FINITE DIMENSIONAL APPROXIMATION OF BRANCHES OF SOLUTIONS OF NONLINEAR PROBLEMS NEAR A CUSP POINT*

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Abstract

This paper presents some results on finite dimensional approximation of branches of solutions of nonlinear problems near a cusp point. These results can be applied to numerical methods of solving nonlinear differential equations.

1. Introduction

Consider nonlinear problems of the form

$$F(\lambda, u) = 0$$

where F is a sufficiently smooth function from $R \times V$ into V for some Banach space V. In [1]–[3], finite dimensional approximation of branches of solutions near a simple limit point and a simple bifurcation point were studied respectively. We will consider here the finite dimensional approximation of branches of solutions of problem (1.1) near a cusp point and obtain results similar to that of [3].

Section 2 is devoted to general analysis of the cusp point of branches of solutions of nonlinear problems. In Section 3 we discuss the finite dimensional approximation of branches of solutions near a cusp point of problem (1.1). In Section 4 we apply our results to the Galerkin approximations of nonlinear problems.

2. Local Analysis of the Continuous Problem Near a Cusp Point

Let V, W be real Banach spaces with the norm $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively and G be a C^p mapping from $R \times V$ into $W (p \geq 4)$ and T be a linear compact operator from W into V. We set

$$F(\lambda, u) = u + TG(\lambda, u). \tag{2.1}$$

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We assume that $(\lambda_0, u_0) \in R \times V$ is a simple critical point of F in the sense that

(i) $F^0 \equiv (\lambda_0, u_0) = 0$;

(ii) $D_u F^0 \equiv D_u F(\lambda_0, u_0) = I + TD_u G(\lambda_0, u_0) \in \mathcal{L}(V; V)$ is singular and -1 is an eigenvalue of the compact operator $TD_u G(\lambda_0, u_0)$ with the algebraic multiplicity 1;

(iii) $D_\lambda F^0 \equiv D_\lambda F(\lambda_0, u_0) \in$ Range $(D_u F^0)$.

We want to solve the equation

$$F(\lambda, u) = 0 \tag{2.3}$$

in a neighborhood of the simple critical point (λ_0, u_0).

As a consequence of (2.2) (ii) and the theory of linear operators; there exists $\varphi_0 \in V$ such that

$$\begin{aligned} &D_u F^0 \cdot \varphi_0 = 0, \quad \|\varphi_0\|_V = 1, \\ &V_1 \equiv \operatorname{Ker}(D_u F^0) = R \cdot \varphi_0. \end{aligned} \tag{2.4}$$

We denote by V' the dual space of V and by $< \cdot, \cdot >$ the duality pairing between the spaces V and V'. Then there exists $\varphi_0^* \in V'$ such that

$$\begin{aligned} &(D_u F^0)^* \cdot \varphi_0^* = 0, \quad < \varphi_0, \varphi_0^* >= 1, \\ &V_2 \equiv \text{ Range } (D_u F^0) = \{v \in V; < v, \varphi_0^* >= 0\}. \end{aligned} \tag{2.5}$$

Finally, we have

$$V = V_1 \oplus V_2$$

and $D_u F^0$ is an isomorphism of V_2. We denote by $L = (D_u F^0|_{V_2})^{-1} \in \mathcal{L}(V_2; V_2)$ the inverse isomorphism of $D_u F^0|_{V_2}$.

Let us define the projection operator $Q : V \to V_2$ by

$$Qv = v - \langle v, \varphi_0^* \rangle \varphi_0, \qquad \forall v \in V. \tag{2.6}$$

Then Eq. (2.3) is equivalent to the system

$$\begin{aligned} &QF(\lambda, u) = 0, \\ &(I - Q)F(\lambda, u) = 0. \end{aligned} \tag{2.7}$$

By the implicit function theorem, there exist two positive constants ξ_0, α_0 and a unique C^p mapping $V : [-\xi_0, \xi_0] \times [-\alpha_0, \alpha_0] \to V_2$ such that

$$\begin{aligned} &QF(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v(\xi, \alpha)) = 0, \\ &v(0, 0) = 0. \end{aligned} \tag{2.8}$$

Hence, solving Eq. (2.3) in a neighborhood of the critical point (λ_0, u_0) is equivalent to solving the bifurcation equation

$$f(\xi,\alpha) =< F(\lambda_0+\xi, u_0+\alpha\varphi_0+v(\xi,\alpha)), \varphi_0^* >= 0 \tag{2.9}$$

in neighborhood of the origin in R^2.

