A NEW METHOD FOR COMPUTING THE WEIGHTED GENERALIZED INVERSION OF PARTITIONED MATRICES*

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Abstract

In this paper, we shall give a new method for computing the weighted generalized inversion of partitioned matrices, i.e. solve the weighted problems with the aid of the unweighted ones, which is quite efficient and convenient for either dealing with the inconsistent equations or computing the generalized inverses, now we discuss some problems about the latter. The theorem 1 below is an extension of the main result in [2], especially, the proof is quite succinct here.

§1. Preliminaries

Let $F \in C^{m\times n}$. $P = P_1^* P_1, Q = Q_1^* Q_1$ are the triangular decompositions of the positive definite matrices $P \in C^{m\times m}, Q \in C^{n\times n}$, respectively.

Lemma. *If $G = P_1 F Q_1^{-1}$, then the weighted generalized inverse is*

$$Q_1^{-1} G^+ P_1 = Q^{-1} F^* (F Q^{-1} F^*)^+ F (F^* P F)^+ F^* P \hat{=} F_{PQ}^{\pm}.$$

Proof. Let BC be any rank factorization of F, and $G = P_1 F Q_1^{-1} = P_1 B C Q_1^{-1}$, then the unweighted generalized inverse

$$G^+ = (CQ_1^{-1})^+ (P_1 B)^+ = Q_1^{*-1} C^* (C Q^{-1} C^*)^{-1} (B^* P B)^{-1} B^* P_1^*$$

in view of $B^+ B = I = CC^+$, we then have

$$Q_1^{-1} G^+ P_1 = Q^{-1} F^* (F Q^{-1} F^*)^+ BC (F^* P F)^+ F^* P.$$

§2. Main Results

Let $A \in C^{m\times s}, B \in C^{m\times t}$. P, q, u are positive definite matrices of order m, s, t, respectively, and

$$Q = \begin{bmatrix} q & l \\ l^* & u \end{bmatrix}.$$

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Theorem 1. *If $M = (A\ B)$, then the weighted inverse*

$$M^{\pm}_{PQ} = \begin{bmatrix} A^{\pm} - ((I - A^{\pm}A)q^{-1}l + A^{\pm}B)f \\ f \end{bmatrix}. \quad (1)$$

where, $A^{\pm} = A^{\pm}_{pq}, d^*d = u - l^*q^{-1}l, J = (I - AA^{\pm})B, p = P$ and

$$f = \begin{cases} d^{-1}(P_1 J d^{-1})^+ P_1 \hat{=} f_1, & J \neq 0, \\ (d^*d + (B - Aq^{-1}l)^* A^{*\pm} q A^{\pm}(B - Aq^{-1}l))^{-1}(qA^{\pm}B - l)^* A^{\pm} \hat{=} f_2, & J = 0, \end{cases}$$

$f_1 = (J^*PJ)^{-1}J^*P$ when J is a full column rank.

Proof. We may write

$$Q = \begin{bmatrix} q_1^* & 0 \\ l^* q_1^{-1} & d^* \end{bmatrix} \begin{bmatrix} q_1 & q_1^{*^{-1}} l \\ 0 & d \end{bmatrix} \hat{=} Q_1^* Q_1.$$

where, $q = q_1^* q_1$ is the triangular decomposition of matrix q. Moreover, let $L = -q^{-1}ld^{-1}, G = P_1 M Q_1^{-1}$, it is easy to see that

$$Q^{-1} = \begin{bmatrix} q_1^{-1} & L \\ 0 & d^{-1} \end{bmatrix}.$$

By the lemma,

$$G = (P_1 A q_1^{-1} \ \ P_1(AL + Bd^{-1})) \hat{=} (C\ D).$$

now we computate $\beta_1 = [(I - CC^+)D]^+$ to find G^+, where,

$$C^+ = q_1 A^{\pm} p_1^{-1}, \qquad D = P_1(B - Aq^{-1}l)d^{-1}.$$

Because of

$$\beta_1^+ = (I - CC^+)D = P_1(I - AA^{\pm})(B - Aq^{-1}l)d^{-1} = P_1(I - AA^{\pm})Bd^{-1} = P_1 J d^{-1},$$

