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GENERALIZED INVERSION AND ITS APPLICATION IN INVERSE SCATTERING PROBLEMS *

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Abstract

In this paper, the generalized inversion theory and its application in inverse scattering problems are discussed. An iterative solution of joint inversion of parameters describing the earth structures and sources is given and a numerical example is also shown.

Introduction

The inversion problem is a quite active field of research in geophysics. Many geophysical problems are regarded as reconstruction of the spatial distribution of some physical parameters from the images of the model space in the data space [1], [2], [3]. The reconstruction of the earth structures from the reflections observed at the surface is just one of this kind of problems. Based on the generalized inversion and by applying Born approximation, an iterative solution of linear inversion of reflections has been obtained by A. Tarantola et al. [3], [4], [5], [7]. Their result of the first iteration just corresponds to the classical migration. It is also possible to regard the source function as an unknown parameter. An iterative solution of simultaneous inversion of parameters describing the earth structures and sources is obtained in this paper.

§1. Theory

The inversion problems in geophysics are generally ill-posed. In particular, when we attempt to discretise model parameters not at the beginning of formula establishment, but at the last step of calculations, the general inversion theory based on the matrix algorithm would no longer be sufficient. The generalized inversion method provides the basis for solving this kind of inversion problems [3], [6].

1. Data space and model space

The functional space which consists of all acceptable models is called the model space and represented by M; the vector space which consists of all observable data is called the data space and denoted by D. The real line is represented by R.

Let the weight functions of M and D be $W_m(r,r')$ and $W_d(r,r')$ respectively.

The norm of the model space, $\| \ \|_M : M \to R$, is determined by

$$\|\underline{m}\|_M = \left\{ \int dr \int dr' m(r) W_m(r,r') m(r') \right\}^{1/2}, \quad \underline{m} \in M. \tag{1.1}$$

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The norm of the data space, $\| \|_D : D \to R$, is determined by

$$\|\underline{d}\|_D = \left\{ \int dr \int dr' d(r) W_d(r,r') d(r') \right\}^{1/2}, \quad \underline{d} \in D. \tag{1.2}$$

Define the scalar products of the model space and the data space as $\langle \ , \ \rangle_M : M \times M \to R$, and

$$\langle \underline{m}_1, \underline{m}_2 \rangle_M = \int dr \int dr' m_1(r) W_m(r,r') m_2(r'), \quad \underline{m}_1, \underline{m}_2 \in M; \tag{1.3}$$

$\langle \ , \ \rangle_D : D \times D \to R$, and

$$\langle \underline{d}_1, \underline{d}_2 \rangle = \int dr \int dr' d_1(r) W_d(r,r') d_2(r'), \quad \underline{d}_1, \underline{d}_2 \in D. \tag{1.4}$$

We have

$$\begin{aligned} \|\underline{d}\|_D^2 &= \langle \underline{d}, \underline{d} \rangle_D, \quad \underline{d} \in D, \\ \|\underline{m}\|_M^2 &= \langle \underline{m}, \underline{m} \rangle_M, \quad \underline{m} \in M. \end{aligned} \tag{1.5}$$

Let the dual spaces of model and data spaces be denoted by $\hat{M}$ and $\hat{D}$ respectively. From the definition of scalar products, the elements in the dual spaces can be correlated with the elements in the original spaces.

$$\begin{aligned} \hat{m}_1(r) &= \int dr' W_m(r,r') m_1(r), \quad \underline{m}_1 \in M, \quad \underline{\hat{m}}_1 \in \hat{M}, \\ \hat{d}_1(r) &= \int dr' W_d(r,r') d_1(r), \quad \underline{d}_1 \in D, \quad \underline{\hat{d}}_1 \in \hat{D}. \end{aligned} \tag{1.6}$$

Introduce the definition of transposed operator [3], [6]: the transposed operator $\underline{G}^T$ of the linear operator $\underline{G} : M \to D$, maps $\hat{D}$ into $\hat{M}$ and meets the following relation:

$$\langle \underline{Gm}, \underline{d} \rangle_D = \langle \underline{\hat{m}}, \underline{G}^T \underline{\hat{d}} \rangle_{\hat{M}}, \quad \underline{m} \in M, \underline{d} \in \underline{D}, \quad \underline{\hat{m}} \in \hat{M}, \quad \underline{\hat{d}} \in \hat{D}. \tag{1.7}$$

The introduction of the transposed operator will play an important role in the generalized inversion theory. It allows us to compare the inversion problem of functionals with that of discrete parameters in many cases.

If for $\underline{G} : M \to D$ there is an operator $\underline{G}^* : D \to M$ and it meets $\langle \underline{d}, \underline{Gm} \rangle_D = \langle \underline{G}^* \underline{d}, \underline{m} \rangle_M, \underline{d} \in D, \underline{m} \in M, \underline{G}^*$ is called the adjoint operator of $\underline{G}$.

If a symmetric linear and positive definite operator $\underline{C}_m : \hat{M} \to M$ exists, it is called the covariance operator of the model space; in the same way, $\underline{C}_d : \hat{D} \to D$ is called the covariance operator of the data space. Their inverse operators always exist. If the weight functions of M and $D, W_m(r,r')$ and $W_d(r,r')$, are just the kernels of $\underline{C}_m^{-1}$ and $\underline{C}_d^{-1}$ respectively, we call the scalar products defined in this way the natural scalar products [6]. Throughout this paper, we shall define the scalar products in this way. Thus, we have

$$\underline{\hat{m}} = \underline{C}_m^{-1} \underline{m}, \quad \underline{m} \in M, \quad \underline{\hat{m}} \in \hat{M}, \quad \underline{\hat{d}} = \underline{C}_d^{-1} \underline{d}, \quad \underline{d} \in D, \quad \underline{\hat{d}} \in \hat{D}. \tag{1.8}$$

Consequently, $\hat{M}$ and $\hat{D}$ can be regarded as the image space of $\underline{C}_m^{-1}$ and $\underline{C}_d^{-1}$.