By calculations, we obtain

$$f(0,0) = \frac{\partial f}{\partial \xi}(0,0) = \frac{\partial f}{\partial \alpha}(0,0) = 0 \tag{2.10}$$

so that the origin is a critical point of the function f. We set

$$\frac{\partial^2 f}{\partial \xi^2}(0,0) = C_0, \quad \frac{\partial^2 f}{\partial \xi \partial \alpha}(0,0) = B_0, \quad \frac{\partial^2 f}{\partial \alpha^2}(0,0) = A_0 \tag{2.11}$$

where

$$\begin{aligned} A_0 &=< D^2_{uu}F^0 \cdot (\varphi_0)^2, \varphi_0^* >, \\ B_0 &=< D^2_{\lambda u}F^0\varphi_0 + D^2_{uu}F^0 \cdot (\varphi_0, -LD_\lambda F^0), \varphi_0^* >, \\ C_0 &=< D^2_{\lambda\lambda}F^0 + 2D^2_{\lambda u}F^0 \cdot (-LD_\lambda F^0) + D^0_{uu}F^0 \cdot (-LD_\lambda F^0)^2, \varphi_0^* > . \end{aligned} \tag{2.12}$$

In [3], the author discussed the case where (λ_0, u_0) is a simple bifurcation point of F, i.e. (λ_0, u_0) is a simple critical point of F satisfying

$$\Delta \equiv B_0^2 - A_0C_0 > 0. \tag{2.13}$$

For the case $\Delta < 0$, the author also have given some remarks.

Let us consider the case that (λ_0, u_0) is a simple critical point satisfying

$$\Delta = 0. \tag{2.14}$$

Now the origin is a degenerate critical point of f (see [4]–[6]).

By calculations, we have

$$D^2 f^0 = \begin{pmatrix} C_0 & B_0 \\ B_0 & A_0 \end{pmatrix}, \qquad \Delta = \det D^2 f^0. \tag{2.15}$$

From now on, we shall assume that

$$\mathrm{rank} D^2 f^0 = 1. \tag{2.16}$$

Then, from the matrix theory, there exist real eigenvalues $\lambda_1 = 0, \lambda_2 \neq 0$ and normalized orthogonal eigenvectors h_1, h_2 of $D^2 f^0$ corresponding to λ_1 and λ_2 respectively. Therefore, $H = (h_1\ h_2)$ is an orthogonal matrix such that

$$H^T D^2 f^0 H = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.17}$$

Lemma 1. *Assume that (2.16) and (2.17) hold. In addition, we assume that*

$$\delta \equiv \frac{\partial^3 f^0}{\partial \xi^3} h_{11}^3 + 3\frac{\partial^3 f^0}{\partial \xi^2 \partial \alpha} h_{11}^2 h_{12} + 3\frac{\partial^3 f^0}{\partial \xi \partial \alpha^2} h_{11} h_{12}^2 + \frac{\partial^3 f^0}{\partial \alpha^3} h_{12}^3 \neq 0 \tag{2.18}$$

where $h_1 = (h_{11}\ h_{12})$. Then, the set S of solutions of the bifurcation equation (2.9) in a neighborhood of the origin is diffeomorphic to a part of a semi-cubical parabola

$$y^2 = x^3.$$

Proof. It is easy to see that the set S of solutions of bifurcation equation (2.9) is diffeomorphic to the set of solutions of the equation

$$g(x_1, x_2) = f(\xi, \alpha) = 0, \tag{2.19}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = H^T \begin{pmatrix} \xi \\ \alpha \end{pmatrix}. \tag{2.20}$$

It follows from $Dg^0 = H^T Df^0 = 0$ that

$$D^2 g^0 = H^T D^2 f^0 H = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and $\frac{\partial^2 g^0}{\partial x_2^2} = 0$. Therefore, by the implicit function theorem, there exists a C^{p-1} function $x_2 = \varphi(x_1)$ near the origin such that

$$\frac{\partial g}{\partial x_2}(x_1, \varphi(x_1)) = 0$$

and

$$\varphi(0) = 0, \quad \varphi'(0) = 0.$$

Introducing diffeomorphic transformation of the coordinates

$$(\bar{x}_1, \bar{x}_2) = (x_1, x_2 - \varphi(x_1)),$$

we have

$$g(x_1, x_2) = g(\bar{x}_1, \varphi(\bar{x}_1)) + \bar{x}_2^2 \int_0^1 (1-s) \frac{\partial^2}{\partial x_2^2} g(\bar{x}_1, \varphi(\bar{x}_1) + s\bar{x}_2) ds.$$

