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# A NUMERICAL METHOD OF THE RAMM INTEGRAL EQUATION *

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**Abstract**

A numerical method for solving the ill-posed Ramm integral equation is presented in this paper. It is found that the method is stable and more accurate. Particularly, when the given data is contaminated by noise, satisfactory results are obtained by using the algorithm of this paper.

## §1. Introduction

In this paper we consider the numerical solution of the following Fredholm first kind integral equation which was derived by A.G.Ramm in [1], [2]:

$$\int_B \frac{V(z)}{|x'-z|\,|y'-z|}dz = f(x',y') \tag{1.1}$$

where

$$B=\Big\{z\big|z\in R^3,|z-z_0|\le d,z_0=(0,0,-a),a>d\Big\},$$

$$P=\Big\{z\big|z=(z_1,z_2,z_3)\in R^3,z_3=0\Big\},$$

$$x'=(x_1,x_2)\in P,\quad y'=(y_1,y_2)\in P,$$

$f(x',y')$ is the datum which can be measured, $V(z)$ is the wanted solution,

$$K(z;x',y')=\frac{1}{|x'-z|\,|y'-z|}$$

is the kernel function of (1.1). Clearly, $K(z;x',y')\in C^\infty(B)$, so equation (1.1) is ill-posed. We will present an efficient numerical method for solving (1.1) and give satisfactory computing results.

## §2. The Numerical Method

2.1. Super-Isoparametric Finite Element [4] Discretization of (1.1)

Now we present the super-isoparametric finite element discretization scheme for the numerical solution of equation (1.1). The method is as follows.

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Subdivide $B$ into a set of nonoverlapping hexahedron $e_i, i = 1, 2, \cdots, p$, i.e. $B = \bigcup_{i=1}^{p} e_i, e_i \bigcap e_j = \phi, i \neq j$. The subdivision points are $z_1, z_2, \cdots, z_n$. And get a parametric mapping for each element

$$e_i \longleftrightarrow \hat{e}. \tag{2.1}$$

$$z_1 = f_1(\xi, \eta, \varsigma), \quad z_2 = f_2(\xi, \eta, \varsigma), \quad z_3 = f_3(\xi, \eta, \varsigma), \tag{2.2}$$

where $\hat{e}$ is the standard element–cube, and $f_1, f_2, f_3$ are polynomials. We find that, for the ill-posed problem, a very wonderful form of element mapping is the so-called super-isoparametric element in which the order of the mapping functions $f_i, i = 1, 2, 3$, is greater than that of the shape functions in $\hat{e}$. Here, we use 20-node two order serendiping-type interpolation for coordinates $(z_1, z_2, z_3)$, and use 8-node trilinear interpolation for the unknown function $V$. If $\phi_j^e(\xi, \eta, \varsigma)$ represents a standard type of the finite element shape function for a 20–node element in the local domain [4], we can write the mapping relationship of (2.2) for each element as

$$z_1 = \sum_{j=1}^{20} z_{1j}\phi_j^e, \tag{2.3}$$

$$z_2 = \sum_{j=1}^{20} z_{2j}\phi_j^e, \tag{2.4}$$

$$z_3 = \sum_{j=1}^{20} z_{3j}\phi_j^e, \tag{2.5}$$

in which $\phi_j^e = \phi_j^e(\xi, \eta, \varsigma)$.

Let $\psi_j^e(\xi, \eta, \varsigma), j = 1, 2, \cdots, 8$, be a system of trilinear interpolation base function[4] in the local domain. Then the finite element approximate solution of $V$ in $\hat{e}$ can be written in the following form

$$V^h = \sum_{j=1}^{8} V_j \Psi_j^e(\xi, \eta, \varsigma). \tag{2.6}$$

For given $x_i' = (x_{1i}, x_{2i}), y_i' = (y_{1i}, y_{2i})$, we have

$$\int_B K(z; x_i', y_i')V(z)dz = \sum_{l=1}^{p} \int_{e_l} K(z; x_i', y_i')V(z)dz. \tag{2.7}$$

In any element $e_l$,

$$\begin{aligned} \int_{e_l} K(z; x_i', y_i')V(z)dz &\sim \int_{\hat{e}} K(\xi, \eta, \varsigma; x_i', y_i') \sum_j V_j \Psi_j^e |J| d\xi d\eta d\varsigma \\ &= \sum_j \Big( \int_{\hat{e}} K(\xi, \eta, \varsigma; x_i', y_i') \Psi_j^e |J| d\xi d\eta d\varsigma \Big) V_j = \sum_j a_{ij}^{e_l} V_j \end{aligned} \tag{2.8}$$

where $j$ is the number of net points for element $e_l$, $J$ is the Jacobi matrix of mapping (2.3)–(2.5), and

