MAX-NORM ESTIMATES FOR GALERKIN APPROXIMATIONS OF ONE-DIMENSIONAL ELLIPTIC, PARABOLIC AND HYPERBOLIC PROBLEMS WITH MIXED BOUNDARY CONDITIONS*1)

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Abstract

The Galerkin methods are studied for two-point boundary value problems and the related one-dimensional parabolic and hyperbolic problems. The boundary value problem considered here is of non-adjoint form and with mixed boundary conditions. The optimal order error estimate in the max-norm is first derived for the boundary problem for the finite element subspace $M \subset S_{k+1,s+1}(I)$ with $0 \le k \le s$. This result then gives optimal order max-norm error estimates for the continuous and discrete time approximations for the evolution problems described above.

§1. Introduction

Galerkin methods for the two-point boundary value problems with Dirichlet boundary have been studied intensively in [2], [3], [4], [7], etc. and a series of significant results have been achieved. In this paper, our emphasis is on the boundary condition of mixed-type. In Section 2 an optimal order L^∞ estimate for Galerkin approximations is derived. This result is then applied in Sections 3 and 4 to the single space variable parabolic and hyperbolic equations, respectively, to get the optimal order L^∞ estimates for continuous and discrete time Galerkin approximations.

Consider the following boundary value problems

$$\begin{aligned}&Lu \equiv -(a(x)u')' + b(x)u' + d(x)u = f(x), \quad x \in I = (0,1),\\ &a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0;\end{aligned} \tag{1.1}$$

and the initial-boundary value problems

$$\begin{aligned}&\frac{\partial u}{\partial t} + Lu = f_1(x,t), \quad (x,t) \in I \times (o,T],\\ &a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0,\\ &u(x,0) = u_0(x), \quad x \in I,\end{aligned} \tag{1.2}$$

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and

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} + Lu = f_2(x,t), \quad (x,t) \in I \times (0,T],\\&a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0,\\&u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in I.\end{aligned} \tag{1.3}$$

For problem (1.1), assume that

(i) $a(x) \in C^1(I), b(x) \in C^0(I)$ and $b'(x), d(x) \in L^\infty(I)$;

(ii) $\sigma_0, \sigma_1 \geq 0$ with $\sigma_0^2 + \sigma_1^2 > 0$ and there exist constants $a_0, a_1 > 0$ such that

$$0 < a_0 \leq a(x) \leq a_1, \quad \forall x \in I; \tag{1.4}$$

(iii) for each $f \in L^2(I)$, Problem (1.1) has a unique solution $u(x)$.

Problem (1.1) can be posed as

$$B(u,v) = (f,v), \quad \forall v \in H^1(I), \tag{1.5}$$

where

$$\begin{aligned}B(\phi,\psi) &= (a\phi',\psi') + (b\phi',\psi) + (d\phi,\psi) + \langle\phi,\psi\rangle,\\(\phi,\psi) &= \int_I \phi\psi dx,\\\langle\phi,\psi\rangle &= \sigma_0\phi(0)\psi(0) + \sigma_1\phi(1)\psi(1).\end{aligned} \tag{1.6}$$

The adjoint problem of Problem (1.1) is the following:

$$\begin{aligned}&L^*w = -(a(x)w')' - (b(x)w)' + d(x)w = g,\\&a(0)w'(0) - (b(0) + \sigma_0)w(0) = 0,\\&a(1)w'(1) + (b(1) + \sigma_1)w(1) = 0.\end{aligned} \tag{1.7}$$

Problem (1.7) can be posed as

$$B^*(w,v) = (g,v), \quad \forall v \in H^1(I) \tag{1.8}$$

where

$$B^*(\phi,\psi) = B(\psi,\phi). \tag{1.9}$$

From the theory of O.D.E.'s and Green's function expression of the solution of the boundary-value problem ([1]), we can assert that there exist C and C^* such that the following hold:

1. For each $f \in L^2(I)$, the solution, $u(x)$, of Problem (1.1) satisfies (1.5) and

$$\|u\|_{H^2(I)} \leq C\|f\|_{L^2(I)}.$$

2. For each $g \in L^2(I)$, Problem (1.7) and thus Problem (1.8) have a unique solution W and

$$\|W\|_{H^2(I)} \leq C^*|g|_{L^2(I)}.$$

3. Green's functions $G(x;z)$ and $G^*(x;z)$ for Problem (1.1) and Problem (1.7) exist and $G^*(x;z) = G(z;x)$. Also , $\forall z \in [0,1]$, we have

$$B(\phi, G^*(\cdot, z)) = \phi(z), \quad \forall \phi \in H^1(I). \tag{1.10}$$

Note that the bilinear form $B(\phi, \psi)$ on $H^1(I) \times H^1(I)$ is not necessarily symmetric or coercive. By use of inequalities

$$v^2(0) + \frac{1}{2}\|v'\|^2_{L^2(I)} \geq \frac{1}{2}\|v\|^2_{L^2(I)}, \quad v^2(1) + \frac{1}{2}\|v'\|^2_{L^2(I)} \geq \frac{1}{2}\|v\|^2_{L^2(I)}, \tag{1.11}$$

we see that the bilinear form

$$\widetilde{B}(\phi, \psi) = (a\phi', \psi') + \langle \phi, \psi \rangle \tag{1.12}$$

is symmetric, positive definite and bounded on $H^1(I) \times H^1(I)$.