so that $\beta_1 = [(I - CC^+)D]^+ = df_1 P_1^{-1}$, from this, $\beta^+ = 0$ or not, according as $J = 0$ or not. Especially,

$$\beta_1 = (P_1 J d^{-1})^+ = d(J^*PJ)^{-1}J^*P_1^* \quad (2)$$

when J is a full column rank.

In addition, we may obtain

$$\beta_2 = (I + D^*C^{*+}C^+D)^{-1}D^*C^{*+}C^+ = df_2 P^{-1}. \quad (3)$$

Notice that the lemma and the main results in [5], it follows that

$$M^{\pm}_{PQ} = \begin{bmatrix} q_1^{-1} & L \\ 0 & d^{-1} \end{bmatrix} G^+ P_1 = \begin{bmatrix} q_1^{-1} & L \\ 0 & d^{-1} \end{bmatrix} \begin{bmatrix} C^+ - C^+ D\beta \\ \beta \end{bmatrix} P_1. \quad (4)$$

where,

$$\beta=\begin{cases}\beta_1, & \beta_1^+\neq 0,\\ \beta_2. & \beta_1^+=0\end{cases}\quad \text{or}\quad f=d^{-1}\beta P_1=\begin{cases}d^{-1}\beta_1 P_1,\\ d^{-1}\beta_2 P_1.\end{cases}$$

Substituting the above expressions in (4), we thus complete the proof of (1). Of course, equality (2) holds when $B\in C_1^{m\times 1}$.

Let $A\in C_r^{m\times n}, U\in C_{m-r}^{m\times(m-r)}, V\in C_{n-r}^{n\times(n-r)}$. By the method above, it follows that the important result in [3].

Theorem 2. *If $A^*U=0, AV=0.p,q$ are the positive definite matrices of order m,n, respectively, then*

$$\begin{bmatrix} A & p^{-1}U\\ V^*q & 0\end{bmatrix}^{-1}=\begin{bmatrix} A_{pq}^{\pm} & V(V^*qV)^{-1}\\ (U^*p^{-1}U)^{-1}U^* & 0\end{bmatrix}.$$

Proof. Let $p=p_1^*p_1, q=q_1^*q_1$,

$$M=\begin{bmatrix} p_1Aq_1^{-1} & p_1^{*^{-1}}U\\ V^*q_1^* & 0\end{bmatrix}.$$

It is clear that

$$q_1^{*^{-1}}A^*p_1^*(p_1^{*^{-1}}U)=0,\quad p_1Aq_1^{-1}(q_1V)=0,$$

Using the theorem 4 in [5] and the lemma, it follows that

$$M^{-1}=\begin{bmatrix} (p_1Aq_1^{-1})^+ & (V^*q_1^*)^+\\ (p_1^{*^{-1}}U)^+ & 0\end{bmatrix}$$

$$=\begin{bmatrix} q_1 & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} A_{pq}^{\pm} & V(V^*qV)^{-1}\\ (U^*p^{-1}U)^{-1}U^* & 0\end{bmatrix}\begin{bmatrix} p_1^{-1} & 0\\ 0 & I\end{bmatrix},$$

we thus have

$$\begin{bmatrix} A & p^{-1}U\\ V^*q & 0\end{bmatrix}^{-1}=\left\{\begin{bmatrix} p_1^{-1} & 0\\ 0 & I\end{bmatrix}M\begin{bmatrix} q_1 & 0\\ 0 & I\end{bmatrix}\right\}^{-1}=\begin{bmatrix} A_{pq}^{\pm} & V(V^*qV)^{-1}\\ (U^*p^{-1}U)^{-1}U^* & 0\end{bmatrix}.$$

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