THE WAVE EQUATION APPROACH TO ROBBIN INVERSE PROBLEMS FOR A DOUBLY-CONNECTED REGION: AN EXTENSION TO HIGHER DIMENSIONS*

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Abstract

The spectral function $\hat{\mu}(t) = \sum_{j=1}^{\infty} e^{-it\lambda_j^{1/2}}$ where $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of the three-dimensional Laplacian is studied for a variety of domains, where $-\infty < t < \infty$ and $i = \sqrt{-1}$. The dependence of $\hat{\mu}(t)$ on the connectivity of a domain and the impedance boundary conditions (Robbin conditions) are analyzed. Particular attention is given to the spherical shell together with Robbin boundary conditions on its surface.

1. Historical Remarks

Let $D \subseteq R^3$ be a simply connected bounded domain with a smooth bounding surface S. Then, there exist eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and corresponding eigenfunctions $\{\phi_j(\underset{\sim}{x})\}_{j=1}^{\infty}$ of the Laplace operator $\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in xyz-space, under the impedance boundary conditions (Robbin boundary conditions), such that $\{\phi_j(\underset{\sim}{x})\}_{j=1}^{\infty}$ is a complete orthonormal system in $L^2(D)$. That is, we have the following impedance problem (Robbin problem):

$$-\Delta_3\phi_j = \lambda_j\phi_j \quad \text{in } D, \tag{1.1}$$

$$(\frac{\partial}{\partial n} + \gamma)\phi_j = 0 \quad \text{on } S, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to S and γ is a positive constant. We may assume that each ϕ_j is real-valued and that the eigenvalues λ_j are enumerated in the order of magnitude

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_j \le \cdots \to \infty \quad \text{as } j \to \infty. \tag{1.3}$$

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There are numerous works treating the asymptotic behaviour of the number of eigenvalues, $N(\lambda)$, as $\lambda \to \infty$. It has been shown that (H. Weyl, 1912)

$$N(\lambda) \sim \frac{V}{6\pi^2}\lambda^{3/2} \quad \text{as } \lambda \to \infty, \tag{1.4}$$

and that (R. Courant, 1920)

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + O(\lambda \log \lambda) \quad \text{as } \lambda \to \infty, \tag{1.5}$$

where V is the volume of D.

In order to obtain further information about the geometry of D, one studies certain functions of the spectrum. The most useful to date comes from the study of the heat equation or the wave equation.

Accordingly, let $e^{-t\Delta_3}$ denote the heat operator. Then, we can construct the trace function

$$\theta(t) = \operatorname{tr}(e^{-t\Delta_3}) = \sum_{j=1}^{\infty} e^{-t\lambda_j}, \tag{1.6}$$

which converges for all positive t.

Suppose that $e^{-it\Delta_3^{1/2}}$ is the wave operator. Then an alternative to (1.6) is to study the trace function

$$\hat{\mu}(t) = \operatorname{tr}(e^{-it\Delta_3^{1/2}}) = \sum_{j=1}^{\infty} e^{-it\lambda_j^{1/2}}, \tag{1.7}$$

which represents a tempered distribution for $-\infty < t < \infty$ and $i = \sqrt{-1}$. The applications of (1.6) to problem (1.1) and (1.2) and to more general ones can be found in Gottlieb [1], Pleijel [4], Waechter [5], Zayed [6, 7] and the references given there. thus, Pleijel has investigated problem (1.1)–(1.2) by using the heat equation approach and has shown that: if $\gamma \to \infty$ (Dirichlet problem),

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\int_S H ds + O(t^{1/2}) \quad \text{as } t \to 0, \tag{1.8}$$

and if $\gamma = 0$ (Neumann problem),

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\int_S H ds + O(t^{1/2}) \quad \text{as } t \to 0, \tag{1.9}$$

where V and S are respectively the volume and the surface area of the domain D while $H = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2})$, R_1 and R_2 are the principal radii of curvature.

Zayed [7] has investigated problem (1.1)– (1.2) for either large or small impedance γ, by using the heat equation approach, and has shown that, if $\gamma >> 1$,

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{1}{16\pi t}\{S - 2\gamma^{-1}\int_S H ds\} + \frac{1}{12\pi^{3/2}t^{1/2}}\int_S H ds + O(t^{1/2}) \quad \text{as } t \to 0, \tag{1.10}$$

and if $0 < \gamma << 1$,

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S (H - 3\gamma) ds + O(t^{1/2}), \quad \text{as } t \to 0. \tag{1.11}$$

The asymptotic expansion (1.10) may be interpreted as:

(i) D =convex domain and we have an impedance boundary condition with large impedance, or

(ii) D =convex domain has volume V and its surface has the area $\{S - 2\gamma^{-1} \int_S H ds\}$ together with the Dirichlet boundary condition.

