# SENSITIVITY ANALYSIS OF MULTIPLE EIGENVALUES (I)*

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## Abstract

The technique described in [9] is used to discuss the sensitivity of multiple eigenvalues and corresponding eigenspaces of symmetric eigenproblems analytically dependent on several parameters. The results may be useful for investigating problems of stability and response analysis of linear structures.

## § 1. Introduction

Throughout this paper we use the following notation. The symbol $\mathbb{R}^{m\times n}$ denotes the set of real $m\times n$ matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$, $\mathbb{R}=\mathbb{R}^1$ and

$$\mathbb{R}_r^{m\times n}=\{A\in\mathbb{R}^{m\times n}: \operatorname{rank}(A)=r\},\ S\mathbb{R}^{n\times n}=\{A\in\mathbb{R}^{n\times n}: A^T=A\},$$

in which the superscript $T$ is for transpose. $I^{(n)}$ is the $n\times n$ identity matrix, and 0 is the null matrix. $A>0$ denotes that $A$ is a positive definite matrix. We use $\rho(\ )$ for the spectral radius and $\|\ \|_2$ the usual Euclidean vector norm. The set of the eigenvalues of an eigenproblem $Ax=\lambda x$ is denoted by $\lambda(A)$ and the set of the eigenvalues of the eigenproblem $Ax=\lambda Bx$ is denoted by $\lambda(A, B)$. Besides, let $\lambda_1(A), \cdots, \lambda_n(A)$ denote an eigenvalues of an $n\times n$ matrix $A$.

Let $p=(p_1, p_2, \cdots, p_N)^T\in\mathbb{R}^N$. Suppose that $A(p)=(\alpha_{ij}(p))$, $B(p)=(\beta_{ij}(p))\in S\mathbb{R}^{n\times n}$ are real analytic functions in some neighbourhood $\mathscr{B}(p^*)$ of the point $p^*\in\mathbb{R}^N$ and $B(p)>0\ \forall p\in\mathscr{B}(p^*)$. Without loss of generality we may assume that the point $p^*$ is the origin of $\mathbb{R}^N$. It is well known that the eigenproblem

$$A(p)x(p)=\lambda(p)B(p)x(p),\quad \lambda(p)\in\mathbb{R},\ x(p)\in\mathbb{R}^n,\ p\in\mathscr{B}(0) \tag{1.1}$$

arises frequently in structural design, and it is often desirable to be able to estimate the sensitivity of the available designs to changes in system parameters. Although investigation of the sensitivity of eigenvalues and eigenvectors has a long history[3,4,9], the case of multiple eigenvalues is rarely treated in the literature (see [8, p. 606], [6, p. 133], [9, p. 362], [5]). The object of this paper is to discuss the sensitivity of multiple eigenvalues and corresponding eigenspaces of the eigenproblem (1.1) with respect to the parameters $p_1, p_2, \cdots, p_N$. The results of this paper may be useful for investigating problems of stability and response analysis of linear structures.

The following example is very interesting, from which we shall gain a good deal of enlightenment.

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*Example* 1.1[1, p.386], [9, p.362]. We consider an eigenproblem

$$A(p)x(p)=\lambda(p)x(p) \tag{1.2}$$

with

$$A(p)=\begin{pmatrix} 1+2p_1+2p_2 & p_2 \\ p_2 & 1+2p_2 \end{pmatrix},\ \lambda(p)\in\mathbb{R},\ x(p)\in\mathbb{R}^2,\ p=(p_1,p_2)^T\in\mathbb{R}^2. \tag{1.3}$$

Obviously, the matrix $A(p)$ is a real analytic function of $p\in\mathbb{R}^2$, $A(0)$ has eigenvalue 1 with multiplicity 2 and the eigenvalues of $A(p)$ are

$$\lambda_1(p)=1+p_1+2p_2+\sqrt{p_1^2+p_2^2},\quad \lambda_2(p)=1+p_1+2p_2-\sqrt{p_1^2+p_2^2}. \tag{1.4}$$

It is easy to see the following facts:

$$\begin{aligned}&\left(\frac{\partial\lambda_1(p)}{\partial p_1}\right)_{p=0,\ p_1=+0}=\left(\frac{\partial\lambda_2(p)}{\partial p_1}\right)_{p=0,\ p_1=-0}=2,\\ &\left(\frac{\partial\lambda_1(p)}{\partial p_1}\right)_{p=0,\ p_1=-0}=\left(\frac{\partial\lambda_2(p)}{\partial p_1}\right)_{p=0,\ p_1=+0}=0,\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\left(\frac{\partial\lambda_1(p)}{\partial p_2}\right)_{p=0,\ p_2=+0}=\left(\frac{\partial\lambda_2(p)}{\partial p_2}\right)_{p=0,\ p_2=-0}=3,\\ &\left(\frac{\partial\lambda_1(p)}{\partial p_2}\right)_{p=0,\ p_2=-0}=\left(\frac{\partial\lambda_2(p)}{\partial p_2}\right)_{p=0,\ p_2=+0}=1;\end{aligned} \tag{1.6}$$

$$\lambda\left(\left(\frac{\partial A(p)}{\partial p_1}\right)_{p=0}\right)=\{2,\ 0\},\quad \lambda\left(\left(\frac{\partial A(p)}{\partial p_2}\right)_{p=0}\right)=\{3,\ 1\}. \tag{1.7}$$

Here we define

$$\left(\frac{\partial\lambda_s(p)}{\partial p_1}\right)_{p=0,\ p_1=+0}=\lim_{p_1\to+0}\frac{\lambda_s(p_1,0)-\lambda_s(0,0)}{p_1}$$

and

$$\left(\frac{\partial\lambda_s(p)}{\partial p_1}\right)_{p=0,\ p_1=-0}=\lim_{p_1\to-0}\frac{\lambda_s(p_1,0)-\lambda_s(0,0)}{p_1},\quad s=1,\ 2.$$

The partial derivatives $\left(\frac{\partial\lambda_s(p)}{\partial p_2}\right)_{p=0,\ p_2=+0}$ and $\left(\frac{\partial\lambda_s(p)}{\partial p_2}\right)_{p=0,\ p_2=-0}$ $(s=1,\ 2)$ are defined similarly.

