REPRESENTATIONS FOR THE WEIGHTED MOORE-PENROSE INVERSE OF A PARTITIONED MATRIX*

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Abstract

The weighted moore-Penrose inverse of a partitioned matrix $A = (U\ V)$ is discussed. Representations for the weighted Moore-Penrose inverse of the matrix A are derived, which extend some earler results.

§1. Introduction

Various expressions for the generalized inverse have been developed by a number of authors. Greville [4] has developed a representation for the generalized inverse of a partitioned matrix $A_k = (A_{k-1}, a_k)$ and presented a famous recursive method for computing the M–P inverse of A.

Wang and Chen[5] extended Greville's result to compute the weighted M–P inverse of $A_k = (A_{k-1}, a_k)$. The result is as follows: Let A_k be the submatrix of A consisting of the first k columns and A_k be partitioned as $A_k = (A_{k-1}, a_k)$. The matrix $N_k \in C^{k\times k}$ is the leading principal submatrix of N, and N_k is partitioned as $N_k = \begin{pmatrix} N_{k-1} & l_k \\ l_k^* & n_{kk} \end{pmatrix}$. Let $X_{k-1} = (A_{k-1})^+_{MN_{k-1}}$, $X_k = (A_k)^+_{MN_k}$, $d_k = X_{k-1}a_k$, and $c_k = a_k - A_{k-1}d_k = (I - A_{k-1}X_{k-1})a_k$. Then

$$X_k = \begin{pmatrix} X_{k-1} - d_k b_k^* - (I - X_{k-1}A_{k-1})N_{k-1}^{-1} l_k b_k^* \\ b_k^* \end{pmatrix}, \tag{1.1}$$

where

$$b_k^* = \begin{cases} (c_k^* M c_k)^{-1} c_k^* M, & \text{if } c_k \neq 0, \\ \delta_k^{-1}(d_k^* N_{k-1} - l_k^*) X_{k-1}, & \text{if } c_k = 0, \end{cases} \tag{1.2}$$

and

$$\delta_k = n_{kk} + d_k^* N_{k-1} d_k - (d_k^* l_k + l_k^* d_k) - l_k^*(I - X_{k-1}A_{k-1})N_{k-1}^{-1} l_k \tag{1.3}$$

is a positive real scalar.

Cline[3] discussed the M–P inverse of any matrix A partitioned as $A = (U\ V)$, in which U and V are submatrices, and presented an expression for the M–P inverse of A under some conditions (see [3], Theorem 1).

It is the purpose of this paper to develop representations for the weighted M–P inverse of a partitioned matrix $A = (U\ V)$. A more general result is given without any conditions.

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Lemma 1. *Let the columns of the matrices R and S consist of a basis for $N(A^*)$ and $N(A)$ respectively, then*

$$I - AA^+_{MN} = M^{-1}R(R^*M^{-1}R)^{-1}R^*, \quad I - A^+_{MN}A = S(S^*NS)^{-1}S^*N.$$

§2. Main Results

We begin by combining expressions in (1.2) into a single expression. Since c_k is a single column vector, $c_k \neq 0$ implies $b_k^* = (c_k^*Mc_k)^{-1}c_k^*M = (c_k)^+_{M,\delta_k}$, and thus $(c_k)^+_{M,\delta_k}c_k = 1$. Further, $c_k = 0$ implies $(c_k)^+_{M,\delta_k} = 0$. Then we can rewrite b_k^* as

$$b_k^* = (c_k)^+_{M,\delta_k} + [1 - (c_k)^+_{M,\delta_k}c_k]\delta_k^{-1}(d_k^*N_{k-1} - 1_k^*)X_{k-1}. \tag{2.1}$$

Now consider an arbitrary matrix $A = (U\ V)$, where $U \in C^{m\times n_1}$ and $V \in C^{m\times(n-n_1)}$, the hermitian positive definite matrix N is partitioned as

$$N = \begin{pmatrix} N_1 & L \\ L^* & N_2 \end{pmatrix}, \tag{2.2}$$

where $N_1 \in C^{n_1\times n_1}$. Corresponding to d_k, c_k and δ_k, let

$$D = U^+_{MN_1}V, \tag{2.3}$$

$$C = (I - UU^+_{MN_1})V, \tag{2.4}$$

$$K = N_2 + D^*N_1D - (D^*L + L^*D) - L^*(I - U^+_{MN_1}U)N_1^{-1}L. \tag{2.5}$$

We shall prove that K is hermitian positive definite.

