ERROR EXPANSION FOR FEM AND SUPERCONVERGENCE UNDER NATURAL ASSUMPTION*

Lin Qun Xie Rui-feng
(*Institute of Systems Science, Academia Sinica, Beijing, China*)

In this paper, we derive the error expansion for finite element method under natural assumption and discuss the superconvergence as a special case of error expansion.

§1. Introduction

In a survey by Krisek and Neittaanmaki various types of superconvergence for FEM were discussed at some cases. But we can not expect the superconvergence for the displacement of linear finite element solution. At that case we can raise the convergence accuracy considerably using Richardson extrapolation. On extrapolation for FEM Chinese - German group has obtained a lot of results under some assumptions. See a survey by Rannacher. In this paper, we try to unit the discussion of superconvergence with the one of extrapolation. We deduce the error expansion under natural assumptions and discuss the superconvergence as a special case of error expansion. Consider the model problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \quad \text{on} \partial\Omega \tag{1.1}$$

where $\Omega \subset R^2$ is a convex domain and has smooth or piecewise smooth boundary. We use the finite element space over piecewise uniform or piecewise almost uniform triangulation to construct the finite element solution of problem (1.1). Let u^h and u^I be the linear finite element solution and interpolation of the true solution respectively. We derive the following expansion

$$u^h - u^I = \sum_{K=1}^{n-1} h^{2k} e_k^h + r^h \tag{1.2}$$

where the coefficients e_k^h are the finite element projections of the weak solution of problem (1.1) with defferent right hand side, and the remainder r^h satisfies

$$\|r^h\|_{1,\infty,\Omega_0} \le c\ |\ln h| h^{\alpha_n}$$

where α_n depends on the situation and Ω_0 is the subdomain of Ω which we shall discribe in details.

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As a special case with $n = 1$ we have the superconvergence estimate

$$\|u^h - u^I\|_{1,\infty,\Omega_0} \le ch^{\alpha_1}|\ln h|.$$

Replacing ∇u^h by some kind of average gradient $\bar{\nabla} u^h$ we obtain

$$(\bar{\nabla} u^h - \nabla u)(p) = O(h^{\alpha_1}|\ln h|)$$

for any nodal point p in Ω_0.

With $n = 2$ in (1.2) we get

$$u^h - u^I = h^2 w^I + r^h \tag{1.3}$$

from which we have

$$\begin{aligned} &\frac{1}{3}(4u^{h/2} - u^h)(p) - u(p) = O(h^{\alpha_2}|\ln h|), \\ &\frac{1}{3}\bar{\nabla}(4u^{h/2} - u^h)(p) - \nabla u(p) = O(h^{\alpha_2}|\ln h|). \end{aligned} \tag{1.4}$$

For the proof of (1.3) we emphasize the situation when Ω is a polygonal domain and u is of usual smoothness. At that case $\alpha_2 = 2\bar{\beta} - \varepsilon$ where ε is any positive number and $\bar{\beta}$ depends on the interior angle of Ω. In the last section we prove (1.4) with $\alpha_2 = 4$ when Ω is a smooth domain.

§2. Error Expansion on a Convex Polygonal Domain

Let $\Omega \subset R^2$ be a convex polygonal domain with the corner points $\{0_j\}$. We consider the problem (1.1) and its finite element solution. Choosing an arbitrary point o in $\bar{\Omega}$ and linking o with each corner point o_j, we subdivide Ω into several macro-triangles $\{\Omega_j\}$ with edges $\Gamma_j = \overline{oo_j}$. Let $T_h = \{K\}$ be a regular triangulation of Ω. Suppose that the restriction T_j to each Ω_j of T_h is uniform, i.e. each side of each triangle in T_j is parallel to one of three fixed direction vector. Assume that $S_h \subset H_0^1$ is the standard piecewise linear finite element space and u^h is the finite element solution. In order to evaluate the error expansion we consider the integral

$$\begin{aligned} I(u,v) &= \int_\Omega \nabla(u^h - u^I)\nabla v dx = \int_\Omega \nabla(u - u^I)\nabla v dx \\ &= \sum_j \int_{\Omega_j} \nabla(u - u^I)\nabla v dx, \quad \forall v \in S^h. \end{aligned} \tag{2.1}$$