The relations (1.4)—(1.7) show that the functions $\lambda_1(p)$ and $\lambda_2(p)$ are not differentiable (thus they are not analytic) at $p=0$, but there exist two permutations $\pi$ and $\pi'$ of 1, 2 such that one has the relations

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\ p_i=+0}=\lambda_{\pi(s)}\left(\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}\right) \tag{1.8}$$

and

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\ p_i=-0}=\lambda_{\pi'(s)}\left(\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}\right),\ s=1,\ 2,\ i=1,\ 2. \tag{1.9}$$

In the next section we shall prove that the relations (1.8) and (1.9) are of universal significance, on the basis of which we may define the sensitivity of multiple eigenvalues dependent on several parameters, and give some formulae for computing the sensitivity (see § 3).

**Remark 1.1.** Let $A(p)$ be the matrix described in (1.3). If we set

$$\hat{A}(p_1)=(A(p))_{p=(p_1,0)^T},\quad \tilde{A}(p_2)=(A(p))_{p=(0,p_2)^T},$$

then $\hat{A}(0)$ and $\tilde{A}(0)$ have eigenvalue 1 with multiplicity 2, the eigenvalues of $\hat{A}(p_1)$ are

$$\hat{\lambda}_1(p_1)=1+2p_1, \quad \hat{\lambda}_2(p_1)=1 \tag{1.10}$$

and the eigenvalues of $\tilde{A}(p_2)$ are

$$\tilde{\lambda}_1(p_2)=1+3p_2, \quad \tilde{\lambda}_2(p_2)=1+p_2. \tag{1.11}$$

Clearly, the functions $\hat{\lambda}_1(p_1)$, $\hat{\lambda}_2(p_1)$, $\tilde{\lambda}_1(p_2)$ and $\tilde{\lambda}_2(p_2)$ are all real analytic. Comparing (1.10), (1.11) with (1.4) we find an important phenomenon:

$$(\lambda_1(p))_{p=(p_1,0)^T}=\begin{cases}\hat{\lambda}_1(p_1) & p_1\geqslant 0,\\ \hat{\lambda}_2(p_1) & p_1\leqslant 0,\end{cases} \quad (\lambda_2(p))_{p=(p_1,0)^T}=\begin{cases}\hat{\lambda}_2(p_1) & p_1\geqslant 0,\\ \hat{\lambda}_1(p_1) & p_1\leqslant 0,\end{cases}$$

$$(\lambda_1(p))_{p=(0,p_2)^T}=\begin{cases}\tilde{\lambda}_1(p_2) & p_2\geqslant 0,\\ \tilde{\lambda}_2(p_2) & p_2\leqslant 0,\end{cases} \quad (\lambda_2(p))_{p=(0,p_2)^T}=\begin{cases}\tilde{\lambda}_2(p_2) & p_2\geqslant 0,\\ \tilde{\lambda}_1(p_2) & p_2\leqslant 0.\end{cases}$$

## § 2. Some Results about Partial Derivatives

First we cite the following implicit function theorem ([3, p. 277]).

**Implicit Function Theorem.** *If the real-value functions*

$$f_i(\xi_1, \cdots, \xi_s; \eta_1, \cdots, \eta_t), \quad i=1, \cdots, s$$

*are real analytic functions of $s+t$ real variables in some neighbourhood of the origin of $R^{s+t}$, if $f_i(0;0)=0$, $i=1,\cdots,s$, and if*

$$\det\frac{\partial(f_1,\cdots,f_s)}{\partial(\xi_1,\cdots,\xi_s)}\neq 0, \text{ for } \xi_1=\cdots=\xi_s=\eta_1=\cdots=\eta_t=0,$$

*then the equations*

$$f_i(\xi_1, \cdots, \xi_s; \eta_1, \cdots, \eta_t)=0, \quad i=1, \cdots, s$$

*have a unique solution*

$$\xi_i=g_i(\eta_1, \cdots, \eta_t), \quad i=1, \cdots, s$$

*vanishing for $\eta_1=\cdots=\eta_t=0$ and real analytic in some neighbourhood of the origin of* $\mathbb{R}^t$.

Secondly, we introduce the following definition.

**Definition 2.1.** *Let $A, B\in S\mathbb{R}^{n\times n}$, $B>0$. A subspace $\mathscr{X}\subset\mathbb{R}^n$ is an eigenspace of the matrix pair $\{A, B\}$ if $A\mathscr{X}\subset B\mathscr{X}$.*

Clearly, any set of eigenvectors of the eigenproblem $Ax=\lambda Bx$ spans an eigenspace; Conversely, any eigenspace is spanned by a set of eigenvectors. Moreover, if there is a matrix $X_1\in\mathbb{R}_r^{n\times r}$ and a matrix $C_1\in\mathbb{R}^{r\times r}$ such that

$$AX_1=BX_1C_1, \tag{2.1}$$

then the set of column vectors of $X_1$ spans an eigenspace of $\{A, B\}$. Conversely, if $\mathscr{X}$ is an eigenspace, then there exist matrices $X_1\in\mathbb{R}_r^{n\times r}$ and $C_1\in\mathbb{R}^{r\times r}$ satisfying the relation (2.1) such that the space $\mathscr{X}$ is spanned by column vectors of $X_1$.

The following theorem is the main result of this section.