Similarly, (1.11) may be interpreted as:

(i) D = convex domain and we have an impedance boundary condition with small impedance, or

(ii) D =convex domain has volume V and its surface area S has the mean curvature $(H - 3\gamma)$ together with the Neumann boundary condition.

We note that formulae (1.10) and (1.11) are in agreement with formulae (1.8) and (1.9) if $\gamma \to \infty$ and $\gamma = 0$ respectively.

In this paper, we shall concentrate on a study of the tempered distribution $\hat{\mu}(t)$ and then we can see the differences between the heat equation approach and the wave equation approach.

It is easily seen that $\hat{\mu}(t)$ is just the Fourier transform

$$\int_{-\infty}^{+\infty} e^{-it\lambda^{1/2}} dN(\lambda). \tag{1.12}$$

It is well known that the wave equation methods have given very strong results; the definitive one is that of Hörmander [2]. He has constructed the first term of $N(\lambda)$ for an elliptic positive semidefinite pseudo-differential operator $P \subseteq R^n$ of order m by using the distribution $\mathrm{tr}(e^{-itP})$ near $t = 0$.

Recently, Zayed [8] has investigated problem (1.1)-(1.2) by using the wave equation approach when the domain D is just a sphere of radius a and has shown for small $|t|$ that

(i) if $\gamma \to \infty$ (Dirichlet problem),

$$\hat{\mu}(t) = \frac{V}{4\pi t}\delta(-|t|) - \frac{S}{8\pi^2 t} \text{ sign } t + \frac{2a}{3\pi} \text{ sign } t + O(t \text{ sign } t); \tag{1.13}$$

(ii) if $\gamma = 0$ (Neumann problem),

$$\hat{\mu}(t) = \frac{V}{4\pi t}\delta(-|t|) + \frac{S}{8\pi^2 t} \text{ sign } t + \frac{2a}{3\pi} \text{ sign } t + O(t \text{ sign } t); \tag{1.14}$$

(iii) if $\gamma >> 1$,

$$\hat{\mu}(t) = \frac{V}{4\pi t}\delta(-|t|) - \frac{(S - 8\pi a\gamma^{-1})}{8\pi^2 t} \text{ sign } t + \frac{2a}{3\pi} \text{ sign } t + O(t \text{ sign } t); \tag{1.15}$$

(iv) if $0 < \gamma << 1$,

$$\hat{\mu}(t) = \frac{V}{4\pi t}\delta(-|t|) + \frac{S}{8\pi^2 t}\,\mathrm{sign}\,t + \frac{a(2-3\gamma a)}{3\pi}\,\mathrm{sign}\,t + O(t\,\mathrm{sign}\,t), \tag{1.16}$$

where $\delta(-|t|)$ is the Dirac delta function and

$$\mathrm{sign}\,t = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

In these formulae, $V = \frac{4}{3}\pi a^3$ is the volume of the sphere D, and $S = 4\pi a^2$ is its surface area. We note also that formulae (1.15) and (1.16) are in agreement with formulae (1.13) and (1.14) if $\gamma \to \infty$ and $\gamma = 0$ respectively.

It is of interest to investigate the impedance problem (1.1)–(1.2) by using the wave equation approach when $D \subseteq R^3$ is a general convex domain. This is still an open problem.

The object of this paper is to discuss the following impedance problem by using the wave equation approach.

Let

$$D = \{(r,\theta,\phi) : a \le r \le b, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi\}$$

be a spherical shell. Suppose that the eigenvalues (1.3) are given for the Helmholtz equation

$$(\Delta_3 + \lambda)u = 0 \quad \text{in} \quad D, \tag{1.17}$$

together with the impedance boundary conditions (Robbin boundary conditions)

$$\left(\frac{\partial u}{\partial r} + \gamma_1 u\right)_{r=a} = 0, \quad \left(\frac{\partial u}{\partial r} + \gamma_2 u\right)_{r=b} = 0, \tag{1.18}$$

where γ_1 and γ_2 are positive constants. Determine the geometry of D and the impedances γ_1, γ_2 from the asymptotic expansion of

$$\hat{u}(t) = \sum_{j=1}^{\infty} e^{-it\lambda_j^{1/2}} \quad \text{for small } |t|. \tag{1.19}$$

Recently, Zayed [9] has investigated problem (1.17)–(1.18) in the following special cases and has obtained:

Case 1. $\gamma_1 = \gamma_2 = 0$ (Neumann conditions on $r = a, r = b$),

$$\hat{\mu}(t) = \frac{4\pi(b^3 - a^3)}{3(4\pi t)}\delta(-|t|) + \frac{4\pi(b^2 + a^2)}{8\pi^2 t}\,\mathrm{sign}\,t + \frac{4\pi(b + a)}{6\pi^2}\,\mathrm{sign}\,t + O(t\,\mathrm{sign}\,t) \quad \text{as } |t| \to 0. \tag{1.20}$$