By direct calculations we obtain

$$g^0 = \frac{dg^0}{d\bar{x}_1} = \frac{d^2 g^0}{d\bar{x}_1^2} = 0,$$

$$g(x_1, x_2) = \bar{x}_1^3 \psi_1(\bar{x}_1) + \bar{x}_2^2 \psi_2(\bar{x}_1, \bar{x}_2)$$

and

$$\psi_1(0) = \frac{1}{6}\frac{\partial^3 g^0}{\partial x_1^3} = \frac{\delta}{6} \neq 0,$$

$$\psi_2(0,0) = \frac{1}{2}\lambda_2 \neq 0.$$

Let

$$x = \bar{x}_1(\psi_1(\bar{x}_1))^{1/3}, \quad y = -(\ \mathrm{sgn}\ \frac{\partial^2 g^0}{\partial x_2^2})\bar{x}_2(|\psi_2(\bar{x}_1, \bar{x}_2)|)^{1/2}.$$

Then

$$g(x_1, x_2) = x^3 - y^2.$$

It is easy to check that the transformation $(x_1, x_2) \to (x, y)$ is a diffeomorphism. The proof is completed.

For convenience, we parametrize the branches of solutions of F.

We introduce parameter $t \in R$ and let

$$\lambda = \lambda_0 + \xi(t), \quad u = u_0 + \alpha(t)\varphi_0 + v(\xi(t), \alpha(t)). \tag{2.21}$$

Then, it follows from (2.19) and (2.20) that

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = H^T \begin{pmatrix} \xi(t) \\ \alpha(t) \end{pmatrix}.$$

Let

$$x_1(t) = t^2\sigma(t), \quad x_2(t) = t^3 a(t). \tag{2.22}$$

Introducing a C^{p-3} function $\mathcal{F} : (t, \sigma, a) \in [-1, 1] \times [-\xi_0, \xi_0] \times [-\alpha_0, \alpha_0] \to \mathcal{F}(t, \sigma, a) \in R^2$,

$$\begin{aligned} \mathcal{F}(t, \sigma, a) &= \begin{pmatrix} t^{-6} g(t^2\sigma, t^3 a) \\ \sigma^2 + a^2 - 1 \end{pmatrix}, \quad t \neq 0, \\ \mathcal{F}(0, \sigma, a) &= \lim_{t \to 0} \mathcal{F}(t, \sigma, a), \end{aligned} \tag{2.23}$$

we consider equation

$$\mathcal{F}(t, \sigma, a) = 0. \tag{2.24}$$

Lemma 2. *There exist $t_0 > 0$ and two pairs of C^{p-3} functions $[-t_0, t_0] \to (\sigma_i(t), a_i(t)) \in R^2, i = 1, 2$, such that*

$$\mathcal{F}(t, \sigma_i(t), a_i(t)) = 0.$$

Proof. We consider directly function g introduced by (2.19) and (2.20). It is easy to check that

$$g(t^2\sigma, t^3 a) = t^6(\frac{1}{6}\mu\sigma^3 + \frac{1}{2}\lambda_2 a^2) + O(t^6), \qquad t \to 0,$$

where $\mu = \frac{\partial^3 g^0}{\partial x_1^3} \neq 0, \lambda_2 = \frac{\partial^2 g^0}{\partial x_2^2} \neq 0$. Therefore, the function $\mathcal{F}$ is C^{p-3}. Let

$$\mathcal{F}(0, \sigma, a) = \begin{pmatrix} \frac{1}{6}\mu\sigma^3 + \frac{1}{2}\lambda_2 a^2 \\ \sigma^2 + a^2 - 1 \end{pmatrix} = 0. \tag{2.25}$$

It is easy to see that the solution of the first equation of (2.25) is a semi-cubical parabola and the solution of the second equation of (2.25) is a unit circle. Thus, (2.25) has two solutions $(\sigma_i^0, a_i^0), i = 1, 2$. Moreover,

$$D_{(\sigma,a)}\mathcal{F}(0, \sigma, a) = \begin{pmatrix} \frac{1}{2}\mu\sigma^2 & \lambda_2 a \\ 2\sigma & 2a \end{pmatrix},$$

$$\delta(\sigma, a) \equiv \det(D_{(\sigma,a)}\mathcal{F}(0, \sigma, a)) = \mu\sigma^2 a - 2\lambda_2\sigma a.$$