**Theorem 2.1.** *Let $p=(p_1,\cdots,p_N)^T\in\mathbb{R}^N$, and let $A(p), B(p)\in S\mathbb{R}^{n\times n}$ be real analytic functions of $p$ in some neighbourhood $\mathscr{B}(0)$ of the origin of $\mathbb{R}^N$, in which $B(p)>0$ $\forall p\in\mathscr{B}(0)$. Suppose that there is a nonsingular matrix $X\in\mathbb{R}^{n\times n}$ satisfying*

$$X=(\underset{r}{X_1}, \underset{n-r}{X_2}),\ X^TA(0)X=\begin{pmatrix}\lambda_1 I^{(r)} & 0\\ 0 & A_2\end{pmatrix},\ X^TB(0)X=I,\ \lambda_1\overline{\in}\lambda(A_2). \tag{2.2}$$

*Then the eigenproblem* (1.1) *has r eigenvalues* $\lambda_1(p), \cdots, \lambda_r(p)$ *satisfying*

$$\lambda_s(0)=\lambda_1, \quad s=1, \cdots, r,$$

*and there exist two permutations* $\pi$ *and* $\pi'$ *of* $1, \cdots, r$ *such that one has*

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\, p_i=+0}=\lambda_{\pi(s)}\left(X_1^T\left(\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}-\lambda_1\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=0}\right)X_1\right) \tag{2.3}$$

*and*

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\, p_i=-0}=\lambda_{\pi'(s)}\left(X_1^T\left(\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}-\lambda_1\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=0}\right)X_1\right),$$
$$s=1, \cdots, r, \ i=1, \cdots, N, \tag{2.4}$$

*in which the right partial derivatives* $\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\, p_i=+0}$ *and the left partial derivatives* $\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\, p_i=-0}$ *are defined by*

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\, p_i=+0}=\lim_{p_i\to+0}\frac{\lambda_s(0, \cdots, 0, p_i, 0, \cdots, 0)-\lambda_s(0, \cdots, 0)}{p_i}$$

*and*

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\, p_i=-0}=\lim_{p_i\to-0}\frac{\lambda_s(0, \cdots, 0, p_i, 0, \cdots, 0)-\lambda_s(0, \cdots, 0)}{p_i},$$
$$s=1, \cdots, r, \ i=1, \cdots, N.$$

*Moreover, there exists a real analytic function* $X_1(p)\in\mathbb{R}_r^{n\times r}$ *whose column vectors span the eigenspace of* $\{A(p), B(p)\}$ *corresponding to the eigenvalues* $\lambda_1(p), \cdots, \lambda_r(p)$ *in some neighbourhood* $\mathscr{B}_0(\subset\mathscr{B}(0))$ *of the origin of* $\mathbb{R}^N$ *such that*

$$X_1(0)=X_1 \tag{2.5}$$

*and*

$$\left(\frac{\partial X_1(p)}{\partial p_i}\right)_{p=0}=X_2(\lambda_1I-A_2)^{-1}X_2^T\left(\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}-\lambda_1\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=0}\right)X_1, \ i=1, \cdots, N. \tag{2.6}$$

*Proof.* 1. Let

$$\tilde{A}(p)=X^TA(p)X=\begin{pmatrix}\tilde{A}_{11}(p) & \tilde{A}_{21}(p)^T\\ \tilde{A}_{21}(p) & \tilde{A}_{22}(p)\end{pmatrix}, \ B(p)=X^TB(p)X=\begin{pmatrix}\tilde{B}_{11}(p) & \tilde{B}_{21}(p)^T\\ \tilde{B}_{21}(p) & \tilde{B}_{22}(p)\end{pmatrix}, \tag{2.7}$$

where $\tilde{A}_{11}(p), \tilde{B}_{11}(p)\in S\mathbb{R}^{r\times r}$. We introduce matrix-valued functions

$$F(Z, W, p)=\tilde{A}_{21}(p)+\tilde{A}_{22}(p)Z+W\tilde{A}_{11}(p)+W\tilde{A}_{21}(p)^TZ \tag{2.8}$$

and

$$G(Z, W, p)=\tilde{B}_{21}(p)+\tilde{B}_{22}(p)Z+W\tilde{B}_{11}(p)+W\tilde{B}_{21}(p)^TZ, \tag{2.9}$$

where

$$Z=(\zeta_{ij})\in\mathbb{R}^{(n-r)\times r}, \ W=(\omega_{ij})\in\mathbb{R}^{(n-r)\times r}, \ p=(p_1, \cdots, p_N)^T\in\mathbb{R}^N \tag{2.10}$$

and

$$F(Z, W, p)=(f_{ij}(Z, W, p))\in\mathbb{R}^{(n-r)\times r},$$
$$G(Z, W, p)=(g_{ij}(Z, W, p))\in\mathbb{R}^{(n-r)\times r}. \tag{2.11}$$

Observe that the functions $F(Z, W, p)$ and $G(Z, W, p)$ are analytic for $Z, W\in\mathbb{R}^{(n-r)\times r}$ and $p\in\mathscr{B}(0)$,

$$f_{ij}(0, 0, 0)=0, \ g_{ij}(0, 0, 0)=0, \quad i=1, \cdots, n-r, \ j=1, \cdots, r$$

and

$$\left(\det\frac{\partial(f_{11},\cdots,f_{1r},\cdots,f_{n-r,1},\cdots,f_{n-r,r},g_{11},\cdots,g_{1r},\cdots,g_{n-r,1},\cdots,g_{n-r,r})}{\partial(\zeta_{11},\cdots,\zeta_{1r},\cdots,\zeta_{n-r,1},\cdots,\zeta_{n-r,r},\omega_{11},\cdots,\omega_{1r},\cdots,\omega_{n-r,1},\cdots,\omega_{n-r,r})}\right)_{Z=0,W=0,p=0}$$

$$=\det\begin{pmatrix} I^{(r)}\otimes\tilde{A}_{22}(0) & I^{(r)}\otimes\tilde{B}_{22}(0)\\ \tilde{A}_{11}(0)\otimes I^{(n-r)} & \tilde{B}_{11}(0)^T\otimes I^{(n-r)}\end{pmatrix}=\det\begin{pmatrix} I^{(r)}\otimes A_2 & I^{(r)}\otimes I^{(n-r)}\\ \lambda_1 I^{(r)}\otimes I^{(n-r)} & I^{(r)}\otimes I^{(n-r)}\end{pmatrix}$$