$$a_{ij}^{e_l} = \int_e K(\xi, \eta, \varsigma; x_i', y_i') \Psi_j^e |J| d\xi d\eta d\varsigma \tag{2.9}$$

in which

$$K(\xi, \eta, \varsigma; x_i', y_i') = \left\{ \left[ (x_{1i} - \sum_j x_{1j} \phi_j^e)^2 + (x_{2i} - \sum_j x_{2j} \phi_j^e)^2 + (\sum_j z_{3j} \phi_j^e)^2 \right]^{1/2} \right.$$
$$\left. \times \left[ (y_{1i} - \sum_j x_{1j} \phi_j^e)^2 + (y_{2i} - \sum_j x_{2j} \phi_j^e)^2 + (\sum_j z_{3j} \phi_j^e)^2 \right]^{1/2} \right\}^{-1}. \tag{2.10}$$

The $a_{ij}^{e_l}$ is computed by using the Gauss numerical integration formula.

Properly choose $(x_i', y_i'), i = 1, 2, \cdots m$, substitute (2.8) into (2.7) and add the coefficient $a_{ij}$ to the basis of the number of net points, i.e.

$$a_{ij}^{e_l} + a_{ij}^{e_{l'}} \longrightarrow a_{ij}. \tag{2.11}$$

Now (1.1) can be written as an overdetermined linear algebraic system:

$$AV = F \tag{2.12}$$

where

$$V = (V_1, V_2, \cdots, V_n), \qquad V_i = V(z_i),$$
$$F = (f_1, f_2, \cdots, f_m), \qquad f_i = f(x_i', y_i'),$$

and $A = (a_{ij})$ is an $m \times n$ matrix with elements $a_{ij}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$.

2.2. Regularisation of the Discrete Problem

The discrete problem (2.12), related to the integral equation (1.1), remains "ill-conditioned". It follows that a small random variation of the data $f(x_i', y_i')$, or equivalently a rounding error in the computation of the matrix elements, may induce an unacceptable perturbation on the computed solution of the overdetermined linear system.

To avoid these difficulties, we have adopted a classical Tikhonov regularisation procedure [3]. In practice, we try to compute the regularised solution $V_\alpha$ of (2.12), which minimises the following functional

$$\|AV - F\|_2^2 + \alpha \Omega^2(V), \tag{2.13}$$

and is near to the exact solution $V$ in the sense of the $L_2$ norm. If $\Omega^2$ is appropriately chosen, the second term has a "smoothing" or stabilizing effect on the regularised solution. We take the smoothing functional $\Omega^2$ as

$$\Omega^2(V) = \|LV\|_2^2, \tag{2.14}$$

where $L$ denotes some matrix, normally a discrete approximation to some derivative operator. And $H = L^T L$ is the symmetric positive definite matrix. Especially, we can get $H = I$.

We assume that the right-hand side of (2.12) is affected by a random error $\varepsilon$, such that

$$\|\varepsilon\|_2^2 \leq \delta^2$$

for a given $\delta^2$. From now on, we consider $F = F_e + \varepsilon$, where $F_e$ is the exact (but unknown) right-hand side. Due to the error, we have in practice

$$\|AV - F\|_2^2 = \|\varepsilon\|_2^2 \leq \delta^2. \tag{2.15}$$

Obviously, the minimising problem (2.13) is equivalent to computing the quasi-solution of the following equations ,

$$\begin{pmatrix} A \\ \alpha L \end{pmatrix} V = \begin{pmatrix} F \\ 0 \end{pmatrix}. \tag{2.16}$$

We use the orthogonalising method and properly choose the regularization parameter $\alpha$ for solving (2.16). Thus we obtain satisfactory results.

## §3. Numerical Results

In order to assess the numerical performance of the method outlined in the previous section, we have carried out some numerical experiments. The computer program was structured as follows. First let

$$Q(V_\alpha, \alpha) = \|AV_\alpha - F\|_2^2 - \delta^2. \tag{3.1}$$

It follows that we can determine a value $\bar{\alpha} > 0$ such that $Q(V_{\bar{\alpha}}, \bar{\alpha}) > 0$. For example, as $\delta = 0.1$, we can take $\bar{\alpha} = 0.2$. Therefore, a better value of the regularization parameter can be obtained using s simple optimum seeking method [5]. For given $\alpha \in (0, \bar{\alpha})$, the linear system (2.16) is solved using an orthogonalizing algorithm. Once the solution $V_\alpha$ corresponding to the current value of $\alpha$ is known, the function $Q(V_\alpha, \alpha)$ is evaluated. If $Q(V_\alpha, \alpha) > 0$, we determine a new $\alpha$ in $(0, \bar{\alpha})$ by using the optimum seeking method and solve again the system (2.16).

Repeat the above-mentioned procedure until $Q(V_\alpha, \alpha) < 0$ is obtained. Then $V_\alpha$ is the wanted approximate solution.