We see that solutions of $\delta(\sigma, a) = 0$ consist of three lines

$$\sigma = 0, \quad a = 0, \quad \sigma = \frac{2\lambda_2}{\mu}.$$

It follows from $\sigma_i^0 = -\left(\frac{3\lambda_2}{\mu}\right)^{\frac{1}{3}} (a_i^0)^{\frac{2}{3}}$ that the sign of σ_i^0 is contrary to the sign of $\frac{\lambda_2}{\mu}$ (see Figure 1). Hence,

$$\delta(\sigma_i^0, a_i^0) \neq 0.$$

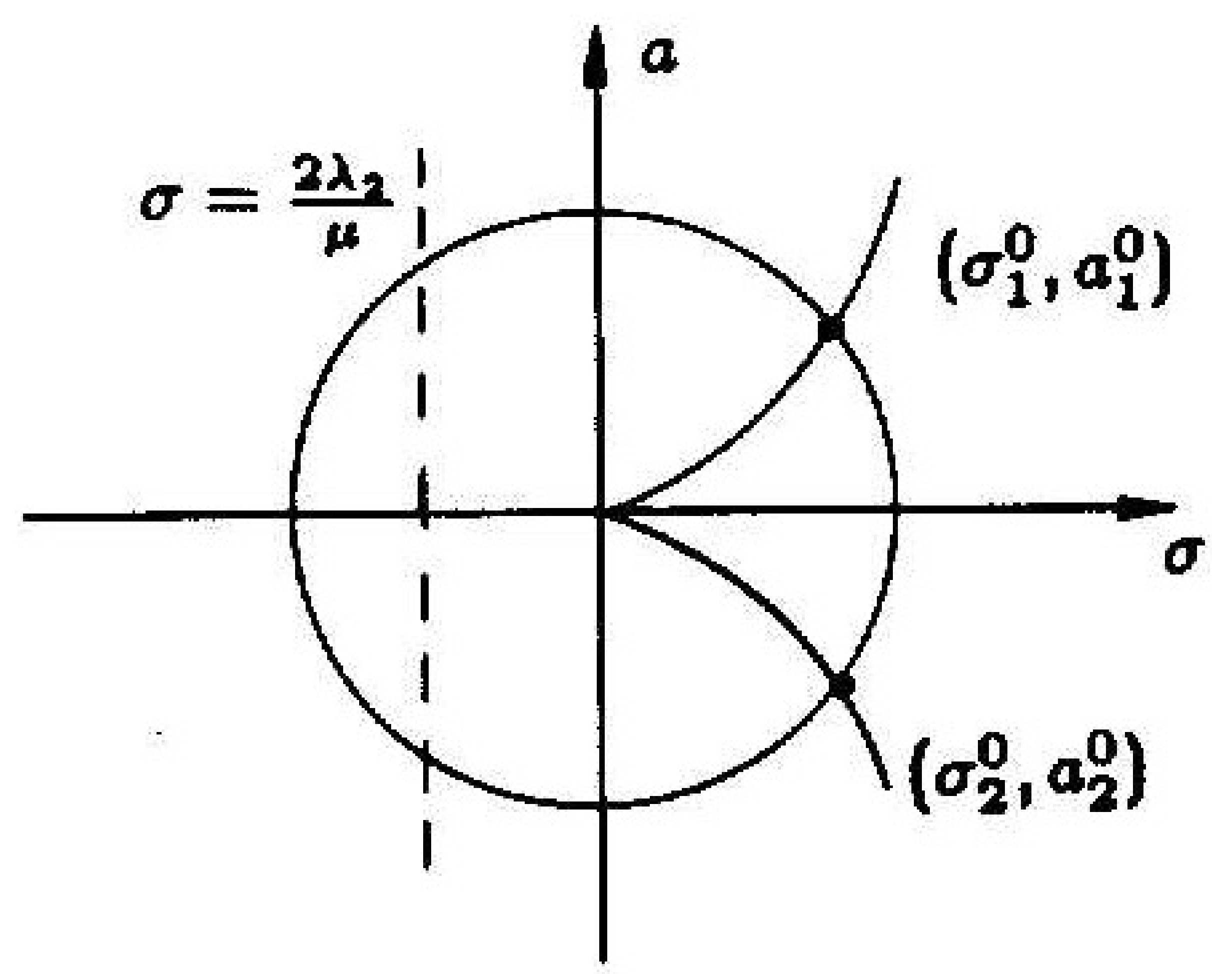

Fig.1. $\lambda_2\mu < 0$.