$$=\left[\det\begin{pmatrix} A_2 & I^{(n-r)}\\ \lambda_1 I^{(n-r)} & I^{(n-r)}\end{pmatrix}\right]^r=\det(A_2-\lambda_1 I)^r\neq 0,$$

where $\otimes$ denotes the Kronecker product symbol (see [7, p. 8—9]). Hence by the Implicit Function Theorem the equations

$$F(Z,W,p)=0,\quad G(Z,W,p)=0 \tag{2.12}$$

have a unique real analytic solution

$$Z=Z(p),\quad W=W(p) \tag{2.13}$$

in some neighbourhood $\mathscr{B}_0(\subset\mathscr{B}(0))$ of the origin of $\mathbb{R}^N$ with $Z(0)=0$ and $W(0)=0$, and

$$\det(I^{(n-r)}-W(p)Z(p)^T)\neq 0\quad \forall p\in\mathscr{B}_0. \tag{2.14}$$

From (2.14) we know that the matrix $\begin{pmatrix} I & W(p)^T\\ Z(p) & I\end{pmatrix}$ is nonsingular for $p\in\mathscr{B}_0$. Therefore we have

$$\begin{pmatrix} I & W(p)^T\\ Z(p) & I\end{pmatrix}^T\tilde{A}(p)\begin{pmatrix} I & W(p)^T\\ Z(p) & I\end{pmatrix}=\begin{pmatrix} A_1(p) & 0\\ 0 & A_2(p)\end{pmatrix},\ A_1(p)\in S\mathbb{R}^{r\times r} \tag{2.15}$$

and

$$\begin{pmatrix} I & W(p)^T\\ Z(p) & I\end{pmatrix}^T\tilde{B}(p)\begin{pmatrix} I & W(p)^T\\ Z(p) & I\end{pmatrix}$$

$$=\begin{pmatrix} B_1(p) & 0\\ 0 & B_2(p)\end{pmatrix}>0,\ B_1(p)\in S\mathbb{R}^{r\times r}, \tag{2.16}$$

in which

$$A_1(p)=\tilde{A}_{11}(p)+Z(p)^T\tilde{A}_{21}(p)+\tilde{A}_{21}(p)^TZ(p)+Z(p)^T\tilde{A}_{22}(p)Z(p), \tag{2.17}$$

$$B_1(p)=\tilde{B}_{11}(p)+Z(p)^T\tilde{B}_{21}(p)+\tilde{B}_{21}(p)^TZ(p)+Z(p)^T\tilde{B}_{22}(p)Z(p). \tag{2.18}$$

From (2.15) and (2.16)

$$\tilde{A}(p)\begin{pmatrix} I\\ Z(p)\end{pmatrix}=\tilde{B}(p)\begin{pmatrix} I\\ Z(p)\end{pmatrix}B_1(p)^{-1}A_1(p). \tag{2.19}$$

Combining with (2.7) and writing

$$X_1(p)=X\begin{pmatrix} I\\ Z(p)\end{pmatrix}, \tag{2.20}$$

we get

$$A(p)X_1(p)=B(p)X_1(p)B_1(p)^{-1}A_1(p) \tag{2.21}$$

and

$$A_1(0)=\lambda_1 I^{(r)},\ B_1(0)=I^{(r)},\ X_1(0)=X_1. \tag{2.22}$$

2. From (2.21)

$$B_1(p)^{-1}A_1(p)=[X_1(p)^TB(p)X_1(p)]^{-1}[X_1(p)^TA(p)X_1(p)]. \tag{2.23}$$

Utilizing (2.2) and (2.20) the first partial derivative of (2.23) with respect to $p_i$ at $p=0$ may be written in the form

$$\left(\frac{\partial(B_1(p)^{-1}A_1(p))}{\partial p_i}\right)_{p=0}=X_1^T\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}X_1-\lambda_1 X_1^T\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=0}X_1,\quad i=1,\cdots,N. \tag{2.24}$$

Let

$$\lambda(A_1(p),\ B_1(p))=\{\lambda_s(p)\}_{s=1}^{r}. \tag{2.25}$$

The relations (2.2), (2.7), (2.15) and (2.16) show that

$$\lambda_s(p)\in\lambda(A(p),\ B(p)),\ \lambda_s(0)=\lambda_1,\ s=1,\cdots,r \tag{2.26}$$

and that the eigenvalues $\lambda_1(p),\cdots,\lambda_r(p)$ are sufficiently near by $\lambda_1$ provided that the point $p$ belong to a sufficiently small neighbourhood $\mathscr{B}_0$ of the origin.

Let $i$ be any fixed index from $1,\cdots,N$. From (2.24) we see that the real matrix $\left(\frac{\partial(B_1(p)^{-1}A_1(p))}{\partial p_i}\right)_{p=0}$ is symmetric. Consequently there is a real orthogonal matrix $Q_i\in\mathbb{R}^{r\times r}$ such that

$$Q_i^T\left(\frac{\partial(B_1(p)^{-1}A_1(p))}{\partial p_i}\right)_{p=0}Q_i=\begin{pmatrix}\delta_1^{(i)} & & \\ & \ddots & \\ & & \delta_r^{(i)}\end{pmatrix}, \tag{2.27}$$

where

$$\begin{aligned}&\delta_1^{(i)}=\cdots=\delta_{r_1}^{(i)}=d_1^{(i)},\ \delta_{r_1+1}^{(i)}=\cdots=\delta_{r_1+r_2}^{(i)}=d_2^{(i)},\ \cdots,\\ &\delta_{r_1+\cdots+r_{q-1}+1}^{(i)}=\cdots=\delta_r^{(i)}=d_q^{(i)},\ d_j^{(i)}\neq d_m^{(i)}\ \text{for}\ j\neq m,\ r_1+\cdots+r_q=r.\end{aligned} \tag{2.28}$$

