THE EVOLUTION OF INITIAL SMALL DISTURBANCE IN DISCRETE COMPUTATION OF CONTOUR DYNAMICS*

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Abstract

In this paper, we mainly discuss the elvolution of initial small disturbance in discrete computation of the contour dynamics method. For one class of smooth contour, we prove the stability of evolution of initial small disturbance based on the analysis of the convergence of the contour dynamics method with Euler's explicit method in time. Namely, at terminal time T, the evolving disturbance is going to zero as initial small disturbance goes to zero. The numerical experiment on the stability of contour dynamics has been given in [5, 6].

§1. Introduction

It is well-known that vortices play a very powerful role in nature. A description of the study on the vortical phenomena is given in detail by H. J. Lugt[1]. But it is not enough for humankind to understand the vortices, and to make use of vortex flows. As the mystery of vortical motion has not been pictured clearly, much work has been done by experiment to simulate the vortical motion. In general, it needs both much time and high cost to complete the experiment. Among the numerous simulations for vortex flows [2], N. J. Zabusky's work for simulating the evolution of piecewise constant vorticity areas in two dimensions for inviscid incompressible flows is most fascinating not only in numerical methods but also in mathematics [3]. Here we discuss the stability of his method in some sense for a class of physical models.

This method, contour dynamics method, is applied to finite area vortex regions (FAVR'S) of piecewise-constant-vorticity for the Euler equation in two-dimensional inviscid incompressible flows.

The incompressible, inviscid Navier-Stokes equation in two dimensions is

$$\omega_t + u\omega_x + v\omega_y = 0, \tag{1}$$

$$\psi_{xx} + \psi_{yy} = -\omega, \tag{2}$$

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and

$$u = \psi_y, \qquad v = -\psi_x, \tag{3}$$

$$\omega = -u_y + v_x \tag{4}$$

where ω is vorticity. Let

$$K_{z'}(z) = \frac{1}{\pi}\begin{bmatrix} -\frac{y-\eta}{r^2} \\ \frac{x-\xi}{r^2} \end{bmatrix} = K(z - z'), \tag{5}$$

$$z = (x, y), \qquad z' = (\xi, \eta), \tag{6}$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2, \tag{7}$$

$$U = (u, v). \tag{8}$$

The velocity can be denoted in terms of vorticity:

$$U(z) = K \times \omega(z) = \iint_{R^2} K(z - z')\omega(z')dz'. \tag{9}$$

For incompressible inviscid flows, Kelvin's theorem enssures that the vorticity is constant along the path of the fluid particle. So we can mark the trace of the fluid particle for simulating the vortex flows:

$$\frac{dz}{dt} = K \times \omega(z). \tag{10}$$

In [3], N.J. Zabusky proposed the contour dynamics method in which the vorticity $\omega(z)$ is approximated by piecewise constant vorticity areas $\bar{\omega}(z)$ with polygonal boundaries. For convenience, we only consider the single constant vorticity area. Chosen N-fluid-particles on the contour are connected by a closed broken line; hence, an N-polygonal-boundary constant vorticity area is constructed. So we can follow the motion of these N-particles to simulate the motion of the contour. The semi-discrete equations of motion of the fluid particles are the following:

$$\frac{d\bar{z}_j}{dt} = K \times \bar{\omega} = K(\bar{z}_j) \times \bar{\omega}(\bar{z}_j), \tag{11}$$

$$j = 1, 2, \cdots, N.$$

Denote

$$H = \max_j |z_{j+1} - z_j|, \tag{12}$$

$$h = \min_j |z_{j+1} - z_j|, \tag{13}$$

$$z_{j+N} = z_j$$

and

$$H/h \le M_1 \tag{14}$$

where M_1 is a positive constant.

In this paper, we consider the smooth contour

$$(*) \qquad \Gamma : z = Z(s(t), t), \tag{15}$$

$$s(t) = S(t, s_0) \tag{16}$$

and

$$0 \le s(t) \le S(t), \qquad 0 \le t \le T. \tag{17}$$

$S(t, s_0)$ is the arc parameter of the contour at time t, s_0 is the arc parameter of the initial contour. We assume that the contour has the following properties.