It suffices to expand the integral

$$\int_{\Omega_j} \nabla(u - u^I)\nabla v dx.$$

For uniform triangulation T_j we introduce the standard notation as in [2]: s_i, p_i, n_i, t_i, h_i, A, ∂_i D_i. Our discussion is based on the following Euler-Maclaurin formula:

$$\int_{s_i}(u-u^I)ds=\sum_{k=1}^{n-1}\beta_k h_i^{2k}\int_{s_i}D_i^{2k}uds+h_i^{2k}\int_{s_i}\beta_n(s)D_i^{2n}uds \tag{2.2}$$

where $\beta_n(s)\in C_0^1(s_i)$ and $\beta_n(s)$ can be extended to a function $B_i\in C(\bar{\Omega})$ such that

$$B_i|_{s_i}(s)=\beta_n(s),\quad D_{i+2}B_i=0,\quad B_i|_{s_{i+2}}=0.$$

By the argument used in [2], we derive

$$\begin{aligned}\int_\Omega\nabla(u-u^I)\nabla vdx &=\sum_{k=1}^{n-1}h^{2k}\sum_j\Big(\int_{\Omega_j}L_j^{2k+1}uvdx+\int_{\Gamma_j}L_j^{2k+1}uvds\Big)\\&+\sum_{n=1}^{n-1}h^{2k}L_0^{2k}u(0)v(0)+h^{2n}\sum_{i,j}c_{ji}\int_{\Omega_j}B_{ji}D_{i+2}D_i^{2n}uD_{i+1}vdx\\&+h^{2n}\sum_j\int_{\Gamma_j}\beta_n(s)L_j^{2n}u\partial_\tau vds\end{aligned}\tag{2.3}$$

where L_j^m are some differential operators of order m, c_{ji} are some constant independent of h, B_{ji} are the function B_i defined on Ω_j and ∂_τ denotes the tangent derivative along Γ_j.

Letting $e_k\in H_0^1, r_n^h\in S_h, z_j^h\in S_h$ such that

$$\begin{aligned}(\nabla e_k,\nabla\varphi)&=I_k(\varphi)&&\text{for}\quad\varphi\in H_0^1,\\(\nabla r_n^h,\nabla v)&=I_n(v)&&\text{for}\quad v\in S^h,\\(\nabla z_j^h,\nabla v)&=F_j(v)&&\text{for}\quad v\in S^h,\end{aligned}$$

we may write

$$u^h-u^I=\sum_{k=1}^{n-1}h^{2k}(e_k^h+c_kG^h)+h^{2n}(r_n^h+\sum_j z_j^h) \tag{2.4}$$

where G is the Green function with singular point at 0. In order to estimate the remainder we need the regularized Green function g_z in [2]. With any fixed directional derivative ∂ we can write

$$\partial r_n^h(z)=(\nabla r_n^h,\nabla g_z^h)=I_n(g_z^h)=\sum_{i,j}c_{ji}\int_{\Omega_j}B_{ji}D_{i+2}D_i^{2n}uD_{i+1}g_z^hdx,$$

$$|\partial r_n^h(z)|\le c\|u\|_{2n+1,\infty}\|g_z^h\|_{1,1}\le c|\ln h|\|u\|_{2n+1,\infty},$$