Partition [0, 1] using $0 = x_0 < x_1 < \cdots < x_N = 1$ and let $I_i = (x_{i-1}, x_i), h_i = x_i - x_{i-1}, i = 1, 2, \cdots, N, h = \max_{1 \leq i \leq N} h_i$. Assume that this mesh is quasi-uniform, i.e., there exists a constant $\beta_0 > 0$ such that

$$\max_{i,j} h_i/h_j \leq \beta_0. \tag{1.13}$$

Let M be a finite element space of the following form

$$M = M_k^S = \{v : v \in C^k(\bar{I}), v \in P_S(I_i), \quad i = 1, 2, \cdots, N\} \tag{1.14}$$

where $P_S(J)$ is the set of polynomials on J of degree at most S, and $0 \leq k < S$. The space $M \subset H^{k+1}(I)$ and it is of class $S_{k+1,S+1}(I)$ ([6]). For $k = 0$ we denote $M \triangleq M_0 = M_0^S$.

The Galerkin approximation Problem (1.1) is defined as follows: Find $U \in M$ such that

$$B(U, V) = (f, V), \quad \forall V \in M. \tag{1.15}$$

Besides the usual Sobolev-Hilbert spaces $H^r(I)$ and their norms $\|\cdot\|_{H^r(I)}$ we need the space ([3], [7])

$$W^q_\infty(I) = \{v : v^{(i)} \in L^\infty(I), \quad i = 1, 2, \cdots, q\}$$

and its norm

$$\|v\|_{W^q_\infty(I)} = \sum_{i=0}^{q} \|v^{(i)}\|_{L^\infty(I)}.$$

We adopt the abbreviated notation

$$\|\cdot\|_r \equiv \|\cdot\|_{H^r(I)}, \quad \|\cdot\| \equiv \|\cdot\|_{H^0(I)}.$$

The letters C, C_i will denote generic constants with different values in different inequalities.

We will give optimal order L^∞ error estimates for the Galerkin solution of Problems (1.1), (1.2) and (1.3), respectively.

Contrasting our discussion with [7] and [8] one sees not only that the boundary condition considered here is more complicated but also that the conditions on the coefficients $b(x)$ and $d(x)$ are weaker and the subspace M is more general. Also, the treatments for Problems (1.2) and (1.3) use a different approach.

§2. The Optimal Order L^∞ Estimate for Galerkin Approximation of Problem (1.1)

The solvability of Galerkin procedure (1.15) is not obvious since $B(\phi, \psi)$ is not positive definite.

Theorem 1. *The Galerkin F.E. equation* (1.15) *has a unique solution* $U(x)$ *for* h *sufficiently small* ([2]).

Proof. It is sufficient to show the uniqueness. Suppose that $U_1(x)$ and $U_2(x)$ are the solutions of Problem (1.15). Set $\xi = U_1 - U_2$. Then

$$B(\xi, V) = 0, \quad \forall V \in M. \tag{2.1}$$

From the preliminary descriptions 2^0 in Section 1, for $\xi \in H^1(I)$ there exists a unique function $\nu \in H^2(I)$ such that

$$B(v, \nu) = (v, \xi), \quad \forall v \in H^1(I). \tag{2.2}$$

Take $v = \xi$ in (2.2). Then

$$\|\xi\|^2 = B(\xi, \nu - \chi), \quad \forall \chi \in M.$$

Thus

$$\|\xi\|^2 \le C\|\xi\|_1 \inf_{\chi \in M} \|\nu - \chi\|_1 \le C_1 h\|\xi\|_1 \|\nu\|_2 \le C_2 h\|\xi\|_1 \|\xi\|, \quad \|\xi\| \le C_2 h\|\xi\|_1. \tag{2.3}$$

Since

$$B(\xi, \xi) = 0,$$

we see that

$$(a\xi', \xi') + \langle \xi, \xi \rangle = -(b\xi', \xi) - (d\xi, \xi),$$

$$\|\xi\|_1^2 \le C[\|\xi'\| \cdot \|\xi\| + \|\xi\|^2],$$

$$\|\xi\|_1 \le C\|\xi\|.$$

Substituting this into (2.3) one gets

$$\|\xi\| \le C_3 \|\xi\| h, \tag{2.4}$$

where C_3 is independent of h. Hence for h sufficiently small we get $\|\xi\| = 0$, i.e., $U_1 \equiv U_2$.

Lemma 1. *Let* $u \in H^{S+1}(I)$. *Then there exists* C *such that*

$$\|u - U\| + h\|u - U\|_1 \le Ch^{S+1}\|u\|_{H^{S+1}(I)} \tag{2.5}$$

for h sufficiently small.

Proof. Since

$$B(u-U,V)=0,\quad \forall V\in M,$$

we have, as the derivation of inequality (2.3),

$$\|u-U\|\le C_2h\|u-U\|_1. \tag{2.6}$$

Note that

$$B(\phi,\psi)=\tilde{B}(\phi,\psi)+B_0(\phi,\psi)$$

where

$$B_0(\phi,\psi)=(b\phi',\psi)+(d\phi,\psi).$$

We see that

$$\begin{aligned}\|u-U\|_1^2 &\le C\tilde{B}(u-U,u-U)=C[B(u-U,u-U)-B_0(u-U,u-U)]\\ &=C[B(u-U,u-\chi)-B_0(u-U,u-U)],\quad \forall\chi\in M\\ &\le C_1[\|u-U\|_1\|u-\chi\|_1+\|u-U\|_1\|u-U\|].\end{aligned}$$

By (2.6),

$$\|u-U\|_1\le C_1\inf_{\chi\in M}\|u-\chi\|_1+C_2h\|u-U\|_1.$$

Thus for h small enough,

$$\|u-U\|_1\le C_3\inf_{\chi\in M}\|u-\chi\|_1\le Ch^S\|u\|_{H^{S+1}(I)}. \tag{2.7}$$

The conclusion follows from (2.6) and (2.7).