Suppose the nonlinear operator $\underline{\phi} : M \to R$. The gradient operator of $\underline{\phi}$ is an operator, denoted by $\underline{\gamma}$, which maps M into itself, and the relation between $\underline{\gamma}$ and the Frechet differential operator of $\underline{\phi}$ is

$$\underline{\gamma} = \underline{C}_m \left(\frac{\partial \underline{\phi}}{\partial \underline{m}}\right)^T, \quad \underline{m} \in M. \tag{1.9}$$

The Hessian operator of $\underline{\phi}$ is $\underline{H} : M \to M$,

$$\underline{H}_0 \underline{m} = \left(\frac{\partial \underline{\gamma}}{\partial \underline{m}}\right)_{\underline{m}=\underline{m}_0}, \quad \underline{m}, \underline{m}_0 \in M. \tag{1.10}$$

2. Generalized inversion

1) Generalised least square criterion

With a nonlinear operator $\underline{f} : M \to D$, the forward problem can be written generally as

$$\underline{d} = \underline{f}(\underline{m}) \quad \underline{d} \in D, \quad \underline{m} \in M. \tag{1.11}$$

$\underline{d}_{\text{obs}} \in D$ represents the observed data and $\underline{C}_d$ is the covariance operator describing the measured errors. Because the inversion problem which we meet is essentially ill-posed, it is necessary to introduce a priori knowledge for models. Such a priori knowledge for models can be described by an a priori model $\underline{m}_p$ and a covariance operator $\underline{C}_m$ representing the discrepancy between $\underline{m}_p$ and the accetable solution [3], [6]. The inversion can be reduced to looking for an $\underline{m}_{est} \in M$, which meets (1.11) and makes minimum the objective functional

$$S(\underline{m}) = \frac{1}{2}\{\|\underline{f}(\underline{m}) - \underline{d}_{\text{obs}}\|_D^2 + \|\underline{m} - \underline{m}_p\|^2\}. \tag{1.12}$$

(1.12) can be rewritten as

$$S(\underline{m}) = \frac{1}{2}\{(\underline{f}(\underline{m}) - \underline{d}_{\text{obs}})^T \underline{C}_d^{-1} (\underline{f}(\underline{m}) - \underline{d}_{\text{obs}}) + (\underline{m} - \underline{m}_p)^T \underline{C}_m^{-1} (\underline{m} - \underline{m}_p)\}. \tag{1.13}$$

(1.13) implies that the solution we are looking for is the nearest to the a priori model $\underline{m}_p$ in the model space and its image in data space is the nearest to the observed data.

2) Optimal approach to inversion solution

From (1.13), the gradient operator $\underline{\gamma}$ and Hessian operator $\underline{H}$ of the objective functional $S(\underline{m})$ can be obtained:

$$\underline{\gamma}_k = \underline{C}_m \underline{G}_k^T \underline{C}_d^{-1} (\underline{f}(\underline{m}_k) - \underline{d}_{\text{obs}}) + (\underline{m}_k - \underline{m}_p), \tag{1.14}$$

where $\underline{G}_k$ is the Frechet differential operator of the operator $\underline{f}$ at $\underline{m}_k$;

$$\underline{H}_k = \underline{I} + \underline{C}_m \underline{G}_k^T \underline{C}_d^{-1} \underline{G}_k + \underline{C}_m \underline{J}_k^T \underline{C}_d^{-1} (\underline{f}(\underline{m}_k) - \underline{d}_{\text{obs}}), \tag{1.15}$$

where

$$\underline{J}_k = \left(\frac{\partial \underline{G}}{\partial \underline{m}}\right)_{\underline{m}=\underline{m}_k}.$$

If $\underline{f}$ is a linear operator, $\underline{J}_k = 0$, (1.15) can be rewritten as

$$\underline{H}_k = \underline{I} + \underline{C}_m \underline{G}_k^T \underline{C}_d^{-1} \underline{G}_k, \tag{1.16}$$

where $\underline{I}$ is the identity operator.

Thus, the inversion solution, which makes the objective functional $S(\underline{m})$ minimum, can be obtained by means of the gradient method,

$$\begin{cases} \underline{m}_{n+1} = \underline{m}_n + \alpha_n \Delta \underline{m}_n, \\ \Delta \underline{m}_0 = -\underline{\gamma}_0, \\ \Delta \underline{m}_n = -\underline{\gamma}_n + \dfrac{\langle \underline{\gamma}_n, \underline{\gamma}_n \rangle}{\langle \underline{\gamma}_{n-1}, \underline{\gamma}_{n-1} \rangle} \Delta \underline{m}_{n-1}, \\ \alpha_n = -\dfrac{\langle \Delta \underline{m}_n, \underline{\gamma}_n \rangle}{\langle \Delta \underline{m}_n, \underline{H}_n \Delta \underline{m}_n \rangle}, \\ \underline{H}_n = \underline{I} + \underline{C}_m \underline{G}_n^T \underline{C}_d^{-1} \underline{G}_n, \\ \underline{\gamma}_n = (\underline{m}_n - \underline{m}_p) - \underline{C}_m \underline{G}_n^T \underline{C}_d^{-1} (\underline{d}_{\text{obs}} - f(\underline{m}_n)). \end{cases} \tag{1.17}$$

Some techniques accelerating the convergence can be adopted in the iterative processes. $S(m)$ is of quadric form for linear problems; therefore, a minimum point will always exist. But the situation would be different for nonlinear problems. Under that condition, even if the iterative processes converge, it is necessary to justify that the solution obtained is what we are looking for.