$$\|r_n^h\|_{1,\infty}\le c|\ln h|\|u\|_{2n+1,\infty}.$$

To estimate z_j^h we write

$$\beta_n(s) = \beta_n + h_i^2\beta''_{n+1}(s), \quad \forall s \in s_i$$

with $\beta'_{n+1}(s) \in c_0^1(s_i)$. Then

$$\begin{aligned}
\partial z_j^h(z) &= F_j(g_z^h) = \int_{\Gamma_j} \beta_n(s) L_j^{2n} u \partial_\tau g_z^h ds \\
&= \beta_n \int_{\Gamma_j} L_j^{2n} u \partial_\tau g_z^h ds - h_i^2 \int_{\Gamma_j} \beta'_{n+1}(s) \partial_\tau L_j^{2n} u \partial_\tau g_z^h ds \\
&= -\beta_n \int_{\Gamma_j} \partial_\tau L_j^{2n} u g_z^h ds - h_i^2 \int_{\Gamma_j} \beta'_{n+1}(s) \partial_\tau L_j^{2n} u \partial_\tau g_z^h ds + \beta_n L_j^{2n} u(0) g_z^h(0), \\
|\nabla z_j^h(z)| &\le c\|u\|_{2n+1,\infty} \Big(\int_{\Gamma_j} |g_z^h| ds + h \int_{\Gamma_j} |\nabla g_z^h| ds + |g_z^h(0)| \Big), \\
|\nabla z_j^h(z)| &\le c\|u\|_{2n+1,\infty} \Big(\int_{\Omega} |\nabla g_z^h| dx + |g_z^h(0)| \Big) \le c\|u\|_{2n+1,\infty} \Big(|\ln h| + |g_z^h(0)| \Big).
\end{aligned}$$

For a fixed domain Ω_o such that $\bar{\Omega}_o$ doesn't contain 0, we have

$$\|z_j^h\|_{1,\infty,\Omega_0} \le c|\ln h|\ \|u\|_{2n+1,\infty}.$$

Hence, we can write (2.4) as

$$u^h - u^I = \sum_{h=1}^{n-1} h^{2k}(e_k^h + c_k G^h) + \bar{r}_n \tag{2.5}$$

with $\bar{r}_h$ satisfying

$$\|\bar{r}_n\|_{1,\infty,\Omega_o} \le ch^{2n}|\ln h|.$$

Taking $n = 1$ we have the superconvergence estimate

$$\|u^h - u^I\|_{1,\infty,\Omega_o} \le ch^2|\ln h|. \tag{2.6}$$

For any interior node p, there are two opposite elements K and K' with p as their common vertice. We define $\bar{\nabla}$ by

$$\bar{\nabla} v(p) = \frac{1}{2}(\nabla v|_K + \nabla v|_{K'}), \quad v \in S_h.$$

Then we have in [3]

$$\begin{aligned}
&|(\bar{\nabla} u^I - \nabla u)(p)\| \le ch^2\|u\|_{3,\infty,K\cup K'}, \\
&(\bar{\nabla} u^I - \nabla u)(p) = h^2\bar{e}(p) + O(h^4)\|u\|_{5,\infty,K\cup K'}
\end{aligned} \tag{2.7}$$

with some function $\bar{e}$ independent of h.

Now we have the superconvergent approximation to the true solution:

$$|(\bar{\nabla} u^h - \nabla u)(p)| \le ch^2|\ln h|.$$

By the localization method (cf. Wahlin), for $\Omega_1 \subset \Omega$ such that $\Omega_o \subset \Omega_1$ and $(\partial\Omega_1 \setminus \partial\Omega) \cup \partial\Omega_o = \phi$, we have

$$\|u^h - u^I\|_{1,\infty,\Omega_o} \le ch^2|\ln h|\|u\|_{3,\infty,\Omega_o} + \|u - u^h\|_{-2,2,\Omega_1}. \tag{2.8}$$

Taking $n = 2$ in (2.5),

$$u^h - u^I = h^2(e_1^h + c_1G^h) + \tilde{r}_2.$$

We assume $\bar{\Omega}_o$ doesn't contain the corner point of Ω in addition. Because e_1 and G are smooth in the neighborhood of Ω_0, an application of (2.8) gives

$$\|e_1^h - e_1^I\|_{1,\infty,\Omega_0} \le ch^2|\ln h|;$$

$$\|G^I - G^h\|_{1,\infty,\Omega_0} \le ch^2|\ln h|.$$

We obtain

$$u^h - u^I = h^2(e_1^I + c_1G^I) + R^h$$

with

$$\|R^h\|_{1,\infty,\Omega_o} \le ch^4|\ln h|.$$

From the above expansion and (2.7) we have

$$(u^h - u)(p) = h^2(e_1 + c_1G)(p) + O(h^4|\ln h|),$$

$$(\nabla u^h - \nabla u)(p) = h^2(\nabla e_1 + c_1\nabla G + \tilde{e})(p) + O(h^4|\ln h|).$$