Lemma 2. *Let $u\in H^{S+1}(I)$ and subspace $M=M_0$. Then at all knots $\{x_i\}_{i=0}^N$,*

$$|(u-U)(x_i)|\le Ch^{S+1}\|u\|_{H^{S+1}(I)} \tag{2.8}$$

provided h is sufficiently small.

Proof. Let $G^*(\cdot,z)$ be the Green function of the adjoint problem (1.7) and $z\in[0,1]$ be an arbitrary point. By (1.10),

$$B(\phi,G^*(\cdot,z))=\phi(z),\quad \forall\phi\in H^1(I). \tag{2.9}$$

Take $z=x_i$ and $\phi=u-U$ in (2.9). Then

$$(u-U)(x_i)=B(u-U;G^*(\cdot,x_i))=B(u-U,G^*(\cdot,x_i)-\chi),\quad \chi\in M.$$

Thus

$$|(u-U)(x_i)|\le C\|u-U\|_1\inf_{\chi\in M}\|G^*(\cdot,x_i)-\chi\|_1. \tag{2.10}$$

Note that $G^*(\cdot,x_i)\in H^1(I)\subset C^0(\bar{I}), G^*(\cdot,x_i)\in H^2(0,x_i)\cap H^2(x_i,1)$ and there is a constant C_1 independent of x_i such that

$$\|G^*(\cdot,x_i)\|_{H^2(0,x_i)}+\|G^*(\cdot,x_i)\|_{H^2(x_i,1)}\le C_1,\quad i=0,1,\cdots,N.$$

We can construct a piecewise linear interpolation χ^* to $G^*(\cdot, x_i)$ so that $\chi^* \in M_0$ and

$$\begin{aligned}\inf_{\chi\in M_0} \|G^*(\cdot,x_i)-\chi\|_1 &\le \|G^*(\cdot,x_i)-\chi^*\|_1 \\ &\le \|G^*(\cdot,x_i)-\chi^*\|_{H^1(0,x_i)} + \|G^*\cdot,x_i)-\chi^*\|_{H^1(x_i,1)} \\ &\le Ch\{\|G^*(\cdot,x_i)\|_{H^2(0,x_i)} + \|G^*(\cdot,x_i)\|_{H^2(x_i,1)}\} \le C_2 h.\end{aligned}$$

From (2.10) the lemma follows.

Noting that, at boundary points $x = x_0 = 0$ and $x = x_N = 1$, Green functions $G^*(\cdot, x_0)$ and $G^*(\cdot, x_N)$ belong to $H^2(I)$, and recalling that $M = M_k^S \subset S_{k+1,S+1}(I)$, one immediately gets from (2.10)

Lemma 3. *Let $u \in H^{S+1}(I)$ and $M = M_k^S$ with $0 \le k < S$. Then*

$$\|(u-U)(x_j)\| \le Ch^{S+1}\|u\|_{H^{S+1}(I)}, \quad j = 0, N \tag{2.11}$$

for h sufficiently small.

In order to derive the estimate of $\|u-U\|_{L^\infty(I)}$, let's define some projections W and Z of u in some sense and estimate $\|U-W\|_{L^\infty(I)}, \|u-Z\|_{L^\infty(I)}$.

Let $W \in M$ and be defined by

$$\widetilde{B}(W-u, V) = 0, \quad \forall V \in M. \tag{2.12}$$

Clearly, W exists and is unique.

Lemma 4. *Let $u \in H^{S+1}(I)$. Then*

$$\|U-W\|_1 \le Ch^{S+1}\|u\|_{H^{S+1}(I)} \tag{2.13}$$

for h small enough .

Proof. Since $\widetilde{B}(\phi,\psi)$ is positive definite,

$$\begin{aligned}\|U-W\|_1^2 &\le C\widetilde{B}(U-W,U-W) = C\widetilde{B}(u-U,W-U) \\ &= C\{B(u-U,W-U) - B_0(u-U,W-U)\} \\ &= CB_0(u-U,U-W).\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}B_0(u-U,U-W) &= \int_I \{b(u-U)'(U-W) + d(u-U)(U-W)\}dx \\ &\le C\{\|u-U\|_{L^\infty(\partial I)}\|U-W\|_{L^\infty(\partial I)} + \|u-U\|\ \|U-W\|_1\} \\ &\le C_1\{\|u-U\|_{L^\infty(\partial I)} + \|u-U\|\}\|U-W\|_1\end{aligned}$$

where ∂I is the boundary of I, i.e., $x = 0$ and $x = 1$. Thus

$$\|U-W\|_1 \le C\{\|u-U\| + \max_{j=0,N} |(u-U)(x_j)|\}. \tag{2.14}$$

The conclusion of Lemma 4 follows from Lemmas 1 and 3.