By the implicit function theorem, there exist $t_0 > 0$ and a unique C^{p-3} function $t \in [-t_0, t_0] \to (\sigma_i(t), a_i(t)) \in R^2$ such that

$$\mathcal{F}(t, \sigma_i(t), a_i(t)) = 0,$$

$$\sigma_i(0) = \sigma_i^0, a_i(0) = a_i^0.$$

The proof is completed.

Note that

$$\mathcal{F}(-t, \sigma, -a) = \mathcal{F}(t, \sigma, a)$$

and

$$\sigma_1^0 = \sigma_2^0, \quad a_1^0 = -a_2^0.$$

Therefore, by the uniqueness of the implicit function, we know that two branhes of solutions of equation (2.24) generate the same branch of solutions of equation (2.19). Let us denote this branch by

$$x_1(t) = t^2\sigma(t), \quad x_2(t) = t^3 a(t).$$

We call (λ_0, u_0) the cusp point of F if (λ_0, u_0) satisfies conditions (2.2), (2.16) and (2.18).

Remark 1. By direct calculations, we have

$$\begin{aligned}
\frac{\partial^3 f^0}{\partial \xi^3} &= \langle D^3_{\lambda\lambda\lambda}F^0 + 3D^3_{\lambda\lambda u}F^0 \cdot \frac{\partial v^0}{\partial \xi} + 3D^3_{\lambda uu}F^0 \cdot (\frac{\partial v^0}{\partial \xi})^2 + D^3_{uuu}F^0 \cdot (\frac{\partial v^0}{\partial \xi})^3 \\
&\quad + 3D^2_{uu}F^0 \cdot (\frac{\partial v^0}{\partial \xi}, \frac{\partial^2 v^0}{\partial \xi^2}) + 3D^2_{\lambda u}F^0 \cdot \frac{\partial^2 v^0}{\partial \xi^2}, \varphi_0^* \rangle, \\
\frac{\partial^3 f^0}{\partial \xi^2 \partial \alpha} &= \langle D^3_{\lambda\lambda u}F^0 \cdot \varphi_0 + 2D^3_{\lambda uu}F^0 \cdot (\varphi_0, \frac{\partial v^0}{\partial \xi}) + 2D^2_{\lambda u}F^0 \cdot \frac{\partial^2 v^0}{\partial \xi \partial \alpha} + D^3_{uuu}F^0 \cdot ((\frac{\partial v^0}{\partial \xi})^2, \varphi_0) \\
&\quad + 2D^2_{uu}F^0 \cdot (\frac{\partial v^0}{\partial \xi}, \frac{\partial^2 v^0}{\partial \xi \partial \alpha}) + D^2_{uu}F^0 \cdot (\varphi_0, \frac{\partial^2 v^0}{\partial \xi^2}), \varphi_0^* \rangle, \\
\frac{\partial^3 f^0}{\partial \xi \partial \alpha^2} &= \langle D^3_{\lambda uu}F^0 \cdot (\varphi_0)^2 + D^3_{uuu}F^0 \cdot (\varphi_0^2, \frac{\partial v^0}{\partial \xi}) + D^2_{\lambda u}F^0 \cdot \frac{\partial^2 v^0}{\partial \alpha^2} + D^2_{uu}F^0 \cdot (\frac{\partial^2 v^0}{\partial \alpha^2}, \frac{\partial v^0}{\partial \xi}) \\
&\quad + 2D^2_{uu}F^0 \cdot (\varphi_0, \frac{\partial^2 v^0}{\partial \xi \partial \alpha}), \varphi_0^* \rangle, \\
\frac{\partial^3 f^0}{\partial \alpha^3} &= \langle D^3_{uuu}F^0 \cdot (\varphi_0)^3 + 3D^2_{uu}F^0 \cdot (\varphi_0, \frac{\partial^2 v^0}{\partial \alpha^2}), \varphi_0^* \rangle,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial v^0}{\partial \xi} &= -LQD_\lambda F^0, \\
\frac{\partial^2 v^0}{\partial \xi^2} &= LQ\big[D^2_{\lambda u}F^0 \cdot LQD_\lambda F^0 - D^2_{uu}F^0 \cdot (LQD_\lambda F^0)^2 - D^2_{\lambda\lambda}F^0 + D^2_{\lambda u}F^0 \cdot LQD_\lambda F^0\big], \\
\frac{\partial^2 v^0}{\partial \xi \partial \alpha} &= LQD^2_{uu}F^0 \cdot (\varphi_0, LQD_\lambda F^0) - LQD^2_{\lambda u}F^0 \cdot \varphi_0, \\
\frac{\partial^2 v^0}{\partial \alpha^2} &= -LQD^2_{uu}F^0 \cdot (\varphi_0)^2.
\end{aligned}$$