1. The contour is a measurable simple closed curve.
2. $Z(s,t)$ has continuous derivatives on s up to third order, and

$$|Z_s^{(\alpha)}(s,t)| \le M_2, \qquad \alpha = 0,1,2,3 \tag{18}$$

where constant M_2 is independent of t.

3. $z\ (s,t)$ is continuously differentiable on t.

We know that the rotating elliptic invariant model with the following elliptical contour

$$\begin{cases} x = \frac{1}{2}k(a+b)\cos(2nt+s) + \frac{1}{2}k(a-b)\cos s, \\ y = \frac{1}{2}k(a+b)\sin(2nt+s) - \frac{1}{2}k(a-b)\sin s \end{cases}$$

has these propertities [7], where $n = \frac{ab}{(a+b)^2}\omega$, and k is some constant.

For $r = 2\cos(\theta_0 - \phi)$, we can get:

$$|r_1 - r_2| \ge \frac{1}{\sqrt[4]{a^4+b^4}} |\sin(\frac{\phi_1+\phi_2}{2} - \theta_0|/|\cos\frac{\phi_1+\phi_2}{2}| \cdot |p_1 - p_2| \tag{19}$$

where

$$p_1 = (r_1, \phi_1), \quad p_2 = (r_2, \phi_2), \tag{20}$$

$$r_1 = 2\cos(\theta_0 - \phi_1), \quad r_2 = 2\sin(\theta_0 - \phi_2). \tag{21}$$

This property is an important condition in [4] for the proof of convergence of the contour dynamics method with Euler's explicit method. There is also a good example analyzed in [8].

§2. The Continuous Case

From now on, we consider the regions of piecewise constant vorticity $\omega(z,t)$ with FAVR's, which are contained in a ball

$$B_R(0) = \{z \mid |z| \le R\}. \tag{22}$$

For the initial value problem:

$$\begin{cases} \frac{dZ}{dt} = K \times \omega(Z,t), \\ Z|_{t=0} = Z_0 \end{cases} \tag{23}$$

we study its initial small disturbance problem:

$$\begin{cases} \frac{d\bar{Z}}{dt} = K \times \omega(\bar{Z},t), \\ \bar{Z}|_{t=0} = \bar{Z}_0. \end{cases} \tag{24}$$

We get the following conclusions as paper [6] does.

Lemma 1. *Suppose $Z(s,t)$ is the smooth contour of equation (23), and $\bar{Z}(s,t)$ the solution of equation (24). Then, there exists t_1, such that*

$$|Z(s,t) - \bar{z}(s,t)| \leq R/e, \quad \text{for } 0 \leq t \leq t_1. \tag{25}$$

Lemma 2. *Let $\delta(t) = \max\limits_s |Z(s,t) - \bar{Z}(s,t)|, \delta_0 = \delta(0)$. Then we have got the estimate of error $\delta(t)$:*

$$\delta(t) \leq \delta_0 + \int_0^t \delta(\tau)(\eta_1 + \eta_2 \ln \frac{R}{\delta(\tau)})d\tau, \quad 0 \leq t \leq t_1 \tag{26}$$

where η_1, η_2 are positive constants independent of t.

Lemma 3. *Suppose $\delta(t)$ is a nonnegative continuous function, $\lambda(t)$ a positive continuous function, η, k, m, R_1, R are nonnegative constants, and*

$$k < R, \quad R_1 < R, \qquad \delta(t) \leq R_1.$$

If

$$\delta(t) \leq k + m \int_{t_0}^t \lambda(\tau)(1 + \eta \ln \frac{ek}{\delta(\tau)})d\tau, \quad t_0 \leq t \leq t_1 \tag{27}$$

then there exists $t_2 \leq t_1$, for $t_0 \leq t \leq t_2$,

$$\delta(t) \leq R \exp\left\{1 + \frac{1}{\eta}[1 - \exp(-m \int_{t_0}^t \lambda(\tau)d\tau)]\right\}[\frac{k}{eR}]^{\exp(-m \int_{t_0}^t \lambda(\tau)d\tau)}. \tag{28}$$

Theorem 4. *Let the initial small disturbance of (24) be so small that*

$$|Z(s,t) - \bar{Z}(s,t)| \leq R/e \tag{29}$$

for $0 \leq t \leq t_1$. We have the expression

$$\delta(t) \leq R \exp\left[(\eta_1/\eta_2)(1 - e^t)\right] \left(\frac{\delta_0}{R}\right)^{e^{-t}}, \quad \forall \ 0 \leq t \leq t_2 \leq t_1 \leq T, \tag{30}$$

where t_2 is the maximum value of t which satisfies the following inequality:

$$\delta_0 + \int_0^t \delta(t)(\eta_1 + \eta_2 \ln \frac{R}{\delta(\tau)})d\tau \leq R/e \tag{31}$$

and the smaller the initial disturbance δ_0 is, the larger the terminal time t_2 is.