Now we use another projection, Z, of u into M.Denote

$$\overset{\circ}{M}=\{v : v \in M \text{ and } v(0) = v(1) = 0\}.$$

Let $Z \in M$ such that $Z = u$ on ∂I and

$$\widetilde{B}(Z-u, V) = 0, \quad \forall V \in \overset{\circ}{M}. \tag{2.15}$$

It is easy to see that Z exists uniquely.

Lemma 5. *Let $u \in H^{S+1}(I)$. Then*

$$\|U - Z\|_1 \leq Ch^{S+1}\|u\|_{H^{S+1}(I)}. \tag{2.16}$$

Proof. Set

$$Z - W = \eta + \theta$$

where $\eta(x) \in P_1(I)$ and $\eta(0) = Z(0) - W(0) = u(0) - W(0)$, $\eta(1) = Z(1) - W(1) = u(1) - W(1)$. Thus $\theta \in \overset{\circ}{M}$. Note that

$$\begin{aligned}\|\theta\|_1^2 \leq C\widetilde{B}(\theta,\theta) &= C\widetilde{B}(Z-W-\eta,\theta) = C\widetilde{B}(u-W-\eta,\theta)\\ &= -C\widetilde{B}(\eta,\theta) \leq C_1\|\eta\|_1\|\theta\|_1\end{aligned}$$

and

$$\|\eta\|_1 \leq C_2(|\eta(1)| + |\eta(0)|) \leq C_3h^{S+1}\|u\|_{H^{S+1}(I)}. \tag{2.17}$$

Then we get

$$\|\theta\|_1 \leq C_4h^{S+1}\|u\|_{H^{S+1}(I)}. \tag{2.18}$$

Hence from (2.14), (2.17), and (2.18),

$$\|U - Z\|_1 \leq \|U - W\|_1 + \|W - Z\|_1 \leq \|U - W\|_1 + \|\eta\|_1 + \|\theta\|_1 \leq Ch^{S+1}\|u\|_{H^{S+1}(I)}.$$

Since Z is a Dirichlet-type projection of u into M, applying the result given in [3] we see that under the assumption (1.13), the following conclusion is true:

Lemma 6. *Let $u \in W_\infty^{S+1}(I)$. Then*

$$\|u - Z\|_{L^\infty(I)} \leq Ch^{S+1}\|u\|_{W_\infty^{S+1}(I)}. \tag{2.19}$$

Using Lemma 5, Lemma 6 and the triangle inequality we obtain the following optimal order L^∞ estimate.

Theorem 2. *Let u and U be the solution of Problem (1.1) and Problem (1.15) respectively. Suppose that $u \in W_\infty^{S+1}(I)$. Then there is a constant C independent of h, u and U such that*

$$\|u - U\|_{L^\infty(I)} \leq Ch^{S+1}\|u\|_{W_\infty^{S+1}(I)} \tag{2.20}$$

provided the mesh parameter h is small enough.

Remark 1. If we take $M = M_0^S$, then the condition (1.13) can be dropped out ([7]).

Remark 2. If the coefficients $b(x)$ and $d(x)$ in (1.1) satisfy the additional assumptions ([7])

$$d(x) \geq d_0 > 0, \quad \|b^2\|_{L^\infty(I)} \leq 2d_0 a_0,$$

then $B(\phi, \psi)$ is positive definite on $H^1(I) \times H^1(I)$. In this case the restriction on the size of h is not needed.

§3. Application For Parabolic Equations

The results obtained in Section 2 can be applied to Galerkin approximations for the one-dimensional parabolic equation with mixed-type boundary conditions to get the optimal order L^∞ estimates.

Rewrite Problem (1.2):

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big) - b(x)\frac{\partial u}{\partial x} - d(x)u + f_1(x,t), \quad (x,t) \in I \times (0,T], \\ &a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0, \quad t \in [0,T], \\ &u(x,0) = u_0(x), \quad x \in I. \end{aligned} \tag{3.1}$$

Retain the assumptions given in Section 1. In addition, assume that $f_1 \in L^2(I \times [0,T])$ and $u_0 \in H^1(I)$.

For simplicity we use notation $u(t) \equiv u(x,t)$, $f_1(t) \equiv f_1(x,t)$ and $L^p(X) \equiv L^p(0,T;X)$.

The continuous-time Galerkin approximation for Problem (3.1) is defined to be a differentiable mapping $U(t) : [0,T] \to M$ satisfying

$$\begin{aligned} &\Big(\frac{\partial U}{\partial t}, V\Big) + B(U,V) = (f_1(t), V), \forall V \in M, \\ &B(U(0) - u_0, V) = 0, \quad \forall V \in M. \end{aligned} \tag{3.2}$$

It is easy to see that Problem (3.2) has a unique solution $U(t)$ for h sufficiently small.