We write

$$(Q_i^TB_1(p)^{-1}A_1(p)Q_i)_{p=(0,\cdots,0,p_i,0,\cdots,0)^T}=(\theta_{kl}(p_i))_{1\le k,l\le r}, \tag{2.29}$$

in which the functions $\theta_{kl}(p_i)$ are real analytic and so may be written as convergent power series:

$$\theta_{kl}(p_i)=\theta_{kl}^{(i,0)}+\theta_{kl}^{(i,1)}p_i+\theta_{kl}^{(i,2)}p_i^2+\theta_{kl}^{(i,3)}p_i^3+\cdots,\ k,\ l=1,\ \cdots,\ r. \tag{2.30}$$

From

$$\begin{gathered}(Q_i^TB_1(p)^{-1}A_1(p)Q_i)_{p=0}=\lambda_1 I^{(r)},\\ \left[\frac{\partial}{\partial p_i}(Q_i^TB_1(p)^{-1}A_1(p)Q_i)\right]_{p=0}=Q_i^T\left(\frac{\partial(B_1(p)^{-1}A_1(p))}{\partial p_i}\right)_{p=0}Q_i\end{gathered} \tag{2.31}$$

and (2.27) it follows that

$$\theta_{kl}^{(i,0)}=\begin{cases}\lambda_1 & \text{if } k=l,\\ 0 & \text{if } k\neq l,\end{cases}\qquad \theta_{kl}^{(i,1)}=\begin{cases}\delta_k^{(i)} & \text{if } k=l,\\ 0 & \text{if } k\neq l.\end{cases}$$

Therefore

$$\theta_{kl}(p_i)=\begin{cases}\lambda_1+\delta_k^{(i)}p_i+\theta_{kk}^{(i,2)}p_i^2+\theta_{kk}^{(i,3)}p_i^3+\cdots & \text{if } k=l,\\ \theta_{kl}^{(i,2)}p_i^2+\theta_{kl}^{(i,3)}p_i^3+\cdots & \text{if } k\neq l.\end{cases} \tag{2.32}$$

Let

$$\hat{A}(p_i)=(A_1(p))_{p=(0,\cdots,0,p_i,0,\cdots,0)^T},\quad \hat{B}(p_i)=(B_1(p))_{p=(0,\cdots,0,p_i,0,\cdots,0)^T}, \tag{2.33}$$

$$\lambda(\hat{B}(p_i)^{-1}\hat{A}(p_i))=\{\hat{\lambda}_t(p_i)\}_{t=1}^{r} \tag{2.34}$$

and

$$S(p_i)=\hat{B}(p_i)^{1/2}(\hat{B}(p_i)^{-1}\hat{A}(p_i))\hat{B}(p_i)^{-1/2}. \tag{2.35}$$

Obviously $S(p_i)^T=S(p_i)$ and $\lambda(S(p_i))=\lambda(\hat{B}(p_i)^{-1}\hat{A}(p_i))$. It is easy to prove that

$S(p_i)$ is a real analytic function of $p_i$ in some neighbourhood $\hat{\mathscr{B}}_0$ of the origin of $\mathbf{R}$. By Rellich's theorem (see [8, Satz 1]) the eigenvalues $\hat{\lambda}_1(p_i), \cdots, \hat{\lambda}_r(p_i)$ are all real analytic functions provided that $p_i$ belongs to a sufficiently small neighbourhood $\hat{\mathscr{B}}_0$. On the other hand, by the Gerschgorin Theorem from (2.28), (2.29) and (2.32) we know that there are precisely $q$ circular disks $D_1, \cdots, D_q$ with centers $\lambda_1+d_1^{(i)}p_i, \cdots, \lambda_1+d_q^{(i)}p_i$ and with radii of magnitude $O(p_i^2)$, respectively, and the union $\bigcup_{j=1}^{q} D_j$ contains the set of all eigenvalues $\hat{\lambda}_1(p_i), \cdots, \hat{\lambda}_r(p_i)$. Besides, the disks $D_1, \cdots, D_q$ are mutually disjoint provided that $p_i$ belong to a sufficient small $\hat{\mathscr{B}}_0$, and in such a case every disk $D_j$ contains exactly $r_j$ eigenvalues $\hat{\lambda}_{r_1+\cdots+r_{j-1}+1}(p_i), \cdots, \hat{\lambda}_{r_1+\cdots+r_{j-1}+r_j}(p_i)$ which may be written as convergent power series

$$\hat{\lambda}_{r_1+\cdots+r_{j-1}+k}(p_i)=\lambda_1+d_j^{(i)}p_i+\varphi^{(i,2)}_{r_1+\cdots+r_{j-1}+k}p_i^2+\varphi^{(i,3)}_{r_1+\cdots+r_{j-1}+k}p_i^3+\cdots,$$
$$1\leqslant k\leqslant r_j,\ j=1, \cdots, q,\ p_i\in\hat{\mathscr{B}}_0 \tag{2.36}$$

(here $r_0=0$).