3. Finite Dimensional Approximation

In this section, we discuss the finite dimensional approximation of the problem (2.3). Let $\{V_h\}_{h>0}$ be a family of finite dimensional subspaces of V, and T_h be an approximate operator of operator $T, T_h \in \mathcal{L}(W : V_h)$. Set

$$F_h(\lambda, u) = u + T_h G(\lambda, u), \qquad \lambda \in R,\ u \in V. \tag{3.1}$$

Then, we consider the approximate problem:

Find $(\lambda, u_h) \in R \times V_h$, such that

$$F_h(\lambda, u_h) = 0. \tag{3.2}$$

If $(\lambda, u) \in R \times V$ satisfies $F_h(\lambda, u) = 0$, it follows from $u = -T_h G(\lambda, u) \in V_h$ that we can equivalently find the solution of equation (3.2) in $R \times V$.

Just as in the above section, (3.2) is equivalent to the system

$$\begin{aligned} QF_h(\lambda, u) &= 0, \\ (I - Q)F_h(\lambda, u) &= 0. \end{aligned} \tag{3.3}$$

We assume that $D^p G$ is bounded in all bounded subsets of $R \times V, p \geq 4$ (3.4) and

$$\lim_{h \to 0} \|T - T_h\|_{\mathcal{L}(W;V)} = 0. \tag{3.5}$$

Let

$$\lambda = \lambda_0 + \xi, \quad u_h = u_0 + \alpha\varphi_0 + v_h. \tag{3.6}$$

Then we have

Lemma 3. *Assume that (2.2) (i), (ii) and (3.4), (3.5) hold. Then there exist three positive constants ξ_0, α_0, a and , for $0 < h \leq h_0$ small enough, a unique C^p mapping $v_h : (\xi, \alpha) \in R(\xi_0, \alpha_0) \to v_h(\xi, \alpha) \in V_2$ such that*

$$\begin{aligned} &QF_h(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v_h(\xi, \alpha)) = 0, \\ &\|v_h(\xi, \alpha) - v(\xi, \alpha)\|_V \leq a \end{aligned} \tag{3.7}$$

where $R(\xi_0, \alpha_0) = [-\xi_0, \xi_0] \times [-\alpha_0, \alpha_0]$. Moreover, we estimates

$$\begin{aligned} &\text{(i)} \quad \|D^m v_h(\xi^*, \alpha^*) - D^m v(\xi, \alpha)\|_{\mathcal{L}_l(R^2;V)} \\ &\qquad \leq K \cdot \Big\{|\xi^* - \xi| + |\alpha^* - \alpha| + \textstyle\sum_{l=0}^{m} \|(T - T_h)D^l J(\xi, \alpha)\|_{\mathcal{L}_l(R^2;V)}\Big\}, \end{aligned} \tag{3.8}$$

$$\text{(ii)} \quad \|D^p v_h(\xi^*, \alpha^*)\|_{\mathcal{L}_p(R^2;V)} \leq K$$

where $(\xi^*, \alpha^*), (\xi, \alpha) \in R(\xi_0, \alpha_0), 0 \leq m \leq p - 1, \mathcal{L}_l(R^2; V)$ is the space of all continuous l-linear mappings from R^2 into V and $J(\xi, \alpha) = G(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v(\xi, \alpha)), v(\xi, \alpha)$ is definded by (2.8) and K is a constant independent of h.

Proof. See Theorem 2 in [2].

By Lemma 3, we can say that solving (3.2) in some neighborhood of (λ_0, u_0) is equivalent to solving an approximate bifurcation equation

$$f_h(\xi, \alpha) = \langle F_h(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v_h(\xi, \alpha)), \varphi_0^* \rangle = 0 \tag{3.9}$$

in some neighborhood of the origin in R^2.