In order to estimate $\|u - U\|_{L^\infty(L^\infty(I))}$, we use an auxiliary function $\tilde{u}(t)$ which is defined to be a mapping: $[0,T] \to M$ for each $t \in [0,T]$ and satisfying

$$B(\tilde{u}(t) - u(t), V) = 0, \quad \forall V \in M \tag{3.3}$$

where $u(t)$ is the solution of Problem (3.1). From Theorem I we see that Problem (3.3) has a unique solution $\tilde{u}(t)$ for each $t \in [0,T]$ provided h is small enough. Also, $\tilde{u}(t)$ is smooth with respect to t if $u(t)$ is. Note that, from (3.2) and (3.3),

$$\tilde{u}(0) = U(0). \tag{3.4}$$

Let $\xi = U - \tilde{u}, \eta = \tilde{u} - u$. Then

$$\Big(\frac{\partial \xi}{\partial t}, V\Big) + B(\xi, V) = -\Big(\frac{\partial \eta}{\partial t}, V\Big), \quad V \in M, \quad t \in (o, T]. \tag{3.5}$$

Take $V = \frac{\partial \xi}{\partial t}$ in (3.5). Then

$$\left\|\frac{\partial \xi}{\partial t}\right\|^2 + \left(a\frac{\partial \xi}{\partial x}, \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right)\right) + \left\langle \xi, \frac{\partial \xi}{\partial t}\right\rangle = -\left(b\frac{\partial \xi}{\partial x} + d\cdot\xi + \frac{\partial \eta}{\partial t}, \frac{\partial \xi}{\partial t}\right).$$

From this we have

$$\frac{1}{2}\frac{d}{dt}\left\{\left\|a^{1/2}\frac{\partial \xi}{\partial x}\right\|^2 + \langle\xi,\xi\rangle\right\} \le C\left\{\|\xi\|_1^2 + \left\|\frac{\partial \eta}{\partial t}\right\|^2\right\}.$$

Hence

$$\left\|a^{1/2}\frac{\partial \xi}{\partial x}\right\|^2(t) + \sigma_1\xi^2(t)|_{x=1} + \sigma_0\xi^2(t)|_{x=0} \le C\left[\int_0^t \|\xi\|_1^2(s)ds + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(L^2(I))}\right]. \quad (3.6)$$

Here we have used the fact $\xi(0) = 0$.

Noting that $a(x) \ge a_0 > 0$ and $\sigma_0^2 + \sigma_1^2 > 0$, one gets from (3.6)

$$\|\xi\|_1^2(t) \le C_1\int_0^t \|\xi\|_1^2(s)ds + C_2\left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(L^2(I))}. \quad (3.7)$$

Thus

$$\|\xi\|_{L^\infty(L^\infty(I))} \le C\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(L^2(I))}. \quad (3.8)$$

Since the coefficients a, b, d are independent of t,

$$B(\eta_t, V) = 0, \quad \forall V \in M. \quad (3.9)$$

Suppose that $\frac{\partial u}{\partial t} \in L^2(H^{S+1}(I))$. Then, from Lemma 1,

$$\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(L^2(I))} \le Ch^{S+1}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^{S+1}(I))} \quad (3.10)$$

for h small enough.

By the definition (3.3), $\tilde{u}(t)$ is a Galerkin approximation of $u(t)$. Apply Theorem 2

$$\|\eta(t)\|_{L^\infty(I)} = \|u(t) - \tilde{u}(t)\|_{L^\infty(I)} \le Ch^{S+1}\|u(t)\|_{W_\infty^{S+1}(I)}$$

and assume that $u \in L^\infty(W_\infty^{S+1}(I))$. Then

$$\|\eta\|_{L^\infty(L^\infty(I))} \le Ch^{S+1}\|u\|_{L^\infty(W_\infty^{S+1}(I))}. \quad (3.11)$$

Combining (3.8), (3.10) and (3.11) we have

Theorem 3. *Let $u(t)$ and $U(t)$ be the solution of Problem (3.1) and Problem (3.2) respectively. Suppose that $u \in L^\infty(W_\infty^{S+1}(I))$ and $\frac{\partial u}{\partial t} \in L^2(H^{S+1}(I))$. Then there exists a constant C such that*

$$\|u - U\|_{L^\infty(L^\infty(I))} \le Ch^{S+1}\left[\|u\|_{L^\infty(W_\infty^{S+1}(I))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^{S+1}(I))}\right] \quad (3.12)$$

provided h is small enough.

Now consider a discrete-time Galerkin approximation for Problem (3.1).

Let $\Delta t = T/J$, J a positive integer, and $t_j = j\Delta t, j = 0, 1, \cdots, J$. Adopt the notation for the function $g(t)$:

$$g_j = g(t_j), \quad g_{j+1/2} = (g_{j+1} + g_j)/2, \quad \partial_t g_{j+1/2} = (g_{j+1} - g_j)/\Delta t.$$

Set $\tilde{J} = \{0, 1, \cdots, J\}$. The discrete-time Crank-Nicolson approximation for Problem (3.1) is defined to be a mapping $\{U_j\}_0^J : \tilde{J} \to M$ satisfying

$$\begin{gathered}(\partial_t U_{j+1/2}, V) + B(U_{j+1/2}, V) = (f_{j+1/2}, V), \quad \forall V \in M, \quad j = 0, 1, \cdots, J-1, \\ B(U_0 - u_0, V) = 0, \quad \forall V \in M.\end{gathered} \tag{3.13}$$