§2. Application of the theory in the inverse scattering problem—simultaneous inversion of parameters describing the earth structures and sources

1. Forward problem

We start from the accoustic equation

$$\left[\frac{1}{k(\underline{r})}\frac{\partial^2}{\partial t^2} + \operatorname{div}\left(\frac{1}{\rho(\underline{r})}\operatorname{grad}\right)\right] U(\underline{r}, t) = S(\underline{r}, t), \tag{2.1}$$

where

$k(\underline{r})$: incompressible modulus, $\rho(\underline{r})$: density,
$S(\underline{r}, t)$: source function, $U(\underline{r}, t)$: displacement function .

When grad $\rho(\underline{r}) \approx 0$, we have

$$\left[\frac{1}{C^2(\underline{r})}\frac{\partial^2}{\partial t^2} - \nabla^2\right] U(\underline{r}, t) = S(\underline{r}, t) \tag{2.2}$$

where $C(\underline{r})$ is the velocity distribution function.

Considering a point source at $\underline{r} = \underline{r}_S$, the source function can be written as

$$S(\underline{r}, t; \underline{r}_s) = 8\pi^2 C^3 \delta(\underline{r} - \underline{r}_s) S(t) \tag{2.3}$$

where δ is the Dirac sign.

With a velocity perturbation close to a constant velocity C_0,

$$C(\underline{r}) = C_0 + \delta c(\underline{r}) \tag{2.4}$$

and $\|\delta c(\underline{r})\| << C_0$, the reflections generated by the velocity perturbation $\delta c(\underline{r})$ can be obtained by replacing (2.2) with (2.3), (2.4) and by using Born approximation [4], [7]

$$\delta \underline{u} = \underline{G}_c \delta \underline{c} + \underline{G}_s \delta \underline{s}. \tag{2.5}$$

The kernels of the operator $\underline{G}_c$ and $\underline{G}_s$ are respectively [7], [8]

$$G_c(\underline{r}_i, t; \underline{r}_s|\underline{r}) = \frac{1}{\|\underline{r} - \underline{r}_i\| \cdot \|\underline{r} - \underline{r}_s\|} S''(t - \frac{\|\underline{r} - \underline{r}_i\| + \|\underline{r} - \underline{r}_s\|}{C_0}), \tag{2.6}$$

$$G_s(r_i, t; r_s|t') = \int d\underline{r} \frac{\delta c(\underline{r})}{\|\underline{r} - \underline{r}_i\| \cdot \|\underline{r} - \underline{r}_s\|} \delta''(t - t' - \frac{\|\underline{r} - \underline{r}_i\| + \|\underline{r} - \underline{r}_s\|}{C_0}) \tag{2.7}$$

where $\underline{r}_i$ is the position vector of receivers. The kernels of corresponding transposed operators $\underline{G}_C^T$ and $\underline{G}_S^T$ are respectively

$$\underline{G}_C^T(\underline{r}'|\underline{r}_i; t; \underline{r}_s) = \frac{1}{\|r' - \underline{r}_i\| \cdot \|\underline{r}' - \underline{r}_s\|} S''(t - \frac{\|\underline{r}' - \underline{r}_i\| + \|\underline{r}' - \underline{r}_s\|}{C_0}, \tag{2.8}$$

$$\underline{G}_S^T(t'|\underline{r}_i, t; \underline{r}_s) = \int d\underline{r}' \frac{\delta c(\underline{r}')}{\|\underline{r}' - \underline{r}_i\| \cdot \|\underline{r}' - \underline{r}_s\|} \delta''(t' - (t - \frac{\|\underline{r}' - \underline{r}_i\| + \|\underline{r}' - \underline{r}_s\|}{C_0})). \tag{2.9}$$

With

$$\delta \underline{m} = \begin{pmatrix} \delta \underline{c} \\ \delta \underline{s} \end{pmatrix}, \tag{2.10}$$

$$\underline{C}_m = \begin{pmatrix} \underline{C}c & 0 \\ 0 & \underline{C}_s \end{pmatrix}, \tag{2.11}$$

$$\underline{G}_m = (\underline{G}_c, \underline{G}_s), \tag{2.12}$$

(2.5) can be rewritten in a more compact form

$$\delta \underline{u} = \underline{G}_m \delta \underline{m}. \tag{2.13}$$

(2.13) gives the solution of the forward problem.

2. Inverse problem

The objective functional is

$$S(\delta \underline{m}) = \frac{1}{2}\left\{(\delta \underline{u}^0 - \underline{G}_m \delta \underline{m})^T \underline{C}_u^{-1} (\delta \underline{u}^0 - \underline{G}_m \delta \underline{m}) + \delta_{\underline{m}}^T \underline{C}_m^{-1} \delta \underline{m}\right\} \tag{2.14}$$

where $\delta \underline{u}^0$ is the observed wave field, and $\underline{C}_u$ is the covariance operator of observed data. Using the results of §1 , the gradient and Hessian operators of $S(\delta \underline{m})$ are

$$\underline{\gamma} = \delta \underline{m} - \underline{C}_m \underline{G}_m^T \underline{C}_u^{-1} (\delta \underline{u}^0 - \underline{G}_m \delta \underline{m}), \tag{2.15}$$

$$\underline{H} = \underline{I} + \underline{C}_m \underline{G}_m^T \underline{C}_u^{-1} \underline{G}_m \tag{2.16}$$

respectively, where

$$\underline{H} = \begin{pmatrix} \underline{H}_{cc} & \underline{H}_{cs} \\ \underline{H}_{sc} & \underline{H}_{ss} \end{pmatrix}, \tag{2.17}$$

$$\underline{H}_{cc} = \underline{I} + \underline{C}_c \underline{G}_c^T \underline{C}_u^{-1} \underline{G}_c,$$

$$\begin{cases} \underline{H}_{cs} = \underline{C}_c \underline{G}_c^T \underline{C}_u^{-1} \underline{G}_s, \\ \underline{H}_{sc} = \underline{C}_s \underline{G}_s^T \underline{C}_u^{-1} \underline{G}_c, \\ \underline{H}_{ss} = \underline{I} + \underline{C}_s \underline{G}_s^T \underline{C}_u^{-1} \underline{G}_s. \end{cases} \tag{2.21}$$

It is easy to find that $\underline{H}_{cs}$ and $\underline{H}_{sc}$ are operators adjoining each other.