§3. Local Analysis of Error Expansion on a Piecewise Uniform Mesh With Less Smooth Solution

In section 2 we obtained the error expansion and derived superconvergence as special case when the solution u is very smooth globally on $\bar{\Omega}$. But the solution is not so smooth usually even if the right hand side f of the problem (1.1) is analytic on $\bar{\Omega}$. In this section, we investigate the error expansion provided only that f is sufficiently smooth. As in section 2, we let $\{o_j\}$ denote the corner points of Ω. Denotes the maximal interior angle of Ω by $\bar{\alpha}$. Set $\bar{\beta} = \pi/\bar{\alpha}$. Without loss of generality, we assume $\bar{\beta} \le 2$. Let $d(x) = \prod_j |x - o_j|$. For any real α and integer k define the space

$$H_\alpha^k = \{u/d^{\alpha+|\beta|}\partial^\beta u \in L^2 \qquad \text{for any} \quad |\beta| \le k\}$$

with the norm

$$|||u|||_{k,\alpha,\Omega} = \Big(\sum_{|\beta|\le k}\int_\Omega d^{2\alpha+2|\beta|}|\partial^\beta u|^2 dx\Big)^{\frac{1}{2}}.$$

It is known that

$$f \in H_\alpha^{k-2} \quad \text{implies } u \in H_\alpha^k \quad \text{for} \quad \alpha > -1-\bar\beta \quad \text{and} \quad k \ge 2.$$

We also use the piecewise uniform triangulation described in §2 to construct the finite element solution. Using the standard analysis for the interpolation function, we can prove

$$|||u-u^I|||_{1,\alpha,\Omega} \le ch^{\alpha-\beta}|||u|||_{2,\beta,\Omega}$$

for $\alpha - 1 \le \beta < \alpha$.

Let $u \in H_\alpha^2 \cap C^3(\Omega)$ for some α. In order to apply the local estimate (2.8), we need to estimate $\|u-u^h\|_{-1,2}$. Using dual argument we have

$$\|u-u^h\|_{-1,2} \le ch^{1+2/q-2/q_0}|u-u^I|_{1,p}$$

pvovided $1 \le p \le 2, q = \frac{p}{p-1}$ and $q_0 \le \min(q, \frac{2}{2-\bar\beta})$. Taking p, r such that $r-1 \le \alpha < r < \frac{2}{p} - 2$, we have

$$|u-u^I|_{1,p} \le |||u-u^I|||_{1,r} \le ch^{r-2}|||u|||_{2,\alpha}.$$

Then choosing the numbers p, r, q_0 appropriatly, we can obtain

$$\|u-u^h\|_{-1,2} \le ch^{\bar\beta-\alpha-1-\epsilon}|||u|||_{2,\alpha}$$

with $\bar\beta - 3 < \alpha < 0$ and $\epsilon > 0$. Inserting the above inequality in (2.8) we derive

$$\|u^I-u^h\|_{1,\infty,\Omega_0} \le ch^{\bar\beta-\alpha-1-\epsilon}|||u|||_{2,\alpha} + ch^2|\ln h|\|u\|_{3,\infty,\Omega_1} \tag{3.1}$$

with $\bar\beta - 3 < \alpha < 0, \epsilon > 0$ and $u \in H_\alpha^2 \cap C^3(\bar\Omega_1)$.