In all the numerical experiments reported in the following, we assume in (1.1):

$$B = \{z | z \in R^3, |z - z_0| \leq \frac{1}{2}, z_0 = (0, 0, -2)\}, \tag{3.2}$$

and subdivide $B$ into seven nonoverlapping hexahedral elements, with a total of 16 discrete points: $z_1, z_2, \cdots, z_{16}$. The partial numerical results are reported in Tables 1 and 2. Notice that in Table 1 the random error $\varepsilon = 0$, and in Table 2 $\varepsilon = F_e \varepsilon^* \sum_{i=1}^{6} (\nu_{n+2i} - \nu_{n+2i-1})$ in which $\varepsilon^*$ is noise strength, $i = 1, 2, \cdots$ are the well distributed stochastic digits.

**Table 1**

| $x_j$ | Exact Solution | Computed Solution $\alpha = 0.005$ | Error | Computed Solution $\alpha = 0.0025$ | Error |
|---|---|---|---|---|---|
| 1 | 1.000000 | 0.993638 | 0.006362 | 0.961433 | 0.038567 |
| 2 | 1.000000 | 1.018251 | 0.018251 | 1.052691 | 0.052691 |
| 3 | 1.000000 | 0.992883 | 0.007117 | 1.050396 | 0.050396 |
| 4 | 1.000000 | 0.968707 | 0.031293 | 0.960768 | 0.039232 |
| 5 | 1.000000 | 0.837799 | 0.162201 | 0.813886 | 0.186114 |
| 6 | 1.000000 | 0.856785 | 0.143215 | 0.880029 | 0.119971 |
| 7 | 1.000000 | 0.842085 | 0.157915 | 0.877937 | 0.122063 |
| 8 | 1.000000 | 0.910660 | 0.089340 | 0.905877 | 0.094123 |
| 9 | I.000000 | 1.005792 | 0.005792 | 0.966629 | 0.033371 |
| 10 | 1.000000 | 1.044140 | 0.044140 | 1.052121 | 0.052121 |
| 11 | 1.000000 | 1.019308 | 0.019308 | 1.065685 | 0.065685 |
| 12 | 1.000000 | 0.998377 | 0.001623 | 0.978395 | 0.021605 |
| 13 | 1.000000 | 1.204432 | 0.204432 | 1.145275 | 0.145275 |
| 14 | 1.000000 | 1.158520 | 0.158520 | 1.215026 | 0.215026 |
| 15 | 1.000000 | 1.056508 | 0.056508 | 1.068823 | 0.068823 |
| 16 | 1.000000 | 0.960621 | 0.039379 | 0.920464 | 0.079536 |
| $\|AV_\alpha - F\|$ | | $0.14460103 \times 10^{-2}$ | | $0.78487745 \times 10^{-3}$ | |

**Table 2**

| $x_j$ | Exact Solution | Computed Solution $\varepsilon^* = 0.1, \alpha = 0.01$ | Error | Computed Solution $\varepsilon^* = 0.8, \alpha = 0.01$ | Error |
|---|---|---|---|---|---|
| 1 | 1.000000 | 1.010064 | 0.010064 | 1.097289 | 0.097289 |
| 2 | 1.000000 | 0.997519 | 0.002481 | 1.011496 | 0.011496 |
| 3 | 1.000000 | 0.942453 | 0.057547 | 0.971507 | 0.028493 |
| 4 | 1.000000 | 0.954378 | 0.045622 | 0.872994 | 0.127006 |
| 5 | 1.000000 | 0.848778 | 0.151222 | 0.907771 | 0.092229 |
| 6 | 1.000000 | 0.842353 | 0.157647 | 0.856397 | 0.143603 |
| 7 | 1.000000 | 0.811157 | 0.188843 | 0.732659 | 0.267341 |
| 8 | 1.000000 | 0.899413 | 0.100587 | 0.848459 | 0.151541 |
| 9 | 1.000000 | 1.032317 | 0.032317 | 1.164914 | 0.164914 |
| 10 | 1.000000 | 1.048519 | 0.048519 | 1.160798 | 0.160798 |
| 11 | 1.000000 | 0.986703 | 0.013297 | 0.929028 | 0.070972 |
| 12 | 1.000000 | 1.001886 | 0.001886 | 1.009768 | 0.009768 |
| 13 | 1.000000 | 1.250012 | 0.250012 | 1.475600 | 0.475600 |
| 14 | 1.000000 | 1.132012 | 0.132012 | 1.161211 | 0.161211 |
| 15 | 1.000000 | 1.046130 | 0.046130 | 1.048916 | 0.048916 |
| 16 | 1.000000 | 0.979698 | 0.020302 | 1.044212 | 0.044212 |
| $\|AV_\alpha - F\|$ | | $0.3419829 \times 10^{-1}$ | | $0.2719986 \times 10^{0}$ | |

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