Let $\tilde{u}(t)$ be defined still by (3.3) and let $\eta = \tilde{u} - u$, $\xi_j = U_j - \tilde{u}_j$. Then

$$(\partial_t \xi_{j+1/2}, V) + B(\xi_{j+1/2}, V) = -(\partial_t \eta_{j+1/2} + E_j, V), \quad \forall V \in M, \quad j = 0, 1, \cdots, J-1, \tag{3.14}$$

where ([8])

$$E_j = \partial_t u_{j+1/2} - \left(\frac{\partial u}{\partial t}\right)_{j+1/2} = \frac{1}{2\Delta t}\int_{t_j}^{t_{j+1}} \left(\frac{\partial^3 u}{\partial t^3}\right)(t_{j+1} - \tau)(t_j - \tau)d\tau. \tag{3.15}$$

Use $V = \partial_t \xi_{j+1/2}$ in (3.14). Then

$$\begin{gathered}\frac{1}{2\Delta t}\left\{\left\|a^{1/2}\frac{\partial \xi_{j+1}}{\partial x}\right\|^2 - \left\|a^{1/2}\frac{\partial \xi_j}{\partial x}\right\|^2 + \sigma_1(\xi_{j+1}^2 - \xi_j^2)|_{x=1} + \sigma_0(\xi_{j+1}^2 - \xi_j^2)|_{x=0}\right\} \\ \le C\{\|\xi_{j+1}\|_1^2 + \|\xi_j\|_1^2 + \|E_j\|^2 + \|\partial_t \eta_{j+1/2}\|^2\}.\end{gathered} \tag{3.16}$$

Summing the above inequalities up for $j = 0$ to $j = m (\le J - 1)$ and noting that $\xi_0 = \xi|_{t=0} = 0$, one gets

$$\begin{aligned}&\left\|a^{1/2}\frac{\partial \xi_{m+1}}{\partial x}\right\|^2 + \sigma_1 \xi_{m+1}^2|_{x=1} + \sigma_0 \xi_{m+1}^2|_{x=0} \\ &\qquad \le C\left\{\sum_{j=0}^{m+1}\|\xi_j\|_1^2 \Delta t + \sum_{j=0}^{J-1}(\|E_j\|^2 + \|\partial_t \eta_{j+1/2}\|^2)\Delta t\right\}.\end{aligned}$$

Choosing Δt small enough, we apply the Gronwall inequality with discrete form to get

$$\|\xi_{m+1}\|_1^2 \le C\sum_{j=0}^{J-1}(\|E_j\|^2\Delta t + \|\partial_t \eta_{j+1/2}\|^2 \Delta t). \tag{3.17}$$

Note that

$$\sum_{j=0}^{J-1}\|\partial_t \eta_{j+1/2}\|^2 \Delta t \le \left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(L^2(I))}^2,$$

$$\sum_{j=0}^{J-1}\|E_j\|^2 \Delta t \le \left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(L^2(I))}^2 \cdot \overline{\Delta t}^4.$$

Then

$$\|\xi\|_{\tilde{L}^\infty(H^1(I))} \le C\Big\{\Big\|\frac{\partial^3 u}{\partial t^3}\Big\|\overline{\Delta t}^2 + \Big\|\frac{\partial \eta}{\partial t}\Big\|_{L^2(L^2(I))}\Big\} \tag{3.18}$$

where the norm

$$\|\cdot\|_{\tilde{L}^\infty(X)} \equiv \max_{0\le j\le J} \|\cdot\|_X.$$

By estimates (3.10), (3.11) and (3.18) we obtain

Theorem 4. *Let u and $\{U_j\}$ be the solution of Problem* (3.1) *and Problem* (3.13) *respectively. Suppose that $u \in L^\infty(W_\infty^{S+1}(I))$, $\frac{\partial u}{\partial t} \in L^2(H^{S+1}(I))$ and $\frac{\partial^3 u}{\partial t^3} \in L^2(L^2(I))$. Then there exists a constant C such that*

$$\begin{aligned}&\|u-U\|_{\tilde{L}^\infty(L^\infty(I))}\\ &\le C\Big[h^{S+1}\big(\|u\|_{L^\infty(W_\infty^{S+1}(I))} + \Big\|\frac{\partial u}{\partial t}\Big\|_{L^2(H^{S+1}(I))}\big) + \overline{\Delta t}^2\Big\|\frac{\partial^3 u}{\partial t^3}\Big)_{L^2(L^2(I))}\Big]\end{aligned} \tag{3.19}$$

for h and Δt sufficiently small.

§4. Application For Hyperbolic Equations

Consider Problem (1.3), that is, the problem below:

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\big(a(x)\frac{\partial u}{\partial x}\big) - b(x)\frac{\partial u}{\partial x} - d(x)u + f_2(x,t), \quad (x,t)\in I\times[0,T],\\ &a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0, \quad t\in[0,T],\\ &u(x,o) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x\in I.\end{aligned} \tag{4.1}$$

As in Section 3, keep the assumptions given in Section I. In addition, assume that $f_2 \in L^2(I\times[0,T])$ and $u_0, u_1 \in H^1(I)$.