Now we rewrite the expressions of (2.36) as

$$\hat{\lambda}_t(p_i)=\lambda_1+\delta_t^{(i)}p_i+\varphi_t^{(i,2)}p_i^2+\varphi_t^{(i,3)}p_i^3+\cdots,\quad t=1, \cdots, r, \tag{2.37}$$

and observe the following facts:

(i) $\hat{\lambda}_1(p_i), \cdots, \hat{\lambda}_r(p_i)$, as the eigenvalues of $[B_1(p)^{-1}A_1(p)]_{p=(0,\cdots,0,p_i,0,\cdots,0)^T}$, are real analytic functions in some sufficient small neighbourhood $\hat{\mathscr{B}}_0$ of the point $p_i=0$, and for any two different indices $t_1$ and $t_2$ $(1\leqslant t_1, t_2\leqslant r)$ we have

$$\hat{\lambda}_{t_1}(p_i)\neq\hat{\lambda}_{t_2}(p_i)\quad \forall p_i\in\hat{\mathscr{B}}_0\backslash\{0\} \tag{2.38}$$

provided that $\hat{\lambda}_{t_1}(p_i)\not\equiv\hat{\lambda}_{t_2}(p_i)$ for $p_i\in\hat{\mathscr{B}}_0$ and $\hat{\mathscr{B}}_0$ is sufficiently small (the conclusion (2.38) may be derived from the fact that if $\hat{\lambda}(p_i)$ is a real analytic function in $\hat{\mathscr{B}}_0$ and if $\hat{\lambda}(p_i)$ is not identically zero, then all the points of zeros of $\hat{\lambda}(p_i)$ are isolated[2, p.41]).

(ii) Since the eigenvalues $\lambda_1(p), \cdots, \lambda_r(p)$ of $B_1(p)^{-1}A_1(p)$ are continuous functions of $p$, $(\lambda_1(p))_{p=(0,\cdots,0,p_i,0,\cdots,0)^T}, \cdots, (\lambda_r(p))_{p=(0,\cdots,0,p_i,0,\cdots,0)^T}$ are continuous functions of $p_i$ in $\hat{\mathscr{B}}_0$;

(iii) For any fixed point $p_i\in\hat{\mathscr{B}}_0$ the set $\{(\lambda_s(p))_{p=(0,\cdots,0,p_i,0,\cdots,0)^T}\}_{s=1}^r$ and the set $\{\hat{\lambda}_t(p_i)\}_{t=1}^r$ are just identical. Hence for every $\lambda_s(p)$ $(1\leqslant s\leqslant r)$ there are $t'$ and $t''$ ($t'$ and $t''$ may be the same index) depending on $s$ such that

$$(\lambda_s(p))_{p=(0,\cdots,0,p_i,0,\cdots,0)^T}=\begin{cases}\hat{\lambda}_{t'}(p_i) & p_i\geqslant 0,\\ \hat{\lambda}_{t''}(p_i) & p_i\leqslant 0,\end{cases} \tag{2.39}$$

and for every $\hat{\lambda}_t(p_i)$ $(1\leqslant t\leqslant r)$ there are $s'$ and $s''$ ($s'$ and $s''$ may be the same index) depending on $t$ such that

$$\hat{\lambda}_t(p_i)=\begin{cases}(\lambda_{s'}(p))_{p=(0,\cdots,0,p_i,\cdots,0)^T}, & p_i\geqslant 0,\\ (\lambda_{s''}(p))_{p=(0,\cdots,0,p_i,\cdots,0)^T}, & p_i\leqslant 0.\end{cases} \tag{2.40}$$

From (2.39) and (2.37)

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0}=\begin{cases}\delta_{t'}^{(i)}, & p_i=+0,\\ \delta_{t''}^{(i)}, & p_i=-0,\end{cases} \tag{2.41}$$

and from (2.40) and (2.37)

$$\delta_i^{(0)} = \begin{cases} \left(\dfrac{\partial \lambda_{s'}(p)}{\partial p_i}\right)_{p=0,\ p_i=+0,} \\ \left(\dfrac{\partial \lambda_{s''}(p)}{\partial p_i}\right)_{p=0,\ p_i=-0.} \end{cases} \tag{2.42}$$

Combining (2.41), (2.42), (2.26), (2.27) and (2.24) we obtain the relations (2.3) and (2.4).

3. The relations (2.21) and (2.25) show that the set of columm vectors of $X_1(p)$ spans an eigenspace of $\{A(p),\ B(p)\}$ corresponding to the eigenvalues $\lambda_1(p), \cdots, \lambda_r(p)$ in some neighbourhood of the origin of $\mathbf{R}^N$, and $X_1(0)=X_1$. From (2.21) we get

$$\begin{aligned}(\lambda_1 B(0)-A(0))X\begin{pmatrix} 0 \\ \left(\dfrac{\partial Z(p)}{\partial p_i}\right)_{p=0}\end{pmatrix} &= \left[\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0} - \lambda_1\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=0}\right]X_1 + \lambda_1 B(0) X_1\left(\frac{\partial B_1(p)}{\partial p_i}\right)_{p=0} \\ &\quad - B(0)X_1\left(\frac{\partial A_1(p)}{\partial p_i}\right)_{p=0}.\end{aligned} \tag{2.43}$$

Combining (2.43) with (2.2) we get

$$\begin{aligned}\begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 I - A_2\end{pmatrix}\begin{pmatrix} 0 \\ \left(\dfrac{\partial Z(p)}{\partial p_i}\right)_{p=0}\end{pmatrix} &= X\left[\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0} - \lambda_1\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=0}\right]X_1 \\ &\quad + \begin{pmatrix} I^{(r)} \\ 0 \end{pmatrix}\left[\lambda_1\left(\frac{\partial B_1(p)}{\partial p_i}\right)_{p=0} - \left(\frac{\partial A_1(p)}{\partial p_i}\right)_{p=0}\right].\end{aligned} \tag{2.44}$$

Since $\lambda_1 \overline{\in} \lambda(A_2)$, it follows from (2.44) that

$$\left(\frac{\partial Z(p)}{p_i}\right)_{p=0} = (\lambda_1 I - A_2)^{-1} X_2^T\left[\left(\frac{\partial A(p)}{\partial p_i}\right]_{p=0} - \lambda_1\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=0}\right]X_1. \tag{2.45}$$

Substituting (2.45) into $\left(\dfrac{\partial X_1(p)}{\partial p_i}\right)_{p=0} = X_2\left(\dfrac{\partial Z(p)}{\partial p_i}\right)_{p=0}$ we obtain the relations (2.6). ∎

The following theorem may be deduced from Theorem 2.1.