Case 2. $\gamma_1 \to \infty, \gamma_2 = 0$ (Dirichlet condition on $r = a$, Neumann condition on $r = b$).

$$\hat{\mu}(t) = \frac{4\pi(b^3 - a^3)}{3(4\pi t)}\delta(-|t|) + \frac{4\pi(b^2 - a^2)}{8\pi^2 t}\,\mathrm{sign}\,t + \frac{4\pi(b + a)}{6\pi^2}\,\mathrm{sign}\,t + O(t\,\mathrm{sign}\,t) \quad \text{as } |t| \to 0. \tag{1.21}$$

Case 3. $\gamma_1 = 0, \gamma_2 \to \infty$ (Neumann condition on $r = a$, Dirichlet condition on $r = b$).

$$\hat{\mu}(t) = \frac{4\pi(b^3 - a^3)}{3(4\pi t)}\delta(-|t|) + \frac{4\pi(a^2 - b^2)}{8\pi^2 t} \text{ sign } t + \frac{4\pi(b + a)}{6\pi^2} \text{ sign } t + O(t \text{ sign } t) \quad \text{as } |t| \to 0. \tag{1.22}$$

Case 4. $\gamma_1 = \gamma_2 \to \infty$ (Dirichlet conditions on $r = a, r = b$),

$$\hat{\mu}(t) = \frac{4\pi(b^3 - a^3)}{3(4\pi t)}\delta(-|t|) - \frac{4\pi(b^2 + a^2)}{8\pi^2 t} \text{ sign } t + \frac{4\pi(b + a)}{6\pi^2} \text{ sign } t + O(t \text{ sign } t) \quad |t| \to 0. \tag{1.23}$$

An examination of these results shows that the first term of $\hat{\mu}(t)$ determines the volume of the shell D, the second term determines its surface area, and the third term determines the principal radii of curvature.

2. Formulation of the Mathematical Problem

It can be easily seen that the trace function $\hat{\mu}(t)$ is given by

$$\hat{\mu}(t) = \iiint_D G(\underset{\sim}{x}, \underset{\sim}{x}; t) d\underset{\sim}{x}, \tag{2.1}$$

where $G(\underset{\sim}{x}, \underset{\sim}{x}'; t)$ is Green's function for the wave equation

$$(\Delta_3 - \frac{\partial^2}{\partial t^2}) G(\underset{\sim}{x}, \underset{\sim}{x}'; t) = 0 \text{ in } DX\{-\infty < t < \infty\}, \tag{2.2}$$

subject to the impedance boundary conditions (1.18) and the initial conditions

$$\lim_{t \to 0} G(\underset{\sim}{x}, \underset{\sim}{x}'; t) = 0, \quad \lim_{t \to 0} \frac{\partial G(\underset{\sim}{x}, \underset{\sim}{x}'; t)}{\partial t} = \delta(\underset{\sim}{x} - \underset{\sim}{x}'), \tag{2.3}$$

where $\delta(\underset{\sim}{x} - \underset{\sim}{x}')$ is the Dirac delta function located at the source point $\underset{\sim}{x} = \underset{\sim}{x}'$. The points $\underset{\sim}{x} = (x, y, z)$ and $\underset{\sim}{x}' = (x', y', z')$ belong to the spherical shell D. Let us write

$$G(\underset{\sim}{x}, \underset{\sim}{x}'; t) = G_0(\underset{\sim}{x}, \underset{\sim}{x}'; t) + \chi(\underset{\sim}{x}, \underset{\sim}{x}'; t), \tag{2.4}$$

where

$$G_0(\underset{\sim}{x}, \underset{\sim}{x}'; t) = \frac{1}{4\pi t}\delta(|\underset{\sim}{x} - \underset{\sim}{x}'| - |t|) \tag{2.5}$$

is the "fundamental solution" of the wave equation (2.2) while $\chi(\underset{\sim}{x}, \underset{\sim}{x}'; t)$ is the "regular solution" chosen in such a way that $G(\underset{\sim}{x}, \underset{\sim}{x}'; t)$ satisfies the impedance boundary conditions (1.18).

From (2.1), (2.4) and (2.5), we find that

$$\hat{\mu}(t)=\frac{4\pi(b^3-a^3)}{3(4\pi t)}\delta(-|t|)+K(t), \tag{2.6}$$

where

$$K(t)=\iiint_D \chi(\underset{\sim}{x},\underset{\sim}{x};t)d\underset{\sim}{x}. \tag{2.7}$$

In what follows we shall use Fourier transforms with repect to $-\infty < t < \infty$ and use $-\infty < \eta < \infty$ as the Fourier transform parameter. That is, we define

$$\hat{G}(\underset{\sim}{x},\underset{\sim}{x}';\eta)=\int_{-\infty}^{+\infty} e^{-2\pi i\eta t}\quad G(\underset{\sim}{x},\underset{\sim}{x}';t)dt. \tag{2.8}$$

An application of the Fourier transform to the wave equation (2.2) shows that $\hat{G}(\underset{\sim}{x},\underset{\sim}{x}'\eta)$ satisfies the reduced wave equation

$$(\Delta_3+4\pi^2\eta^2)\hat{G}(\underset{\sim}{x},\underset{\sim}{x}';\eta)=0 \quad \text{in } D, \tag{2.9}$$

together with the impedance boundary conditions (1.18).