The continuous-time Galerkin approximation for Problem (4.1) is defined to be a twice differentiable mapping $U(t):[0,T]\to M$ satisfying

$$\begin{aligned}&\Big(\frac{\partial^2 U}{\partial t^2}, V\Big) + B(U,V) = (f_2(t),V), \quad \forall V\in M, \quad t\in(0,T],\\ &B(U(0)-u_0, V) = 0,\\ &B\Big(\frac{\partial U}{\partial t}(0) - u_1, V\Big) = 0, \quad \forall V\in M.\end{aligned} \tag{4.2}$$

Let $\tilde{u}(t)$ be defined by (3.3) and set $\xi = U - \tilde{u}, \eta = \tilde{u} - u$. Then

$$\Big(\frac{\partial^2 \xi}{\partial t^2}, V\Big) + B(\xi, V) = -\Big(\frac{\partial^2 \eta}{\partial t^2}, V\Big). \tag{4.3}$$

Take $V = \frac{\partial \xi}{\partial t}$ in (4.3) to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\Big\|\frac{\partial \xi}{\partial t}\Big\|^2 + \Big\|a^{1/2}\frac{\partial \xi}{\partial x}\Big\|^2 + \sigma_1\xi^2|_{x=1} + \sigma_0\xi^2|_{x=0}\Big)\\ &\le C\Big\{\Big\|\frac{\partial \xi}{\partial t}\Big\|^2 + \|\xi\|_1^2 + \Big\|\frac{\partial^2 \eta}{\partial t^2}\Big\|^2\Big\}.\end{aligned}$$

Note that $\xi|_{t=0}=0$, $\frac{\partial\xi}{\partial t}|_{t=0}=0$. Applying the Gronwall inequality we have

$$\left\|\frac{\partial\xi}{\partial t}\right\|^2+\|\xi\|_1^2\le C\left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{L^2(L^2(I))}. \tag{4.4}$$

Therefore we obtain the following

Theorem 5. *Let $u(t)$ and $U(t)$ be the solution of Problem (4.1) and Problem (4.2) respectively. Suppose that $u\in L^\infty(W_\infty^{S+1}(I))$ and $\frac{\partial^2u}{\partial t^2}\in L^2(H^{S+1}(I))$. Then there exists a constant C such that*

$$\|u-U\|_{L^\infty(L^\infty(I))}\le Ch^{S+1}\left[\|u\|_{L^\infty(W_\infty^{S+1}(I))}+\left\|\frac{\partial^2u}{\partial t^2}\right\|_{L^2(H^{S+1}(I))}\right] \tag{4.5}$$

provided h is small enough.

Finally, consider a discrete-time scheme for problem (4.1) ([5]). Define for the function $g=g(t)$

$$\begin{gathered}g_{j,1/4}=\tfrac14(g_{j+1}+2g_j+g_{j-1}),\\ \delta_tg_j=(g_{j+1}-g_{j-1})/2\Delta t,\quad \partial_t^2g_j=(g_{j+1}-2g_j+g_{j-1})/\overline{\Delta t}^2.\end{gathered} \tag{4.6}$$

A discrete-time Galerkin procedure for Problem (4.1) is ([5], [8])

$$\begin{gathered}(\partial_t^2U_j,V)+B(U_{j,1/4},V)=(f_{j,1/4},V),\quad \forall V\in M,\quad j=1,2,\cdots,J-1,\\ B(U_0-u_0,V)=0,\\ B(U_1-u_1^*,V)=0,\quad \forall V\in M,\end{gathered} \tag{4.7}$$

where

$$u_1^*=u_0+\Delta tu_1+\frac{\overline{\Delta t}^2}{2}\frac{\partial^2u}{\partial t^2}(x,0). \tag{4.8}$$

Here, term $\frac{\partial^2u}{\partial t^2}(x,0)$ can be evaluated by the equation and initial conditions in (4.1).

Using the auxiliary function $\tilde u(t)$ as before we get

$$(\partial_t^2\xi_j,V)+B(\xi_{j,1/4},V)=-(\partial_t^2\eta_j+E_j,V),\quad \forall V\in M,\quad j=1,2,\cdots,J-1, \tag{4.9}$$

where $\xi=U-\tilde u,\eta=\tilde u-u$ and ([5])

$$E_j=\partial_t^2u_j-\left(\frac{\partial^2u}{\partial t^2}\right)_{j,1/4}=\frac{1}{12}\int_{-\Delta t}^{\Delta t}(\Delta t-|\tau|)\left[3-2\left(1-\frac{|\tau|}{\Delta t}\right)^2\right]\frac{\partial^4u}{\partial t^4}(t_j+\tau)d\tau. \tag{4.10}$$

Take $V=\delta_t\xi_j$ in (4.9) and note that

$$\begin{gathered}(\partial_t^2\xi_j,\delta_t\xi_j)=\frac{1}{2\Delta t}(\|\partial_t\xi_{j+1/2}\|^2-\|\partial_t\xi_{j-1/2}\|^2),\\ \left(a\left(\frac{\partial\xi}{\partial x}\right)_{j,1/4},\delta_t\xi_j\right)=\frac{1}{2\Delta t}\left\{\left\|a^{1/2}\left(\frac{\partial\xi}{\partial x}\right)_{j+1/2}\right\|^2-\left\|a^{1/2}\frac{\partial\xi}{\partial x}\right\|_{j-1/2}^2\right\},\\ \sigma_i\xi_{j,1/4}\cdot\delta_t\xi_j|_{x=x_i}=\frac{1}{2\Delta t}\sigma_i\left\{\xi_{j+1/2}^2|_{x=x_i}\right\},\quad i=0,1.\end{gathered}$$