**Theorem 2.2.** *Let $p=(p_1, \cdots, p_N)^T \in \mathbf{R}^N$, and let $A(p)\in S\mathbf{R}^{n\times n}$ be a real analytic function of $p$ in some neighbourhood $\mathscr{B}(0)$ of the origin of $\mathbf{R}^N$. Suppose that there is an orthogonal matrix $X\in \mathbf{R}^{n\times n}$ satisfying*

$$X = (\underset{r}{X_1},\ \underset{n-r}{X_2}),\quad X^T A(0) X = \begin{pmatrix} \lambda_1 I^{(r)} & 0 \\ 0 & A_2\end{pmatrix},\quad \lambda_1 \overline{\in} \lambda(A_2).$$

*Then the eigenproblem (1.2) has $r$ eigenvalues $\lambda_1(p), \cdots, \lambda_r(p)$ satisfying $\lambda_s(0)=\lambda_1$ for $s=1, \cdots, r$, and there exist two permutations $\pi$ and $\pi'$ of $1, \cdots, r$ such that one has*

$$\left(\frac{\partial \lambda_s(p)}{\partial p_i}\right)_{p=0,\ p_i=+0} = \lambda_{\pi(s)}\left(X_1^T\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0} X_1\right)$$

*and*

$$\left(\frac{\partial \lambda_s(p)}{\partial p_i}\right)_{p=0,\ p_i=-0} = \lambda_{\pi'(s)}\left(X_1^T\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0} X_1\right),\quad s=1, \cdots, r,\ i=1, \cdots, N.$$

*Moreover, there exists a real analytic function $X_1(p)\in \mathbf{R}_r^{n\times r}$ whose column vectors span* *an eigenspace of* $A(p)$ *corresponding to the eigenvalues* $\lambda_1(p), \cdots, \lambda_r(p)$ *in some neighbourhood* $\mathscr{B}_0(\subset\mathscr{B}(0))$ *of the origin of* $\mathbb{R}^N$ *such that*

$$X_1(0)=X_1$$

*and*

$$\left(\frac{\partial X_1(p)}{\partial p_i}\right)_{p=0}=X_2(\lambda_1 I-A_2)^{-1}X_2^T\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}X_1, \quad i=1, \cdots, N.$$

## § 3. Applications

3.1. Sensitivity of multiple eigenvalues

According to Theorem 2.1 and Theorem 2.2 we may introduce the following definition.

**Definition 3.1.** *Let* $p=(p_1, \cdots, p_N)^T\in\mathbb{R}^N$, *and let* $A(p),\ B(p)\in S\mathbb{R}^{n\times n}$ *be real analytic functions of* $p$ *in some neighbourhood* $\mathscr{B}(p^*)$ *of the point* $p^*\in\mathbb{R}^N$, *in which* $B(p)>0\ \forall p\in B(p^*)$. *Suppose that there is a nonsingular matrix* $X\in\mathbb{R}^{n\times n}$ *satisfying*

$$X=(\underset{r}{X_1}, \underset{n-r}{X_2}), X^TA(p^*)X=\begin{pmatrix}\lambda_1 I^{(r)} & 0\\ 0 & A_2\end{pmatrix}, \ X^TB(p^*)X=I, \quad \lambda_1\overline{\in}\lambda(A_2).$$

*Then the quantity*

$$s_{p_i}(\lambda_1)=\rho\left(X_1^T\left[\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=p^*}-\lambda_1\left(\frac{\partial B(p)}{\partial p_i}\right)_{p=p^*}\right]X_1\right) \tag{3.1}$$

*is called the sensitivity of the multiple eigenvalue* $\lambda_1$ *with respect to the parameter* $p_i$; *the quantity*

$$s_{p_{i_1},p_{i_2},\cdots,p_{i_m}}(\lambda_1)=\sqrt{\sum_{k=1}^{m}S^2_{p_{i_k}}(\lambda_1)} \tag{3.2}$$

*is called the sensitivity of the multiple eigenvalue* $\lambda_1$ *with respect to the parameters* $p_{i_1}$, $p_{i_2}, \cdots, p_{i_m}$; *the quantity*

$$s_p(\lambda_1)=\sqrt{\sum_{j=1}^{N}S^2_{p_j}(\lambda_1)} \tag{3.3}$$

*is called the sensitivity of the multiple eigenvalue* $\lambda_1$.

**Remark 3.1.** If $B(p)\equiv I^{(n)}$ in Definition 3.1, then the sensitivity of the multiple eigenvalue $\lambda_1$ with respect to the parameter $p_i$

$$s_{p_i}(\lambda_1)=\rho\left(X_1^T\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=p^*}X_1\right). \tag{3.4}$$

*Example* 3.1. The matrix $A(p)$ of Example 1.1 has multiple eigenvalue $\lambda_1=1$ at $p=0\in\mathbb{R}^2$. By Definition 3.1 we have

$$s_{p_1}(\lambda_1)=2,\ s_{p_2}(\lambda_1)=3,\ s_p(\lambda_1)=\sqrt{13}.$$

3.2. Determination of sensitive elements

Determination of sensitive elements or locations of a structure is an important problem in solving inverse problems of structural dynamics. Mathematically, this problem is to determine the sensitive elements of a matrix (or a matrix pair).