Let $\underline{\gamma} = 0$. We can obtain the solution of the inverse problem based on (2.15),

$$\delta \underline{m} = \underline{H}^{-1} \underline{C}_m \underline{G}_m^T \underline{C}_u^{-1} \delta \underline{u}^0 \tag{2.19}$$

where $\underline{H}^{-1}$ is the inverse of the Hessian operator. We have

$$\underline{H}^{-1} = \begin{pmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{pmatrix}, \tag{2.20}$$

$$\begin{cases} \underline{B}_{22} = (\underline{H}_{ss} - \underline{H}_{sc} \underline{H}_{cc}^{-1} \underline{H}_{cs})^{-1}, \\ \underline{B}_{12} = -\underline{H}_{cc}^{-1} \underline{H}_{cs} \underline{B}_{22}, \\ \underline{B}_{21} = -\underline{B}_{22} \underline{H}_{sc} \underline{H}_{cc}^{-1}, \\ \underline{B}_{11} = \underline{H}_{cc}^{-1} - \underline{B}_{12} \underline{H}_{sc} \underline{H}_{cc}^{-1}, \end{cases} \tag{2.21}$$

Thus far, our discussion has not involved the discreteness $\delta c(\underline{r})$ and $\delta s(t)$. It is difficult to use directly the form of the solution obtained above. In continuous inversion problems we will be faced with the calculation of the kernel of the inverse operator of $\underline{H}$, which is nearly impossible. Even if the parameter of the model is made discrete, the kernel of the operator $\underline{H}$ is a high dimensional matrix; the calculation of its inversion takes much computer time and occupyies much memory space. Therefore, it is appropriate to use the optimal method to obtain the iterative solution of the inversion problem. The application of the pre-conditioned conjugate gradient method can obviously accelerate the convergence rate [6], [7]. The formulas for iterations are

$$\begin{aligned} &\delta \underline{m}_{k+1} = \delta \underline{m}_k + \alpha_k \Delta \underline{m}_k, \quad \Delta \underline{m}_0 = -\underline{M} \underline{\gamma}_0, \\ &\Delta \underline{m}_k = -\underline{M} \underline{\gamma}_k + \frac{(\underline{M} \underline{\gamma}_k, \underline{\gamma}_k)_M}{(\underline{M} \underline{\gamma}_{k-1}, \underline{\gamma}_{k-1})_M}, \\ &\underline{\gamma}_0 = \delta \underline{m}_0 - \underline{C}_m \underline{G}_m^T \underline{C}_u^{-1} (\delta \underline{u}^0 - \underline{G}_m \delta \underline{m}_0), \\ &\underline{\gamma}_{k+1} = \underline{\gamma}_k + \alpha_k \underline{H} \Delta \underline{m}_k, \quad \underline{H} = \underline{I} + \underline{C}_m \underline{G}_m^T \underline{C}_u^{-1} \underline{G}_m, \quad \alpha_k = -\frac{(\Delta \underline{m}_k, \underline{\gamma}_k)_M}{(\Delta \underline{m}_k, \underline{H} \Delta \underline{m}_k)_M}, \end{aligned} \tag{2.22}$$

where $\underline{M}$ is the pre-conditional operator, the choice of which should be as near to $\underline{H}^{-1}$ as possible under possible conditions of calculation. For discrete model parameters we can take $\underline{M} = (\text{ diag } \underline{H})^{-1}$.

The problem treated in this paper involves two unknown functions $\delta c(\underline{r})$ and $\delta s(t)$. Therefore, it is suitable to use the relaxation technique in the calculation. In this technique, one parameter, for example $\delta s(t)$, is fixed. Then by applying the pre-conditioned conjngate gradient method, the optimal valuc of the other parameter is searched for. This process is repeated alternately until the satisfactory result is obtained.

3. A Numerical example

A numerical example has been made by applying the method mentioned above. The synthetic seismograms are calculated on the basis of formula (2.13). Then, these seismograms are regarded as the data used for inversion. A pulse function is chosen as an a priori source function.

Figs.1a, 1b and 1c show the real model, real source function and a priori source funciton respectively. Fig.1d shows the synthetic seismograms. Figs.1e, 1f, 1g, 1h, 1i and 1j are the results after the first, fourth and eighth iterations respectively. In every iteration, the source is iterated three times and the model is iterated six times. It can be seen that after four iterations the model and the source function are already quite similar to the real model and the real source function. This fact is partially due to the very short source function in our example.

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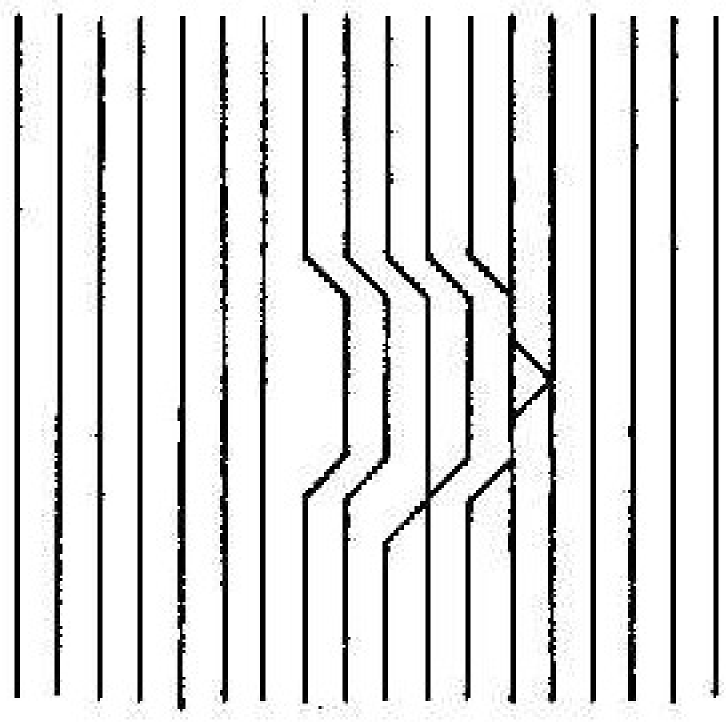