The asymptotic expansion of $K(t)$, for small $|t|$, may then be deduced directly from the asymptotic expansion of $\hat{K}(\eta)$ for large $|\eta|$. On using the spherical polar coordinates (r,θ,ϕ) with $x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi$ and $z=r\cos\theta$, we find that

$$\hat{K}(\eta)=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=a}^{b} r^2\hat{\chi}(r,\theta,\phi,r,\theta,\phi;\eta)\sin\theta dr d\theta d\phi. \tag{2.10}$$

3. Construction of Green's Function

Equation (2.9) has the fundamental solution

$$\begin{aligned}\hat{G}_0(r,\theta,\phi,r',\theta',\phi';\eta) &= \frac{\exp(-2\pi i\eta|\underset{\sim}{x}-\underset{\sim}{x}'|)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|}\\ &= \frac{i\eta}{2\pi^2}\sum_{m=0}^{\infty}(2m+1)j_m(2\pi\eta r')k_m(2\pi\eta r)P_m(\cos\alpha),\end{aligned} \tag{3.1}$$

where α is the angle subtended at the origin by the line joining the field point $\underset{\sim}{x}$ and the source point $\underset{\sim}{x}'$, and P_m is the Legendre polynomial of degree m. We note that

$$j_m(2\pi\eta r)=(\frac{\pi}{2i\eta r})^{1/2}J_{m+1/2}(2\pi\eta r),\quad k_m(2\pi\eta r)=(\frac{\pi}{2i\eta r})^{1/2}Y_{m+1/2}(2\pi\eta r),$$

where $J_{m+1/2}$ and $Y_{m+1/2}$ are Bessel functions of the first and second kinds.

On solving equation (2.9) we deduce that if $r' \le r \le b$,

$$\hat{G}(r,\theta,\phi,r',\theta',\phi';\eta) = \sum_{m=0}^{\infty}\Big\{\frac{i\eta(2m+1)}{2\pi^2}j_m(2\pi\eta r')k_m(2\pi\eta r) + A_m k_m(2\pi\eta r) + B_m j_m(2\pi\eta r)\Big\}P_m(\cos\alpha), \tag{3.2}$$

and if $r' \ge r \ge a$,

$$\hat{G}(r,\theta,\phi,r',\theta',\phi';\eta) = \sum_{m=0}^{\infty}\Big\{\frac{i\eta(2m+1)}{2\pi^2}j_m(2\pi\eta r)k_m(2\pi\eta r') + A_m k_m(2\pi\eta r) + B_m j_m(2\pi\eta r)\Big\}P_m(\cos\alpha), \tag{3.3}$$

where A_m and B_m are constants to be determined.

It is straightforward to show that the impedance boundary conditions (1.18) on $r = a$ and $r = b$ give the following:

$$A_m = \frac{i\eta(2m+1)}{2\pi^2 R_m}\{[2\pi\eta j_m'(2\pi\eta a) + \gamma_1 j_m(2\pi\eta a)][2\pi\eta j_m'(2\pi\eta b)\gamma_2 j_m(2\pi\eta b)]k_m(2\pi\eta r') - [2\pi\eta j_m'(2\pi\eta a) + \gamma_1 j_m(2\pi\eta a)][2\pi\eta k_m'(2\pi\eta b) + \gamma_2 k_m(2\pi\eta b)]j_m(2\pi\eta r')\}, \tag{3.4}$$

and

$$B_m = \frac{i\eta(2m+1)}{2\pi^2 R_m}\{[2\pi\eta k_m'(2\pi\eta a)+\gamma_1 k_m(2\pi\eta a)][2\pi\eta k_m'(2\pi\eta b)+\gamma_2 k_m(2\pi\eta b)]j_m(2\pi\eta r') - [2\pi\eta j_m'(2\pi\eta a) + +\gamma_1 j_m(2\pi\eta a)][2\pi\eta k_m'(2\pi\eta b) + \gamma_2 k_m(2\pi\eta b)]k_m(2\pi\eta r')\}, \tag{3.5}$$

where

$$R_m = [2\pi\eta j_m'(2\pi\eta a) + \gamma_1 j_m(2\pi\eta a)][2\pi\eta k_m'(2\pi\eta b) + \gamma_2 k_m(2\pi\eta b)] - [2\pi\eta k_m'(2\pi\eta a) + \gamma_1 k_m(2\pi\eta a)][2\pi\eta j_m'(2\pi\eta b) + \gamma_2 j_m(2\pi\eta b)] \neq 0. \tag{3.6}$$