Let $A=(\alpha_{ij})$, $B=(\beta_{ij})\in S\mathbb{R}^{n\times n}$ and $B>0$. Assume that the matrix pair $\{A, B\}$ has eigenvalue $\lambda_1$ of multiplicity $r$, i.e., there exists a nonsingular $X\in\mathbb{R}^{n\times n}$ such that

$$X=(\underset{r}{X_1},\ \underset{n-r}{X_2}),\ X^TAX=\begin{pmatrix}\lambda_1 I^{(r)} & 0\\ 0 & A_2\end{pmatrix},\ X^TBX=I,\ \lambda_1\overline{\in}\lambda(A_2).$$

We regard the elements $\alpha_{ij}$ and $\beta_{ij}$ as parameters. By Definition 3.1 the sensitivities of the multiple eigenvalue $\lambda_1$ with respect to $\alpha_{ij}$ and $\beta_{kl}$ are respectively

$$s_{\alpha_{ij}}(\lambda_1)=\rho\Big(X_1^T\frac{\partial A}{\partial\alpha_{ij}}X_1\Big),\quad s_{\beta_{kl}}(\lambda_1)=|\lambda_1|\rho\Big(X_1^T\frac{\partial B}{\partial\beta_{kl}}X_1\Big).\tag{3.5}$$

Let

$$X_1=\begin{pmatrix}x_1'^T\\ \vdots\\ x_n'^T\end{pmatrix},\quad x_j'\in\mathbb{R}^r\quad\forall j.\tag{3.6}$$

Observe that

$$X_1^T\frac{\partial A}{\partial\alpha_{ij}}X_1=x_i'x_j'^T+x_j'x_i'^T,\qquad X_1^T\frac{\partial B}{\partial\beta_{kl}}X_1=x_k'x_l'^T+x_l'x_k'^T.$$

Therefore utilizing the lemma in Appendix we get

$$s_{\alpha_{ij}}(\lambda_1)=\|x_i'\|_2\|x_j'\|_2+|x_i'^Tx_j'|,\quad s_{\beta_{kl}}(\lambda_1)=|\lambda_1|(\|x_k'\|_2\|x_l'\|_2+|x_k'^Tx_l'|).\tag{3.7}$$

Moreover, by Definition 3.1 the sensitivities of the multiple eigenvalue $\lambda_1$ with respect to $\alpha_{i_1,j_1},\cdots,\alpha_{i_m,j_m}$ and $\beta_{k_1,l_1},\cdots,\beta_{k_m,l_m}$ are respectively

$$s_{\alpha_{i_1,j_1},\cdots,\alpha_{i_m,j_m}}(\lambda_1)=\sqrt{\sum_{k=1}^{m}(\|x_{i_k}'\|_2\|x_{j_k}'\|_2+|x_{i_k}'^Tx_{j_k}'|)^2}\tag{3.8}$$

and

$$s_{\beta_{k_1,l_1},\cdots,\beta_{k_m,l_m}}(\lambda_1)=|\lambda_1|\sqrt{\sum_{i=1}^{m}(\|x_{k_i}'\|_2\|x_{l_i}'\|_2+|x_{k_i}'^Tx_{l_i}'|)^2}.\tag{3.9}$$

**Remark 3.2.** The sensitivities of the multiple eigenvalue $\lambda_1$ with respect to $A$ and $B$ are respectively

$$s_A(\lambda_1)=\sqrt{\sum_{i,j=1}^{n}(\|x_i'\|_2\|x_j'\|_2+|x_i'^Tx_j'|)^2}\tag{3.10}$$

and

$$s_B(\lambda_1)=|\lambda_1|\sqrt{\sum_{i,j=1}^{n}(\|x_i'\|_2\|x_j'\|_2+|x_i'^Tx_j'|)^2}.\tag{3.11}$$

*Example* 3.2.

$$A=\begin{pmatrix}4&1&-2&-1\\1&5&1&1\\-2&1&4&-1\\-1&1&-1&5\end{pmatrix}=(\alpha_{ij})\in S\mathbb{R}^{4\times4}.$$

There is an orthogonal matrix

$$X=\left(\begin{array}{rr|rr}-0.7&-0.1&-0.371748&-0.601501\\-0.1&0.7&-0.601501&0.371748\\0.7&0.1&-0.371748&-0.601501\\-0.1&0.7&0.601501&-0.371748\end{array}\right)=(X_1\mid X_2)$$

such that

$$X^TAX=\mathrm{diag}(6,6,3+\sqrt{5},3-\sqrt{5}).$$

Therefore $\lambda_1=6$ is an eigenvalue of multiplicity 2. The sensitivities of the eigenvalue

$\lambda_1$ with respect to $a_{ij}$ are given in Table 1.

**Table 1**

| $S_{a_{ij}}(\lambda_1)$ $i \backslash j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.0 | 0.5 | 1.0 | 0.5 |
| 2 | 0.5 | 1.0 | 0.5 | 1.0 |
| 3 | 1.0 | 0.5 | 1.0 | 0.5 |
| 4 | 0.5 | 1.0 | 0.5 | 1.0 |

## Appendix

**Lemma.** *Let $a, b \in \mathbb{R}^n$. Then*

$$\rho(ab^T+ba^T) = \|a\|_2\|b\|_2 + |a^Tb|.$$

*Proof.* First we may take an orthogonal matrix $U=(u_1, u_2, \cdots, u_n) \in \mathbb{R}^{n\times n}$ such that

$$a = U\begin{pmatrix}\alpha \\ 0 \\ \vdots \\ 0\end{pmatrix}, b = U\begin{pmatrix}\beta_1 \\ \beta_2 \\ \vdots \\ 0\end{pmatrix}, \alpha = \|a\|_2.$$

Therefore

$$\rho(ab^T+ba^T) = \alpha\rho\begin{pmatrix}2\beta_1 & \beta_2 \\ \beta_2 & 0\end{pmatrix}.$$

Since

$$\lambda\begin{pmatrix}2\beta_1 & \beta_2 \\ \beta_2 & 0\end{pmatrix} = \{\beta_1+\sqrt{\beta_1^2+\beta_2^2},\ \beta_1-\sqrt{\beta_1^2+\beta_2^2}\},$$

we get

$$\rho(ab^T+ba^T) = \alpha(|\beta_1| + \sqrt{\beta_1^2+\beta_2^2}) = \|a\|_2|u_1^Tb| + \|a\|_2\|b\|_2 = |a^Tb| + \|a\|_2\|b\|_2.$$

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