Now we come to discuss the error expansion. Assume that Ω_0 is a fixed subdomain of Ω such that $\bar\Omega_0$ doesn't contain 0 or any corner point of Ω. Let $\Omega_0 \subset D_1 \subset D_2 \subset \Omega$ satisfy that $0 \in D_1, D_2$ doesn't contain any corner point of Ω, $(\partial D_2 \setminus \partial\Omega) \cup \bar D_1 = \phi$ and $(\partial D_1 \setminus \partial\Omega_0) \cap \bar\Omega_0 = \phi$. Choose function $\varphi \in C^\infty(\bar\Omega)$ such that $\varphi\big|_{D_1} = 1$ and $\varphi\big|_{\Omega\setminus D_2} = 0$. Suppose f is sufficiently smooth. From the properties of the solution of elliptic problem on a polygonal domain, $u \in C^6(\Omega) \cap H_\alpha^5$ for $\alpha > -1-\bar\beta$. Set $u_1 = \varphi u$ and $u_2 = u - u_1$. Then $u_1 \in C^5(\bar\Omega)$. Using the results derived in section 2, we have the expansion for u_1 :

$$u_1^h - u_1^I = h^2 w^I + r^h, \quad \|r^h\|_{1,\infty,\Omega_0} \le ch^4|\ln h|.$$

For u_2 we use the error expansion (2.4):

$$u_2^h - u_2^I = h^2(e_1^h + c_1 G^h) + h^4(r_2^h + \sum_j z_j^h)$$

with e_1, r_2^h, z_j^h such that

$$(\nabla e_1, \nabla\varphi) = \sum_j \Big(\int_{\Omega_j} L_j^4 u_2 \varphi dx + \int_{\Gamma_j} L_j^3 u_2 \varphi ds\Big) \quad \text{for } \varphi \in C_0^\infty(\Omega),$$

$$(\nabla r_2^h, \nabla v) = \sum_{i,j} c_{ji} \int_{\Omega j} B_{ji} D_{i+2} D_i^4 u_2 D_{i+1} v dx \quad \text{for } v \in S^h,$$

$$(\nabla z_j^h, \nabla v) = \int_{\Gamma_j} \beta_n(S) L_j^4 u_2 \partial_\tau v ds \quad \text{for } v \in S^h.$$

As smooth case in section 2, we have

$$\|G^h - G^I\|_{1,\infty,\Omega_0} \le ch^2 |\ln h|.$$

Because $u \in C^6(\Omega) \cap H_\alpha^5$ for $\alpha > -1 - \bar\beta$, we can prove $e_1 \in C^3(\Omega) \cap H_\alpha^3 \subset C^3(\Omega) \cap H_\alpha^2$ for $\alpha > 1 - \bar\beta$. An application of (3.1) gives

$$\|e_1^h - e_1^I\|_{1,\infty} \le ch^{2\bar\beta - 2 - \epsilon}.$$

In order to estimate the remainders we need some properties for the regularized Green function g_z. By the argument used by Blum and Rannacher, we have

$$|||g_z|||_{1,\alpha,\Omega\setminus D_1} \le |||g_z|||_{2,\alpha,\Omega\setminus D_1} \le c, \quad \text{for } z \in \Omega_0, \alpha > -\bar\beta - 1;$$

$$\|g_z - g_z^h\|_{1,\infty,\Omega\setminus D_1} \le ch^{\bar\beta - 1 - \epsilon} \quad \text{for } z \in \Omega_0 \quad \text{and } \epsilon > 0.$$

Because B_{ji} come from $\beta_2(s)$ which satisfies that $\beta_2(s) \in C_0^1(s_i)$, $\beta_2'(s) \in C_0^1(s_i)$ and $|D^k \beta_2(s)| \le ch^{-k}$ for $k = 0, 1, 2, \cdots$, we have

$$|B_{ji}(x)| \le ch^{-\alpha} d^\alpha \quad \text{for } 0 \le \alpha \le 2.$$

Noting that u_2 vanishes on D_1, we obtain

$$\begin{aligned}
&|\partial r_2^h(z)| = |(\nabla r_2^h, \nabla g_z^h)| \\
&\le ch^{-\alpha} \int_\Omega d^\alpha |\nabla^5 u_2||\nabla g_z| dx + ch^{-r} \|g_z - g_z^h\|_{1,\infty,\Omega\setminus D_1} \int_\Omega d^r |\nabla^5 u_2| dx \\
&\le ch^{-\alpha} |||u_2|||_{5,\alpha+\beta-5} |||g_z|||\Big|_{1,-\beta-1,\Omega\setminus D_1} + ch^{\bar\beta - r - 1 - \epsilon} |||u_2|||_{5,r-4-\epsilon} \\
&\le ch^{-\alpha} |||u_2|||_{5,\alpha+\beta-5} + ch^{\bar\beta - r - 1 - \epsilon} |||u_2|||_{5,r-4-\epsilon}.
\end{aligned}$$