Fig. 1a

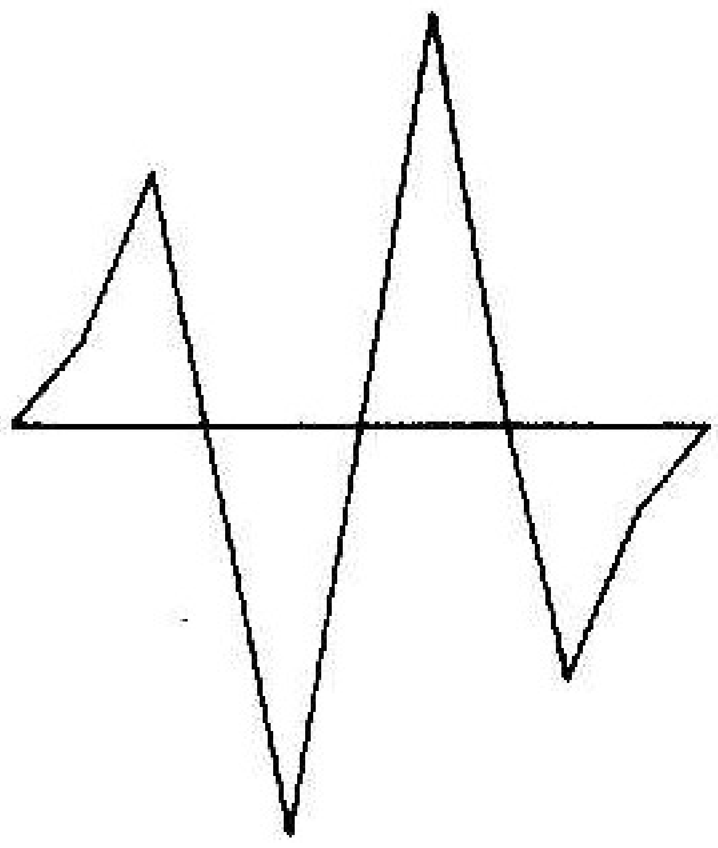


Fig. 1b

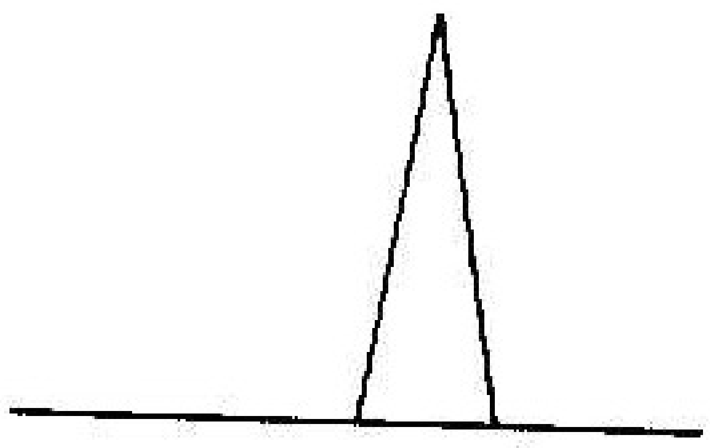


Fig. 1c

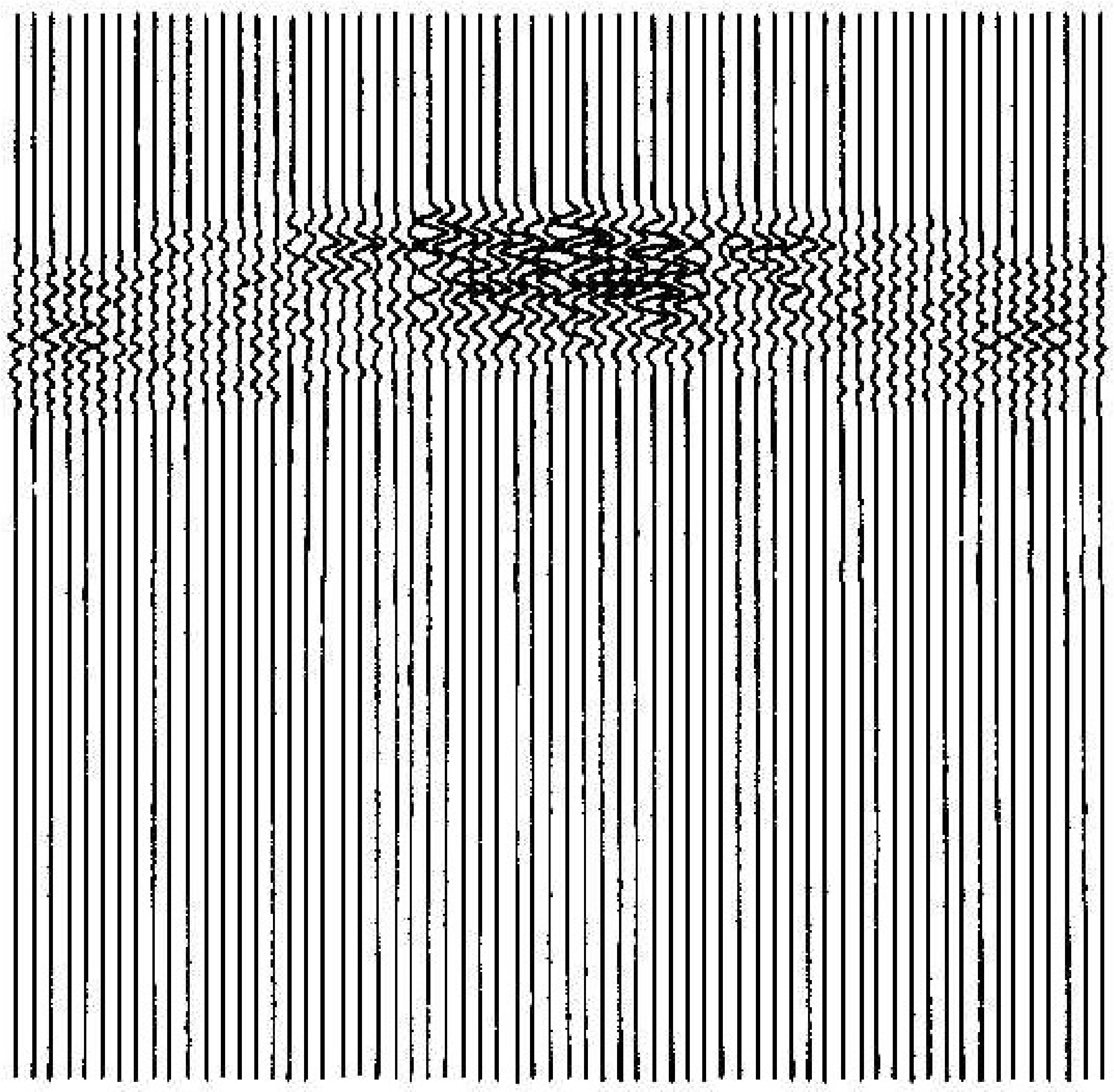
