s_1}(u - i^h u)\partial_1 v\, ds + \sum_K \beta\int_{s_1}(u - i^h u)\partial_2 v\, ds.$$

All line integrals in the first sum over interior sides s_1 are cancelled, since $\partial_1=-\partial'_1$ on the adjacent triangles. And $v=0$ on $\partial\Omega$. It remains to expand the last sum in I_1. By Proposition 2 (with an inverse estimate) and Lemma 9 in [16], we have

$$I_1=c_1\beta h_1^4\sum_K\int_{s_1}\partial_1^4u\,\partial_2v\,ds+c_2\beta h_1^4\sum_{d=1}^3\gamma_d\sum_K\int_{s_1}\partial_1^3u\,\partial_d^2v\,ds+O(h^5)\|v\|'_{2,1}. \tag{17}$$

For line integral in the first sum there holds, by Lemma 10 in [16],

$$\int_{s_1}\partial_1^4u\,\partial_2v\,ds=\frac{h_1}{h_2}\int_{s_2}\partial_1^4u\,\partial_2v\,ds+\frac{h_1h_3}{2a}\int_K\partial_3(\partial_1^4u\,\partial_2v)\,dx.$$

After summation, all line integrals in the first term on the right over interior sides s_2 are cancelled, and the last term gives

$$\frac{h_1h_3}{2a}\Big(\int_\Omega\partial_3\partial_1^4u\,\partial_2v\,ds+\sum_K\int_K\partial_1^4u\,\partial_3\partial_2v\,dx\Big).$$

Dealing with the second term in (17) in the same way we obtain finally

$$I_1=\frac{c_1\beta}{2a}h_1^5h_3\Big(\int_\Omega\partial_3\partial_1^4u\,\partial_2v\,dx+\sum_K\int_K\partial_1^4u\partial_3\partial_2v\,dx\Big)$$
$$+\frac{c_2\beta}{2a}h_1^5\sum_K\Big(h_3\gamma_2\int_K\partial_3\partial_1^3u\,\partial_2^2v\,dx-h_2\gamma_3\int_K\partial_2\partial_1^3u\,\partial_3^2v\,dx\Big)+O(h^5)\|v\|'_{2,1}.$$

We now consider the area integral I_0. Again by Lemma 9 in [16], i.e. $\Delta v=\sum_1^3\eta_d\partial_d^2v$, we have

$$I_0=-\sum_{d=1}^3\eta_d\sum_K\int_K(u-i^hu)\partial_d^2v\,dx.$$

By Proposition 4 in [16],

$$\int_{K\cup K'}(u-i^hu)\,dx=h^4\int_{K\cup K'}D^4u\,dx+O(h^5)\|v\|_{2,1,K\cup K'}.$$

Hence, we have

$$I_0=h^4\sum\int_K D^4u\,D^2v\,dx+O(h^5)\|v\|'_{2,1}.$$

The result is

$$I(v)=h^4\Big(\int_\Omega D^5u\,Dv\,dx+\sum_K\int_K D^4u\,D^2v\,dx\Big)+O(h^5)\|v\|'_{2,1}. \tag{18}$$

Taking $v=g_z^h\in S_0^h$ (the discrete Green function), we have

$$(u^h-i^h)u(z)=I(g_z^h)=O(h^4|\log h|), \tag{19}$$

i.e. the quadratic element approximation u^h has a superconvergence with the same order as the extrapolation from linear elements.

A local result of (19) has been developed, for example, in [26].

We can establish an error expansion with the dominant term of $O(h^4)$.

We can see from this section that the error expansion method is also a powerful tool for observing the superconvergence phenomenon.

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