Provided $\beta < \bar\beta, \alpha + \beta - 5 > -\bar\beta - 1$ and $r - 4 > -\bar\beta - 1$, which yields

$$\|r_2^h\|_{1,\infty,\Omega_0} \le ch^{2\bar\beta - 4 - \epsilon} \quad \text{for } \epsilon > 0.$$

As to z_j^h, we have

$$\begin{aligned}|\partial z_j^h(z)| &= |(\nabla z_j^h, \nabla g_z^h)| \\ &\le ch^{-\alpha}\int_{\Gamma_j} d^\alpha |L_j^4 u_2 \partial_\tau g_z| ds + ch^{-r}\|g_z - g_z^h\|_{1,\infty,\Omega\setminus D_1}\int_{\Gamma_j} d^r |L_j^4 u_2| ds \\ &\le ch^{-\alpha}\int_\Omega |\nabla(d^\alpha L_j^4 u_2 \partial_\tau g_z)| dx + ch^{\bar\beta - r - 1 - \epsilon}\int_\Omega |\nabla(d^r L_j^4 u_2)| dx \\ &\le ch^{-\alpha}|||u_2|||_{5,\alpha+\beta-5}|||g_z|||_{2,-\beta-1,\Omega\setminus D_1} + ch^{\bar\beta - r - 1 - \epsilon}|||u_2|||_{5,r-4-\epsilon} \\ &\le ch^{2\bar\beta - 4 - \epsilon}\end{aligned}$$

with the number $\alpha, \bar\beta, r$ appropriately chosen.

Collecting all above relations, we obtain the error expansion

$$u^h - u^I = h^2 w^I + \tilde r^h$$

with some function $w \in C^3(\Omega_0)$ and $\tilde r^h$ satisfying

$$\|\tilde r^h\|_{1,\infty,\Omega_0} \le ch^{2\bar\beta - \epsilon}.$$

§4. Error Expansion on a Domain With Curved Boundary

Let Ω be a bounded domain with a smooth boundary. For simplicity we assume that Ω is convex. In order to keep the mesh varying regularly, we use a kind of piecewise almost uniform triangulation we are going to describe. In this section we give the outline of the proof of the error expansion at that case. For the details we refer the reader to [6]. Now let us introduce the triangulation. Devide Ω into several parts Ω_j by some straight line such that for each Ω_j there is an invertible transformation ψ_j mapping Ω_j onto the square $[0,1]^2$, which, together with its inverse $\phi_i = \psi_i^{-1}$, is sufficiently smooth. In addition we assume that for each interior straight edge Γ of Ω_i the restriction to Γ of ψ_i is affine. Let $\tilde T^h$ be a uniform triangulation over $[0,1]^2$ with node set $\tilde N^h$, and let $N_j^h = \psi_i(\tilde N^h)$. Linking the points in N_i^h approriatly, we obtain a triangulation T_i^h over Ω_i. Then $T^h = \cup T_i^h$ is a regular triangulation over Ω. Such triangulation may be constructed directly (cf. [6]).

Now we analyze the properties of T^h through $\widehat{T^h}$. As in section 2, for any triangle $\widehat{K} \in \widehat{T^h}$, we introduce the notation $\hat p_d, \hat s_d, \hat h_d, \hat n_d, \hat t_d, \hat T_d$. In addition we introduce the midpoint $\hat q_d$ of $\hat s_d$ and the center $\hat q_0$ of $\hat K$. By h we denote the standard size of $\hat T^h$: $h = \max \hat h_d$. Corresponding to $\hat K$, let $K \in T_j^h$ be a triangular element with vertices