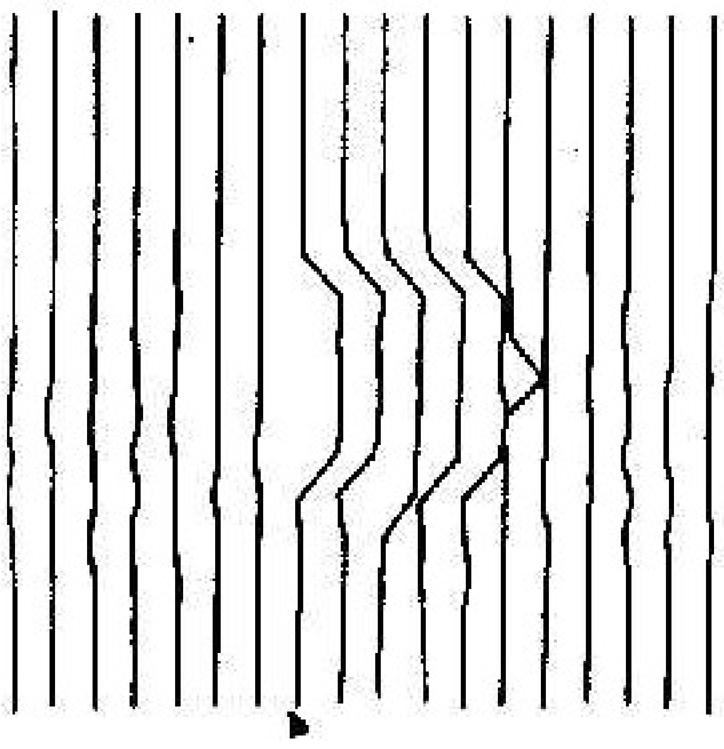


Fig. 1d



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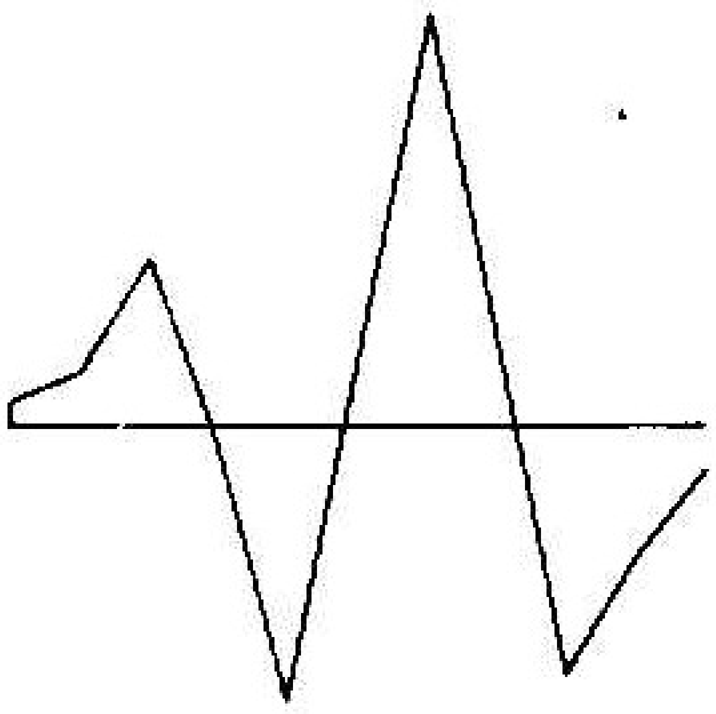