From (3.2)–(3.6) we obtain Green's function $\hat{G}(r,\theta,\phi,r',\theta',\phi';\eta)$, and if we put $r' = r, \theta' = \theta$ and $\phi' = \phi$, we find that equation (2.9) has the regular solution

$$\hat{\chi}(r,\theta,\phi,r,\theta,\phi;\eta) = \sum_{m=0}^{\infty}\frac{i\eta(2m+1)}{2\pi^2 R_m}\{[2\pi\eta j_m'(2\pi\eta a) + \gamma_1 j_m(2\pi\eta a)][2\pi\eta j_m'(2\pi\eta b) + \gamma_2 j_m(2\pi\eta b)]k_m^2(2\pi\eta r) - -2[2\pi\eta j_m'(2\pi\eta a) + \gamma_1 j_m(2\pi\eta a)][2\pi\eta k_m'(2\pi\eta b) + \gamma_2(2\pi\eta b)]j_m(2\pi\eta r)k_m(2\pi\eta r) + [2\pi\eta k_m'(2\pi\eta a) + \gamma_1 k_m(2\pi\eta a)][2\pi\eta k_m'(2\pi\eta b) + \gamma_2 k_m(2\pi\eta b)]j_m^2(2\pi\eta r)\}. \tag{3.7}$$

If we insert (3.7) into (2.10) and integrate, we find after some reduction that

$$\hat{K}(\eta) = b^2\sum_{m=0}^{\infty}(m+\frac{1}{2})f_1(m+\frac{1}{2};\eta) - a^2\sum_{m=0}^{\infty}(m+\frac{1}{2})f_2(m+\frac{1}{2};\eta), \tag{3.8}$$

where

$$f_1(m;\eta) = (1-\frac{m^2}{4\pi^2\eta^2b^2})\Big\{J_m(2\pi\eta b)Y_m(2\pi\eta b)+\frac{2J_m(2\pi\eta b)}{\pi b[2\pi\eta J'_m(2\pi\eta b)+\gamma_2 J_m(2\pi\eta b)]}\Big\}$$
$$+J'_m(2\pi\eta b)Y'_m(2\pi\eta b)-\frac{Y_2 J'_m(2\pi\eta b)}{\pi^2\eta b[2\pi\eta J'_m(2\pi\eta b)+\gamma_2 J_m(2\pi\eta b)]}$$
$$-4(1-\frac{m^2}{4\pi^2\eta^2b^2})\frac{[2\pi\eta J'_m(2\pi\eta a)+\gamma_1 J_m(2\pi\eta a)]}{\pi^2b^2R^*_m[2\pi\eta J'_m(2\pi\eta b)+\gamma_2 J_m(2\pi\eta b)]}$$
$$-\frac{\gamma_2^2[2\pi\eta J'_m(2\pi\eta a)+\gamma_1 J_m(2\pi\eta a)]}{\pi^4\eta^2b^2R^*_m[2\pi\eta J'_m(2\pi\eta b)+\gamma_2 J_m(2\pi\eta b)]}, \tag{3.9}$$

and

$$f_2(m;\eta) = (1-\frac{m^2}{4\pi^2\eta^2a^2})\Big\{J_m(2\pi\eta a)Y_m(2\pi\eta a)-\frac{2J_m(2\pi\eta a)}{\pi a[2\pi\eta J'_m(2\pi\eta a)+\gamma_1 J_m(2\pi\eta a)]}\Big\}$$
$$+J'_m(2\pi\eta a)Y'_m(2\pi\eta a)+\frac{\gamma_1 J'_m(2\pi\eta a)}{\pi^2\eta a[2\pi\eta J'_m(2\pi\eta a)+\gamma_1 J_m(2\pi\eta a)]}$$
$$-4(1-\frac{m^2}{4\pi^2\eta^2a^2})\frac{[2\pi\eta J'_m(2\pi\eta b)+\gamma_2 J_m(2\pi\eta b)]}{\pi^2a^2R^*_m[2\pi\eta J'_m(2\pi\eta a)+\gamma_1 J_m(2\pi\eta a)]}$$
$$-\frac{\gamma_1^2[2\pi\eta J'_m(2\pi\eta b)+\gamma_2 J_m(2\pi\eta b)]}{\pi^4\eta^2a^2R^*_m[2\pi\eta J'_m(2\pi\eta a)+\gamma_1 J_m(2\pi\eta a)]}, \tag{3.10}$$

where

$$R^*_m = [2\pi\eta J'_m(2\pi\eta a)+\gamma_1 J_m(2\pi\eta a)][2\pi\eta Y'_m(2\pi\eta b)+\gamma_2 Y_m(2\pi\eta b)]$$
$$-[2\pi\eta Y'_m(2\pi\eta a)+\gamma_1 Y_m(2\pi\eta a)][2\pi\eta J'_m(2\pi\eta b)+\gamma_2 J_m(2\pi\eta b)].$$