$$p = \psi_i(\hat p_d), \qquad i \le d \le 3.$$

We denote the associate quantities by $s_d, h_d, n_d, t_d, q_d, q_0, A$. Define

$$\delta(K)=\hat{t}_d\cdot\hat{T}_d.$$

Then, for any two adjacent triangles K and K', we have

$$\delta(K)=-\delta(K') \tag{4.1}$$

and the difference of the corresponding quantities between K and K' are of higher order:

$$h_d=h'_d+O(h^2),\quad t_d=-t'_d+O(h)\quad A=A'+O(h^3). \tag{4.2}$$

Using a Taylor expansion, it is easy to check that there exist smooth functions $a_{d,i}, b_{d,i}$ defined on $[0,1]^2$ independent of h such that, for all $K\in T_i^h$,

(i)$h_d=ha_{d,1}(\hat{q}_d)+h^3a_{d,2}(\hat{q}_d)+O(h^4)$,

(ii)$t_d=\delta(b_{d,1}(\hat{q}_d)+h^2b_{a,2}(\hat{q}_d))+O(h^3)$.

By the method used in [6] we can prove that there are some constants c_m and some piecewise smooth functions W and F_i such that

$$(\nabla(u^h-u^I),\nabla v)=h^2\int_\Omega Wvdx+h^2\sum_i\int_{\Gamma_i}F_ivds+h^2\sum_m c_mv(M_m)+\widehat{R}_2\,(u,v)$$

where $\{\Gamma_i\}_i$ denotes the set of all edges in Ω of all Ω_j, $\{M_m\}_m$ denotes the set of all interior corner points of Ω_j and

$$|\,\widehat{R}_2\,(u,v)|\le ch^4(\|v\|_{1,1}+\sum_m|v(M_m)|).$$

As in section 2, we can derive that there exist some locally smooth function e and some function $r_2\in S^h$ such that

$$u^h-u^I=h^2e^I+r_2,\|r_2\|_{1,\infty,\Omega_0}\le ch^4|\ln h| \tag{4.8}$$

where Ω_0 is a domain bounded from $\{M_m\}$. A immediate application of (4.8) gives

$$(u^h-u)(p)=h^2e(p)+O(h^4|\ln h|)$$

for the nodal point $p\in\Omega_0$.

To discuss the derivatives, we have to deal with u^I. By the Taylor expansion, we write

$$\begin{aligned}\partial_1(u^I-u)(p_2)&=\frac{1}{2}h_1\partial_1^2u(p_2)+\frac{1}{6}h_1^2\partial_1^3u(p_2)+\frac{1}{24}h_1^3\partial_1^4u(p_2)+O(h^4),\\ \partial_3(u^I-u)(p_2)&=-\frac{1}{2}h_3\partial_3^2u(p_2)+\frac{1}{6}h_3^2\partial_3^3u(p_2)+\frac{1}{24}h_3^3\partial_3^4u(p_2)+O(h^4).\end{aligned} \tag{4.9}$$

Representing the gradient ∇ by a linear combination of the directional derivatives ∂_3, and using (4.9) and Lemma 5, we can obtain the expansion

$$\nabla(u^I-u)(p_2)=h\delta\sum_{i=0}^{2}h^i\delta^i\mathbb{B}_i(P_2)+O(h^4) \tag{4.10}$$

where $I\!B_i$ are some vector function defined on Ω. For any node p in the interior of some Ω_i, there are two opposite trangles $K, K' \in T^h$ such that

$$p = p_2(K) = p_2(K').$$

We define

$$\bar{\nabla} v(p) = \frac{1}{2}(\nabla v|_K + \nabla v|_{K'}), \quad \text{for } v \in S^h.$$

Using (4.10), we have

$$\bar{\nabla} u^I(p) - \nabla u(p) = h^2 I\!B_1(p) + O(h^4). \tag{4.11}$$

Combining (4.8) with (4.11), we derive

$$\bar{\nabla} u^h(p) - \nabla u(p) = h^2(\nabla e + I\!B)(p) + O(h^4|\ln h|)$$

for any nodal point p in $\Omega_0 \setminus \cup\partial\Omega_j$.

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