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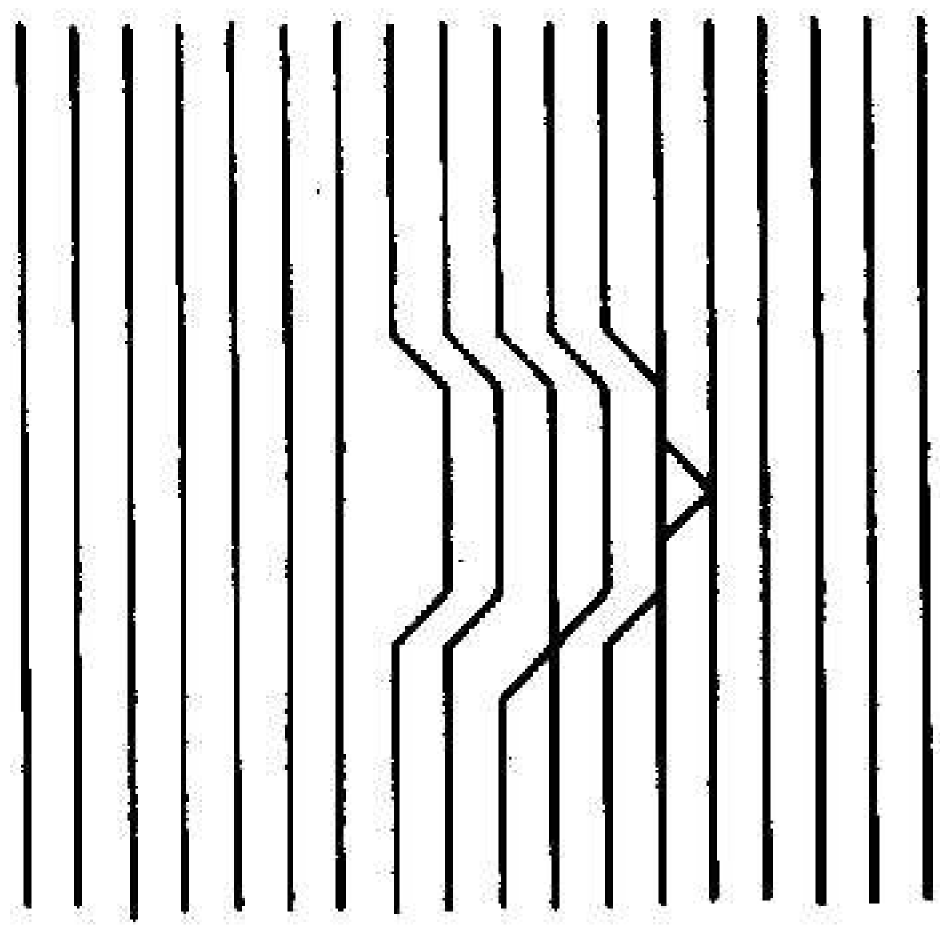
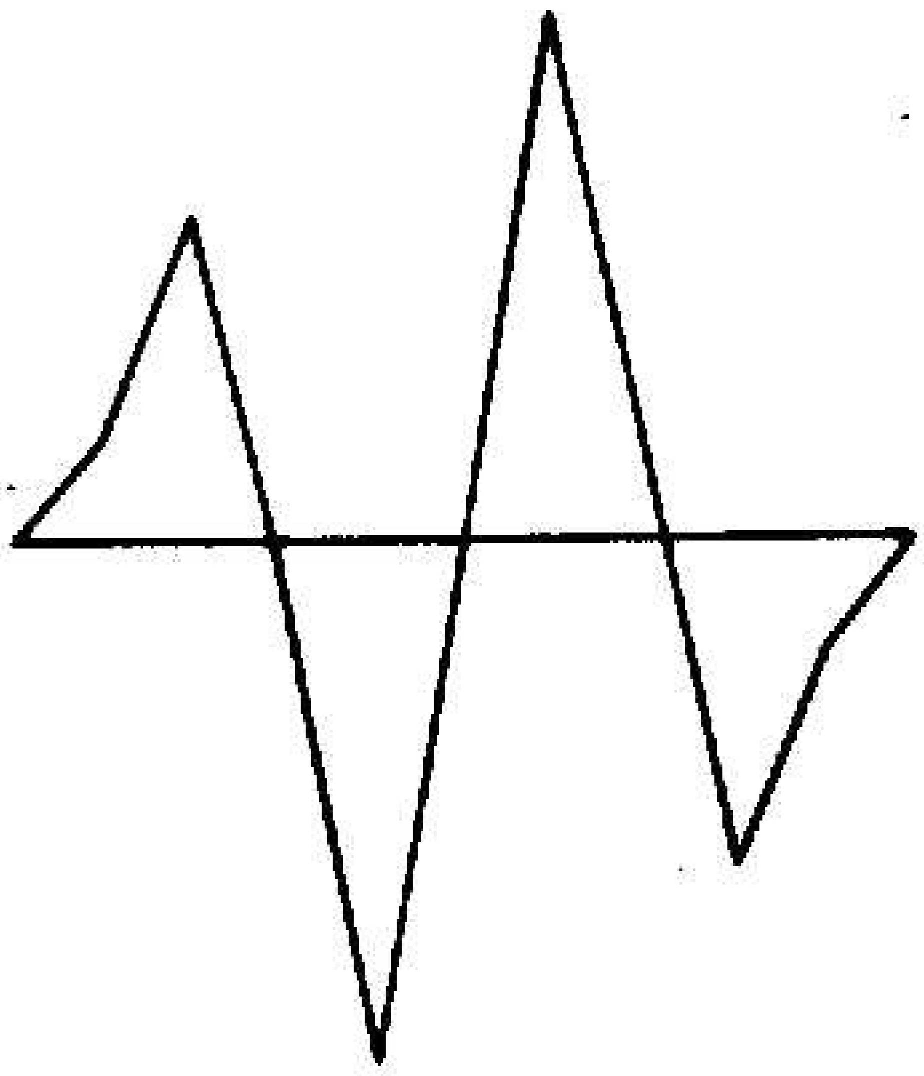