The series (3.8) in fact diverges since $K(t)$ behaves like $\frac{\text{sign}t}{t}$ for small $|t|$; however, this difficulty may easily be removed by deducing the asymptotic expansion for large $|\eta|$ of

$$\hat{K}_N(\eta) = b^2\sum_{m=0}^{N}(m+\frac{1}{2})f_1(m+\frac{1}{2};\eta) - a^2\sum_{m=0}^{N}(m+\frac{1}{2})f_2(m+\frac{1}{2};\eta). \tag{3.11}$$

Inversion of the Fourier transform gives $K_N(t)$ and we may then write

$$K(t) = \lim_{N\to\infty} K_N(t). \tag{3.12}$$

On applying the Watson transformation [5] to (3.11), we deduce for large $|\eta|$ that

$$\hat{K}_N(\eta) \sim b^2\int_0^N \nu f_1(\nu;\eta)d\nu - a^2\int_0^N \nu f_2(\nu;\eta)d\nu. \tag{3.13}$$

It now follows that the functions $f_1(\nu;\eta)$ anf $f_2(\nu;\eta)$ may be expressed in terms of the asymptotic expansions of Bessel functions and their derivatives due to Olver [3]; these expansions for large $|\eta|$ are uniformly valid in ν for $|\arg\nu| < \frac{\pi}{2}$.

4. Some Asymptotic Expansions of the Tempered Distribution $\hat{\mu}(t)$

In this section, we look at the following cases:

Case 1. $0 < \gamma_1, \gamma_2 << 1$.

In this case, we deduce after some reduction that for large $|\eta|$,

$$f_1(\nu;\eta) \sim \frac{(\nu^2 - 4\pi^2 b^2\eta^2)^{1/2}}{4\pi^3 b^2\eta^2} \sum_{n=0}^{\infty} \frac{D_n(\tau_1)}{\nu^n}, \tag{4.1}$$

$$f_2(\nu;\eta) \sim \frac{(\nu^2 - 4\pi^2 a^2\eta^2)^{1/2}}{4\pi^3 a^2\eta^2} \sum_{n=0}^{\infty} \frac{C_n(\tau_2)}{\nu^n}, \tag{4.2}$$

where

$$\tau_1 = \frac{\nu}{(\nu^2 - 4\pi^2 b^2\eta^2)^{1/2}}, \quad \tau_2 = \frac{\nu}{(\nu^2 - 4\pi^2 a^2\eta^2)^{1/2}},$$

and

$$D_0 = 0, \quad D_1 = -\tau_1 + \tau_1^3, \quad D_2 = \tau_1^2(2\gamma_2 b - 1) - \tau_1^4(2\gamma_2 b - 3) - 2\tau_1^6,$$
$$D_3 = -\tau_1^3(\frac{3}{4} - 2\gamma_2 b + 2\gamma_2^2 b^2) - \tau_1^5(-\frac{23}{4} + 6\gamma_2 b - 2\gamma_2^2 b^2) - \tau_1^7(\frac{41}{4} - 4\gamma_2 b) + \frac{21}{4}\tau_1^9,$$
$$C_0 = 0, \quad C_1 = \tau_2 - \tau_2^3, \quad C_2 = \tau_2^2(2\gamma_1 a - 1) - \tau_2^4(2\gamma_1 a - 3) - 2\tau_2^6,$$
$$C_3 = \tau_2^3(\frac{3}{4} - 2\gamma_1 a + 2\gamma_1^2 a^2) + \tau_2^5(-\frac{23}{4} + 6\gamma_1 a - 2\gamma_1^2 a^2) + \tau_2^7(\frac{41}{4} - 4\gamma_1 a) - \frac{21}{4}\tau_2^9.$$

If we insert (4.1) and (4.2) into (3.13) and integrate, then we deduce, after inverting the Fourier transform and letting $N \to \infty$, that

$$K(t) = \frac{4\pi(b^2 + a^2)}{8\pi^2 t} \text{ sign } t + \frac{1}{6\pi^2}\{4\pi b^2(\frac{1}{b} - 3\gamma_2) + 4\pi a^2(\frac{1}{a} - 3\gamma_1)\}\text{sign } t + O(t \text{ sign } t), \quad \text{as } |t| \to 0. \tag{4.3}$$

From (2.6) and (4.3) we have the trace formula

$$\begin{aligned}\hat{\mu}(t) &= \frac{4\pi(b^3 - a^3)}{3(4\pi t)}\delta(-|t|) + \frac{4\pi(b^2 + a^2)}{8\pi^2 t} \text{ sign } t + \frac{1}{6\pi^2}\{4\pi b^2(\frac{1}{b} - 3\gamma_2) \\ &+ 4\pi a^2(\frac{1}{a} - 3\gamma_1)\} \text{ sign } t + O(t \text{ sign } t), \quad \text{as } |t| \to 0.\end{aligned} \tag{4.4}$$

The asymptotic expansion (4.4) may be interpreted as:

(i) D = spherical shell and we have the impedance boundary conditions (1.18) with small impedances γ_1 and γ_2; or

(ii) D = bounded domain has volume $V = \frac{4}{3}\pi(b^3 - a^3)$ and its surface area $S = 4\pi(b^2 + a^2)$ has the mean curvatures $(\frac{1}{b} - 3\gamma_2)$ and $(\frac{1}{a} - 3\gamma_1)$ together with Neumann boundary conditions on its boundaries.