Let the columns of W_1 be consist of a basis for $N(U)$, and $W_2 = \begin{pmatrix} W_1 & -D \\ 0 & I_{n_2} \end{pmatrix}$, then W_2 has full column rank, thus the matrix $W_2^*NW_2$ is hermitian positive definite. Then the second diagonal block of $(W_2^*NW_2)^{-1}$ is also hermitian positive definite, we can show that it is equal to K^{-1} by using Lemma 1. Hence K is hermitian positive definite. Then we have our main theorem.

Theorem 1. *Let $A \in C^{m\times n}, A = (U\ V)$, where $U \in C^{m\times n_1}, n_1 < n, N$ be partitioned as (2.2), D, C and K be defined as before, then*

$$A^+_{MN} = \begin{pmatrix} U^+_{MN_1} - DH - (I - U^+_{MN_1}U)N_1^{-1}LH \\ H \end{pmatrix}, \tag{2.6}$$

where

$$H = C^+_{MK} + (I - C^+_{MK}C)K^{-1}(D^*N_1 - L^*)U^+_{MN_1}. \tag{2.7}$$

Proof. Let the right hand side of (2.6) be X, then we can prove Theorem 1 by verifing

$$AXA = A, \quad XAX = X, \quad (MAX)^* = MAX, \quad (NXA)^* = NXA.$$

Omitted.

Corollary 1.1.

$$A^+ = \begin{pmatrix} U^+ - U^+VC^+_{IK_1} - U^+V(I - C^+_{IK_1}C)K_1^{-1}V^*U^{+*}U^+ \\ C^+_{IK_1} + (I - C^+_{IK_1}C)K_1^{-1}V^*U^{+*}U^+ \end{pmatrix}, \tag{2.8}$$

where

$$K_1 = I + V^*U^{+*}U^+V. \tag{2.9}$$

Note. This result is more general than [3, Th.1], Since no additional condition is imposed.

We shall discuss two special cases when the weighted M–P inverse can be reduced to simpler forms. It is easy to see that $C = 0$ iff $R(V) \subseteq R(U)$; also C is of full column rank iff V is of full column rank and $R(U) \cap R(V) = \{0\}$.

Corollary 1.2. If $C = 0$, then

$$A^+_{MN} = \begin{pmatrix} U^+_{MN_1} - DH - (I - U^+_{MN_1}U)N_1^{-1}LH \\ H \end{pmatrix}, \tag{2.10}$$

where

$$H = K^{-1}(D^*N_1 - L^*)U^+_{MN_1}. \tag{2.11}$$

Corollary 1.3. If C is of full column rank, then

$$A^+_{MN} = \begin{pmatrix} U^+_{MN_1} - D(C^*MC)^{-1}C^*M - (I - U^+_{MN_1}U)N_1^{-1}L(C^*MC)^{-1}C^*M \\ (C^*MC)^{-1}C^*M \end{pmatrix}. \tag{2.12}$$

Clearly, Corollary 1.2 and 1.3 generalize the results of [4] and [5].

Corollary 1.4. If $A \in C^{m\times n}_{n_1}, U \in C^{m\times n_1}_{n_1}$, then

$$A^+_{MN} = \begin{pmatrix} U^+_{MN_1} - DK^{-1}(D^*N_1 - L^*)U^+_{MN_1} \\ K^{-1}(D^*N_1 - L^*)U^+_{MN_1} \end{pmatrix}, \tag{2.13}$$

where

$$U^+_{MN_1} = (U^*MU)^{-1}U^*M, \tag{2.14}$$

and

$$K = N_2 + D^*N_1D - (D^*L + L^*D). \tag{2.15}$$

Proof. In this case, $C = 0$ and $U^+_{MN_1}U = I$, thus Corollary 1.2 is applicable.

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