Introducing transformation (2.20) and setting

$$g_h(x_1, x_2) = f_h(\xi, \alpha),$$

we have

Lemma 4. *Assume the hypotheses of Lemma 3. Then, we have*

(i) $$\|D^m g_h(x_1^*, x_2^*) - D^m g(x_1, x_2)\|_{\mathcal{L}_m(\mathbb{R}^2;\mathbb{R})} \le K\Big\{|x_1^* - x_1| + |x_2^* - x_2| + \sum_{l=0}^{m} \|(T - T_h)D^l J(\xi, \alpha)\|_{\mathcal{L}_l(\mathbb{R}^2;V)}\Big\}, \quad (3.10)$$

(ii) $\|D^p g_h(x_1, x_2)\|_{\mathcal{L}_p(\mathbb{R}^2;\mathbf{R})} \le K$,

where $(x_1^*, x_2^*), (x_1, x_2) \in R(x_1^0, x_2^0), 0 \le m \le p-1$ *and* K, x_1^0, x_2^0 *are positive constants independent of* h.

Proof. Since

$$g(x_1, x_2) = f \circ \psi(x_1, x_2),$$
$$g_h(x_1, x_2) = f_h(\xi, \alpha) = f_h \circ \psi(x_1, x_2),$$

where $\psi(x_1, x_2) : R^2 \to R^2$,

$$\psi(x_1, x_2) = H\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

by the chain rule and the product formula (see [7]), we have

$$\|D^m g_h(x_1^*, x_2^*) - D^m g(x_1, x_2)\|_{\mathcal{L}_m(\mathbb{R}^2;\mathbb{R})} \le C\|D^m f_h(\xi^*, \alpha^*) - D^m f(\xi, \alpha)\|_{\mathcal{L}_m(\mathbb{R}^2;\mathbb{R})},$$

$$\|D^p g_h(x_1, x_2)\|_{\mathcal{L}_p(\mathbb{R}^2;\mathbb{R})} \le C\|D^p f_h(\xi, \alpha)\|_{\mathcal{L}_p(\mathbb{R}^2;\mathbb{R})}.$$

The proof is completed by Lemma 4 of [2] and (2.20).

Let $t \to (x_1(t), x_2(t))$ and $t \to (x_{1h}^*(t), x_{2h}^*(t))$ defined for $|t| \le t_0$ be a pair of C^r functions, $0 \le r \le p$, which satisfy, for $0 < h \le h_0$,

$$\sup_{|t|\le t_0} |x_1(t)| \le x_1^0, \qquad \sup_{|t|\le t_0} |x_{1h}^*(t)| \le x_1^0,$$

$$\sup_{|t|\le t_0} |x_2(t)| \le x_2^0, \qquad \sup_{|t|\le t_0} |x_{2h}^*(t)| \le x_2^0.$$

Then, we have

Lemma 5. *Assume the hypotheses of Lemma 3. Assume in addition that*

$$\text{(i)} \lim_{h\to 0}\sup_{|t|\le t_0}\left\{\left|\frac{d^m}{dt^m}(x_{1h}^*(t)-x_1(t))\right|+\left|\frac{d^m}{dt^m}(x_{2h}^*(t)-x_2(t))\right|\right\}=0,$$

$$\text{(ii)} \sup_{|t|\le t_0}\left\{\left|\frac{d^r}{dt^r}x_{1h}^*(t)\right|+\left|\frac{d^r}{dt^r}x_{2h}^*(t)\right|\right\}\le C. \tag{3.11}$$

Then, we have

$$\left|\frac{d^m}{dt^m}(g_h(X_{1h}^*(t),x_{2h}^*(t))-g(x_1(t),x_2(t)))\right|$$
$$\le K\sum_{l=0}^{m}\left\{\left|\frac{d^l}{dt^l}(x_{1h}^*(t)-x_1(t))\right|+\left|\frac{d^l}{dt^l}(x_{2h}^*(t)-x_2(t))\right|+\|(T-T_h)\frac{d^l}{dt^l}J(\xi(t),\alpha(t))\|_V\right\}, \tag{3.12}$$

where $0\le m\le r-1, |t|\le t_0, K$ *is a constant independent of* h *and* $(\xi(t),\alpha(t))=(x_1(t),x_2(t))H^T$.

Proof. It is similar to the proof of Lemma 4.