We note that the formula (4.4) is in agreement with the formula (1.20) if $\gamma_1 = \gamma_2 = 0$.

Case 2. $\gamma_1 >> 1, 0 < \gamma_2 << 1$.

In this case, we deduce that the function $f_1(\nu;\eta)$ for large $|\eta|$ has the same form as (4.1) while $f_2(\nu;\eta)$ for large $|\eta|$ has the form

$$f_2(\nu;\eta) \sim \frac{(\nu^2 - 4\pi^2 a^2 \eta^2)^{1/2}}{4\pi^2 a^2 \eta^2} \sum_{n=0}^{\infty} \frac{C_n(\tau_2)}{\nu^n}, \tag{4.5}$$

where

$$C_0 = \frac{2(\tau_2^2 - 1)}{\gamma_1 a}, \qquad C_1 = -\tau_2 + \tau_2^3\left(1 - \frac{2}{\gamma_1 a}\right) + \frac{2\tau_2^5}{\gamma_1 a},$$

$$C_2 = -\frac{\tau_2^2}{4\gamma_1 a} - \tau_2^4\left(1 - \frac{19}{4\gamma_1 a}\right) + \tau_2^6\left(1 - \frac{43}{4\gamma_1 a}\right) + \frac{25}{4\gamma_1 a}\tau_2^8,$$

$$C_3 = -\tau_2^3\left(\frac{1}{4} - \frac{1}{2\gamma_1 a}\right) - \tau_2^5\left(-\frac{13}{4} + \frac{27}{2\gamma_1 a}\right) - \tau_2^7\left(\frac{27}{4} - \frac{107}{2\gamma_1 a}\right)$$
$$-\tau_2^9\left(-\frac{15}{4} + \frac{141}{2\gamma_1 a}\right) + \frac{30}{\gamma_1 a}\tau_2^{11}.$$

If we insert (4.1) and (4.5) into (3.13) and integrate, then we deduce, after inverting the Fourier transform and letting $N \to \infty$, that

$$K(t) = \frac{4\pi\{b^2 - (a^2 - 2a\gamma_1^{-1})\}}{8\pi^2 t} \text{ sign } t + \frac{1}{6\pi^2}\{4\pi b^2(\frac{1}{b} - 3\gamma_2) + 4\pi a\} \text{ sign } t + O(t \text{ sign } t), \quad \text{as } |t| \to 0. \tag{4.6}$$

From (2.6) and (4.6) we have the trace formula

$$\hat{\mu}(t) = \frac{4\pi(b^3 - a^3)}{3(4\pi t)}\delta(-|t|) + \frac{4\pi\{b^2 - (a^2 - 2a\gamma_1^{-1})\}}{8\pi^2 t} \text{ sign } t$$
$$+\frac{1}{6\pi^2}\{4\pi b^2(\frac{1}{b} - 3\gamma_2) + 4\pi a\} \text{ sign } t + O(t \text{ sign } t), \text{ as } |t| \to 0. \tag{4.7}$$

The asymptotic expansion (4.7) may be interpreted as:

(i) D = spherical shell and we have the impedance boundary conditions (1.18) with large impedance γ_1 and small impedance γ_2; or

(ii) D = bounded domain has volume $V = \frac{4\pi}{3}(b^3 - a^3)$ and its surface has the mean curvatures $(\frac{1}{b} - 3\gamma_2)$ and $\frac{1}{a}$. A part of this surface has area $4\pi b^2$ with Neumann conditions and the other part of the surface has area $4\pi(a^2 - 2a\gamma_1^{-1})$ with Dirichlet conditions.

We note that the formula (4.7) is in agreement with the formula (1.21) if $\gamma_1 \to \infty$ and $\gamma_2 = 0$.

Case 3. $0 < \gamma_1 << 1, \gamma_2 >> 1$.