We get

$$\frac{1}{2\Delta t}\Big\{ \|\partial_t\xi_{j+1/2}\|^2-\|\partial_t\xi_{j-1/2}\|^2+\Big\|a^{1/2}\Big(\frac{\partial\xi}{\partial x}\Big)_{j+1/2}\Big\|^2-\Big\|a^{1/2}\Big(\frac{\partial\xi}{\partial x}\Big)_{j-1/2}\Big\|^2$$
$$+\sigma_1(\xi^2_{j+1/2}|_{x=1}-\xi^2_{j-1/2}|_{x=1})+\sigma_0(\xi^2_{j+1/2}|_{x=0}-\xi^2_{j-1/2}|_{x=0})\Big\} \tag{4.11}$$
$$\le \frac{1}{2}\|\delta_t\xi_j\|^2+C\Big\{\Big\|\Big(\frac{\partial\xi}{\partial x}\Big)_{j,1/4}\Big\|^2+\|\xi_{j,1/4}\|^2+\|\partial_t\eta_j\|^2+\|E_j\|^2\Big\}.$$

Summing up (4.11) for $j=1$ to $j=m(\le J-1)$ and noticing

$$\|\delta_t\xi_j\|^2\le\frac{1}{2}(\|\partial_t\xi_{j+1/2}\|^2+\|\partial_t\xi_{j-1/2}\|^2)$$

we see that for h sufficiently small

$$\|\xi\|^2_{\bar L^\infty_\Delta(H^1(I))}\le C_1\Big\{\|\partial_t\xi_{1/2}\|^2+\Big\|\Big(\frac{\partial\xi}{\partial x}\Big)_{1/2}\Big\|^2+\xi^2_{1/2}|_{x=1}+\xi^2_{1/2}|_{x=0}\Big\}$$
$$+C_2\Big(\sum_{j=0}^{J-1}\|\partial_t^2\eta_j\|^2\Delta t+\overline{\Delta t}^4\Big\|\frac{\partial^4u}{\partial t^4}\Big\|^2_{L^2(L^2(I))}\Big), \tag{4.12}$$

where the norm

$$\|g\|_{\bar L^\infty_\Delta(X)}=\max_{1\le j\le J-1}\|g_{j,1/2}\|_X.$$

Note that

$$\xi^2_{1/2}|_{x_i}\le\|\xi_{1/2}\|^2_{L^\infty(I)}\le C\|\xi_{1/2}\|_1^2.$$

Also, one can prove that ([8])

$$\|\xi_{1/2}\|_1+\|\partial_t\xi_{1/2}\|\le C\overline{\Delta t}^2\Big\{\Big\|\frac{\partial^2u}{\partial t^2}\Big\|_{L^\infty(H^1)}+\Big\|\frac{\partial^3u}{\partial t^3}\Big\|_{L^\infty(L^2(I))}\Big\}.$$

We have

$$\|\xi\|_{\bar L^\infty_\Delta(H^1(I))}\le C\Big[\Big\|\frac{\partial^2\eta}{\partial t^2}\Big\|_{L^2(L^2(I))}+\overline{\Delta t}^2\Big(\Big\|\frac{\partial^2u}{\partial t^2}\Big\|_{L^\infty(H^1)}+\Big\|\frac{\partial^3u}{\partial t^3}\Big\|_{L^\infty(L^2(I))}+\Big\|\frac{\partial^4u}{\partial t^4}\Big\|_{L^2(L^2(I))}\Big)\Big]. \tag{4.13}$$

Theorem 6. *Let u and $\{U_j\}$ be the solution of Problem (4.1) and Problem (4.7) respectively. Suppose that $u\in L^\infty(W^{S+1}_\infty(I))$, $\frac{\partial^2u}{\partial t^2}\in L^\infty(H^{S+1}(I))$, $\frac{\partial^3u}{\partial t^3}\in L^\infty(L^2(I))$ and $\frac{\partial^4u}{\partial t^4}\in L^2(L^2(I))$. Then there exists a constant C such that*

$$\|u-U\|_{\bar L^\infty_\Delta(L^\infty(I))}\le C\Big[h^{S+1}\Big(\|u\|_{L^\infty(W^{S+1}_\infty(I))}+\Big\|\frac{\partial^2u}{\partial t^2}\Big\|_{L^2(H^{S+1}(I))}\Big)$$
$$+\overline{\Delta t}^2\Big(\Big\|\frac{\partial^2u}{\partial t^2}\Big\|_{L^\infty(H^1(I))}+\Big\|\frac{\partial^3u}{\partial t^3}\Big\|_{L^\infty(L^2(I))}+\Big\|\frac{\partial^4u}{\partial t^4}\Big\|_{L^2(L^2(I))}\Big)\Big] \tag{4.14}$$

provided h and Δt are small enough.

Remark 3. According to Remark 2 in Section 2, we can choose a constant λ large enough such that

$$\lambda \geq \frac{1}{2a_0}\|b^2\|_{L^\infty(I)} - \inf_{x\in I} d(x)$$

and define the auxiliary function $\tilde{u}(t)$ ([8]) by

$$B(\tilde{u}(t) - u(t), V) + \lambda(\tilde{u}(t) - u(t), V) = 0, \quad \forall V \in M.$$

In addition, define the initial functions, say,

$$B(U(0) - u_0, V) + \lambda(U(0) - u_0, V) = 0,$$

$$B\Big(\frac{\partial U}{\partial t}(0) - u_1, V\Big) + \lambda\Big(\frac{\partial U}{\partial t}(0) - u_1, V\Big) = 0, \quad \forall V \in M.$$

Then the restriction on the size of h can be dropped.

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