§3. The Discrete Case

We solve equation (23) or (24) with Euler's one step method:

$$\bar{Z}_j^{n+1} = \bar{Z}_j^n + \Delta t K \times \bar{\omega}^n, j = 1,2,\cdots,N; \quad n = 0,1,2,\cdots,M. \tag{32}$$

Let

$$f_1 = \Delta t g_0 + \left[1 + c_1\Delta t\right]\varepsilon + c_2\Delta t\left[\varepsilon \ln \tfrac{1}{\varepsilon}\right],$$

$$g_n = (D_1 + D_2 \ln \frac{1}{\Delta t})\Delta t + (D_3 + D_4 \ln \frac{1}{H_n})H_n,$$

$$H_n = \max_j |\bar{Z}_{j+1}^n - \bar{Z}_j^n|,$$

$$n = 0,1,2,\cdots,M$$

and

$$g_T = (D_1 + D_2 \ln \frac{1}{\Delta t})\Delta t + (D_5 + D_6 \ln \frac{1}{H_0})H_0^{e^{-K_2 T}},$$

$$r(g_T) = T\Omega g_T^{1-c_3 T}.$$

As the paper [4] did, the estimation of errors at n-th step is defined by induction:

$$f_n \geq \max_j |Z(s_j, n\Delta t) - \bar{Z}_j^n|. \tag{33}$$

Theorem 5. *For smooth contour $Z(s,t), 0 \leq t \leq T$, if the initial disturbance is so small that*

$$r(g_T) + r(\phi_\varepsilon) \leq 1/e, \tag{34}$$

then when $\Delta t, H_0$ are sufficiently small, the evolution of initial disturbance is dependent on both initial disturbance and discrete errors:

$$f_n \leq r(g_T) + r(\phi_\varepsilon), \tag{35}$$

where

$$\phi_\varepsilon = \left[1 + c_1\Delta t\right]\varepsilon + c_2\Delta t\varepsilon \ln \frac{1}{\varepsilon},$$

and ε is the maximum absolute value of the initial small disturbance.

In [4], we proved the convergence of the contour dynamics method. Therefore, when Δt and H_0 go to zero, the discrete solution of equation

$$\begin{cases} \bar{Z}_j^{n+1} = \bar{Z}_j^n + \Delta t K \times \bar{\omega}^n, \\ \bar{Z}_j^0 = Z(s_j, o), \quad j = 1,2,\cdots,M \end{cases} \tag{36}$$

is convergent.

Now we consider the evolution of the initial error in the following Euler's explicit method

$$\begin{cases} \hat{Z}_j^{n+1} = \hat{Z}_j^n + \Delta t K(\hat{Z}_j^n) \times \hat{\omega}^n, \\ \hat{Z}_j^0 = Z(s_j, 0), \\ j = 1, 2, \cdots, N. \end{cases} \tag{37}$$

Consider the scheme with initial errors $\{\varepsilon_j\}$:

$$\begin{cases} \tilde{Z}_j^{n+1} = \tilde{Z}_j^n + \Delta t K(\tilde{Z}_j^n) \times \tilde{\omega}^n, \\ \tilde{Z}_j^0 = Z(s_j, 0) + \varepsilon_j, \\ j = 1, 2, \cdots, N. \end{cases} \tag{38}$$

We know that when $\Delta t, H_0, \varepsilon$ are sufficiently small, the solutions of both schemes (37), and (38) are very close to the solution of equation (10). Hence we get the following result:

Theorem 6. *Let*

$$\bar{f}_n \geq \max_j |\hat{Z}_j^n - \tilde{Z}_j^n|, \tag{39}$$

which is the upper estimate of errors $|\hat{Z}_j^n - \tilde{Z}_j^n|$ *at n-th step by inductive definition So we can get the equation for* $\bar{f}_n$*:*

$$\begin{aligned} &\bar{f}_{n+1} = \bar{f}_n + \Delta t(\eta_1 + \eta_2 \ln \frac{1}{\bar{f}_n})\bar{f}_n, \\ &\bar{f}_0 = \varepsilon, \quad n = 0, 1, 2, \cdots, M. \end{aligned} \tag{40}$$

Then

$$\bar{f}_n \leq \Omega\Big(\frac{1}{\psi(\varepsilon)}\Big)^{\eta_2 T}(\varepsilon + \psi(\varepsilon)) \tag{41}$$

where

$$\begin{aligned} &\Omega = \exp(T\eta_1 + T\eta_2|1 - \ln T|), \\ &\psi(\varepsilon) = \varepsilon(\eta_1 + \eta_2 \ln \tfrac{1}{\varepsilon}), \\ &\varepsilon = \max_j |\varepsilon_j|, \\ &\eta_2 T < 1. \end{aligned}$$

Example. For a rotating circle invariant contour (V-State problem), N fluid particles are chosen on the circle

$$x^2 + y^2 = 1.$$

Put

$$\begin{cases} x_j = \cos(j-1)\theta, \\ y_j = \sin(j-1)\theta, \quad j = 1, 2, \cdots, N, \\ \theta = 2\pi/N. \end{cases}$$

We solve the equation:

$$\begin{cases} \dfrac{d\bar{Z}_j}{dt} = K(\bar{Z}_j) \times \bar{\omega}, \\ \bar{Z}_j|_{t=0} = Z(s_j, 0), \\ \qquad j = 1, 2, \cdots, N \end{cases}$$

with Euler's explicit method:

$$\begin{cases} \bar{Z}_j^{n+1} = \bar{Z}_j^n + \Delta t K(\bar{Z}_j^n) \times \omega^n, \\ \bar{Z}_j^0 = Z(s_j, 0). \end{cases}$$

We get the discrete solution as follows:

$$\begin{cases} x_j^{n+1} = R^{n+1} \cos \theta_j^{n+1}, \\ y_j^{n+1} = R^{n+1} \sin \theta_j^{n+1} \end{cases}$$

where

$$\theta_j^{n+1} = \theta_j^n - \Delta\theta^*,$$
$$\theta_j^0 = (j-1)\theta,$$
$$\sin \Delta\theta^* = c\Delta t / \sqrt{1 + c^2 \Delta t^2},$$
$$R^{n+1} = R^n \sqrt{1 + c^2 \Delta t^2},$$
$$\rho_j = \sqrt{2}\left[\sqrt{1 - \cos j\theta} + \sqrt{1 - \cos(j-1)\theta}\right],$$

$$c = (2\pi)^{-1} \sum_{j=1}^N \frac{2[\cos j\theta - \cos(j-1)\theta] \cdot \frac{1}{2}[\sin j\theta + \sin(j+1)\theta]}{\rho_j \{1 + \frac{1}{2}(1 + \cos\theta) - (\cos j\theta + \cos(j-1)\theta)\}^{1/2}}.$$

So, for the circle contour, the method is convergent and stable.

References

[1] H.J.Lugt, Vortex Flows in Nature and Technology, John Wiley and Sons, 1983.

[2] A.Leonard, Vortex methods for flow simulation, *J.Compt. Phys.*, 37 (1980), 289–335.

[3] N.J.Zabusky, M.H.Hughes, K. V.Roberts, Contour dynamics for Euler equation in two dimensions, *J. Compt. Phys.*, **30** : 1 (1979), 96–106.

[4] WuYu-hua, Wu Hua-mo, The convergence of contour dynamics method (to appear in *JCM*).

[5] E.A.Overman II, N. J. Zabusky, *Phys. Fluids*, 25 (1982), 1291–1305.

[6] Wu Yu-nua, Contour dynamics method (M.S. thesis of Computing Center of Academia Sinica).

[7] H.Lamb, Hydrodynamics, 6th ed. Cambridge University press, Cambridge, 1932.

[8] L.M. Polvani, G.R.Flierl, Generalized Kirchhoff vortices, *Phys. Fluid*, 29 (1986), 2376–2379.