In this case, we deduce that the function $f_2(\nu;\eta)$ for large $|\eta|$ has the same form as (4.2) while $f_1(\nu,\eta)$ for large $|\eta|$ has the form

$$f_1(\nu;\eta) \sim \frac{(\nu^2-4\pi^2b^2\eta^2)^{1/2}}{4\pi^3b^2\eta^2}\sum_{n=0}^{\infty}\frac{D_n(\tau_1)}{\nu^n}, \tag{4.8}$$

where

$$D_0=\frac{2(\tau_1^2-1)}{\gamma_2 b},\qquad D_1=\tau_1-\tau_1^3(1-\frac{2}{\gamma_2 b})-\frac{2\tau_1^5}{\gamma_2 b},$$

$$D_2=-\frac{\tau_1^2}{4\gamma_2 b}-\tau_1^4(1-\frac{19}{4\gamma_2 b})+\tau_1^6(1-\frac{43}{4\gamma_2 b})+\frac{25}{4\gamma_2 b}\tau_1^8,$$

$$D_3=-\tau_1^3(\frac{1}{4}-\frac{1}{2\gamma_2 b})-\tau_1^5(-\frac{13}{4}+\frac{27}{2\gamma_2 b})+\tau_1^7(\frac{27}{4}-\frac{107}{2\gamma_2 b})$$

$$+\tau_1^9(-\frac{15}{4}-\frac{141}{2\gamma_2 b})-\frac{30}{\gamma_2 b}\tau_1^{11}.$$

If we insert (4.2) and (4.8) into (3.13) and integrate, then we deduce, after inverting the Fourier transform and letting $N\to\infty$, that

$$K(t)=\frac{4\pi\{a^2-(b^2-2b\gamma_2^{-1})\}}{8\pi^2 t}\operatorname{sign} t+\frac{1}{6\pi^2}\{4\pi b+4\pi a^2(\frac{1}{a}-3\gamma_1)\}\operatorname{sign} t+O(t\operatorname{sign} t),\quad \text{as } |t|\to 0. \tag{4.9}$$

From (2.6) and (4.9) we have the trace formula

$$\hat{\mu}(t)=\frac{4\pi(b^3-a^3)}{3(4\pi t)}\delta(-|t|)+\frac{4\pi\{a^2-(b^2-2b\gamma_2^{-1})\}}{8\pi^2 t}\operatorname{sign} t$$
$$+\frac{1}{6\pi^2}\{4\pi b+4\pi a^2(\frac{1}{a}-3\gamma_1)\}\operatorname{sign} t+O(t\operatorname{sign} t),\quad \text{as } |t|\to 0. \tag{4.10}$$

The asymptotic expansion (4.10) may be interpreted as:

(i) D = spherical shell and we have the impedance boundary conditions (1.18) with small impedance γ_1 and large impedance γ_2; or

(ii) D = bounded domain has volume $V=\frac{4\pi}{3}(b^3-a^3)$ and its surface has the mean curvatures $\frac{1}{b}$ and $(\frac{1}{a}-3\gamma_1)$. A part of this surface has area $4\pi a^2$ with Neumann conditions and the other part of the surface has area $4\pi(b^2-2b\gamma_2^{-1})$ with Dirichlet conditions.

We note that the formula (4.10) is in agreement with the formula (1.22) if $\gamma_1=0$ and $\gamma_2\to\infty$.

Case 4. $\gamma_1,\gamma_2>>1$.

In this case, we deduce that the function $f_1(\nu;\eta)$ has the same form as (4.8) and the function $f_2(\nu;\eta)$ has the same form as (4.5). If we insert (4.5) and (4.8) into (3.13) and integrate , then we deduce, after inverting the Fourier transform and letting $N \to \infty$, that

$$K(t) = -\frac{4\pi\{b^2 + a^2 - 2(a\gamma_1^{-1} + b r_2^{-1})\}}{8\pi^2 t} \text{ sign } t + \frac{4\pi(b+a)}{6\pi^2} \text{ sign } t + O(t \text{ sign } t), \quad \text{as } |t| \to 0. \tag{4.11}$$

From (2.6) and (4.11), we have the trace formula

$$\begin{aligned}\hat{\mu}(t) = & \frac{4\pi(b^3 - a^3)}{3(4\pi t)}\delta(-|t|) - \frac{4\pi\{b^2 + a^2 - 2(a\gamma_1^{-1} + b\gamma_2^{-1})\}}{8\pi^2 t} \text{ sign } t \\ & + \frac{4\pi(b+a)}{6\pi^2} \text{ sign } t + O(t \text{ sign } t), \quad \text{as } |t| \to 0.\end{aligned} \tag{4.12}$$

The asymptotic expansion (4.12) may be interpreted as:

(i) D = spherical shell and we have the impedance boundary conditions (1.18) with large impedances γ_1 and γ_2; or

(ii) D = bounded domain has volume $V = \frac{4\pi}{3}(b^3 - a^3)$ and its surface area $S = 4\pi\{b^2 + a^2 - 2(a\gamma_1^{-1} + b\gamma_2^{-1})\}$ has the mean curvatures $\frac{1}{b}$ and $\frac{1}{a}$ together with Dirichlet boundary conditions on its boundaries.

Finally, we note that the formula (4.12) is in agreement with the formula (1.23) if $\gamma_1 = \gamma_2 \to \infty$.

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