Fig. 1g



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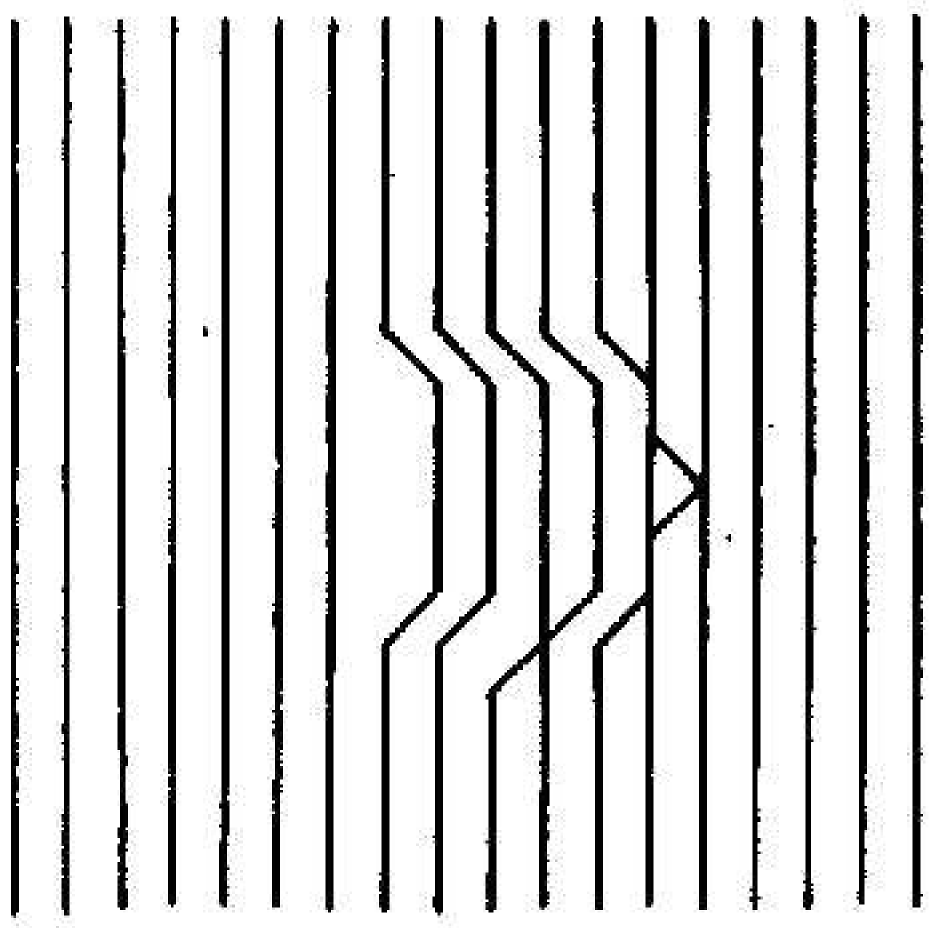

Fig. 1i

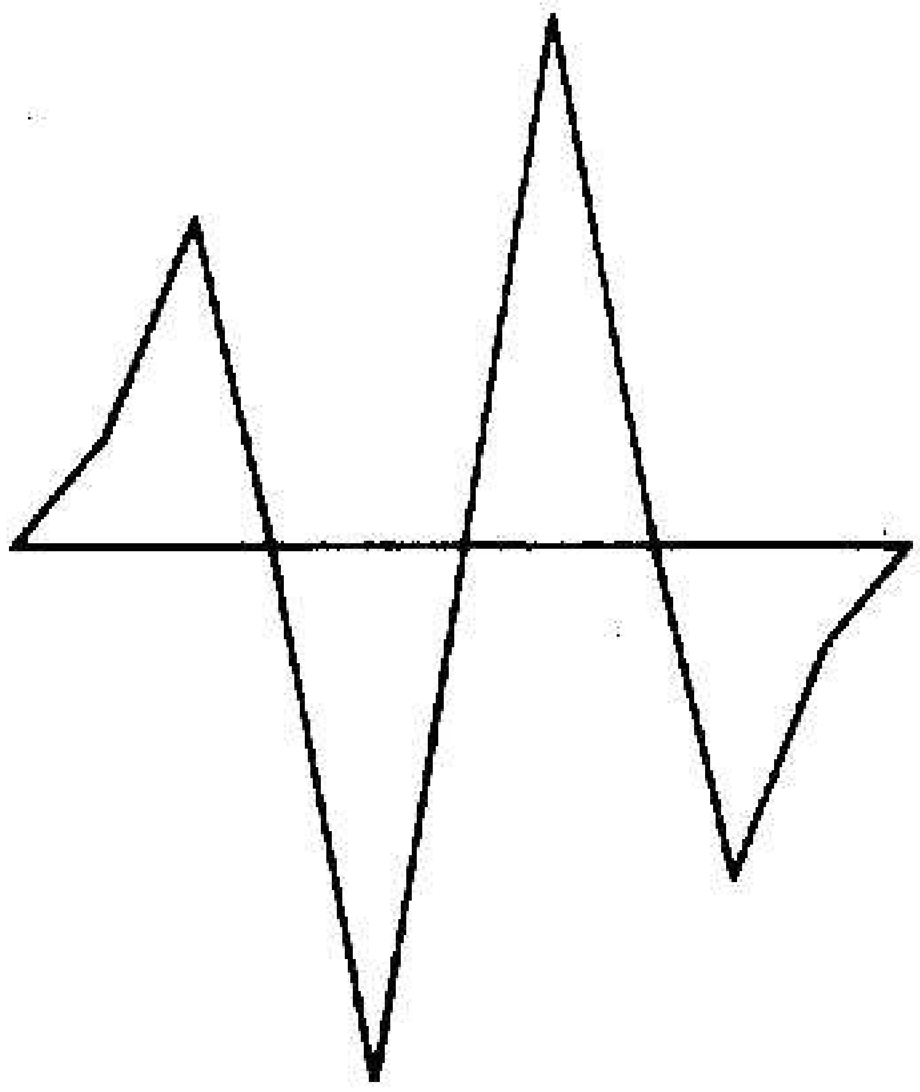

Fig. 1j