Lemma 6. *Assume the hypotheses of all above lemmas. Then, there exist a constant* $\beta_0>0$ *and, for* $0<h\le h_0$ *small enough, a unique point* $(x_{1h}^0,x_{2h}^0)\in R(\beta_0,\beta_0)$ *such that*

$$\frac{\partial g_h(x_{1h}^0,x_{2h}^0)}{\partial x_2}=0,\quad \frac{\partial^3 g_h(x_{1h}^0,x_{2h}^0)}{\partial x_1^3}\ne 0, \tag{3.13}$$

$$\operatorname{rank}D^2g_h(x_{1h}^0,x_{2h}^0)=1 \tag{3.14}$$

and

$$|x_{1h}^0|+|x_{2h}^0|\le K\Big(\sum_{l=0}^{2}\|(T-T_h)D^lJ(0,0)\|_{\mathcal{L}_l(R^2;V)}\Big) \tag{3.15}$$

where K *is a constant independent of* h.

Proof. Let

$$\phi(x_1,x_2)=\frac{\partial g}{\partial x_2}(x_1,x_2),\quad \phi_h(x_1,x_2)=\frac{\partial g_h}{\partial x_2}(x_1,x_2).$$

By Lemma 4, we have

$$D^m\phi_h(x_1,x_2)\to D^m\phi(x_1,x_2),\quad h\to 0,\quad 0\le m\le p-2$$

uniformly in some neighborhood of the origin of R^2. Since

$$\phi(0,0)=0,\quad \frac{\partial\phi}{\partial x_2}(0,0)=\frac{\partial^2 g^0}{\partial x_2^2}=\lambda_2\ne 0,$$

by the implicit function theorem and Th.1 of [2], there exist a constant $\beta_0>0$ and, for $0<h\le h_0$ small enough, a unique C^{p-1} function $x_1\to x_2(x_1),(x_1,x_2(x_1))\in R(\beta_0,\beta_0)$ a unique C^{p-1} function $x_1\to x_{2h}(x_1),(x_1,x_{2h}(x_1))\in R(\beta_0,\beta_0)$ such that

$$\phi(x_1,x_2(x_1))=0, x_2(0)=0, \frac{\partial\phi}{\partial x_2}(x_1,x_2(x_1))\ne 0,$$

and

$$\phi_h(x_1, x_{2h}(x_1)) = 0, \quad \frac{\partial \phi_h}{\partial x_2}(x_1, x_{2h}(x_1)) \neq 0$$

for $|x_1| \leq \beta_0$. Furthermore, we have

$$\left|\frac{d^m}{dx_1^m}(x_2(x_1) - x_{2h}(x_1^*))\right| \leq K\Big\{|x_1 - x_1^*| + \sum_{l=0}^{m+1} \|(T - T_h)D^l J(\xi, \alpha)\|_{\mathcal{L}_l(R^2;V)}\Big\} \tag{3.16}$$

wl re $0 \leq m \leq p-2$, and $(\xi, \alpha)^T = H(x_1, x_2(x_1))^T$.

Let

$$\psi(x_1) = \frac{\partial}{\partial x_1} g(x_1, x_2(x_1))$$

and

$$\psi_h(x_1) = \frac{\partial}{\partial x_1} g_h(x_1, x_{2h}(x_1)).$$

By direct calculations, we have

$$\psi'(x_1) = \Big[\frac{\partial^2}{\partial x_2^2} g(x_1, x_2(x_1))\Big]^{-1} \Delta(x_1, x_2(x_1)),$$

$$\psi_h'(x_1) = \Big[\frac{\partial^2}{\partial x_2^2} g_h(x_1, x_{2h}(x_1))\Big]^{-1} \Delta_h(x_1, x_{2h}(x_1)),$$

where

$$\Delta(x_1, x_2) = \det\{D^2 g(x_1, x_2)\}, \Delta_h(x_1, x_2) = \det\{D^2 g_h(x_1, x_2)\}.$$

Hence, we have

$$\psi'(0) = 0, \quad \psi''(0) = \frac{\partial^3}{\partial x_1^3} g(0,0) = \delta \neq 0.$$

It follows from Lemma 4 and (3.16) that, for some $\beta_0 > 0$, we have

$$\psi_h^{(m)}(x_1) \to \psi^{(m)}(x_1), \quad 0 \leq m \leq p-2, \quad \text{for } |x_1| \leq \beta_0 \text{ uniformly.}$$

By Th.1 of [2], there exist $h_0 > 0$ and, for $0 < h < h_0$, a unique point x_{1h}^0 such that

$$\psi_h'(x_{1h}^0) = 0, \quad |x_{1h}^0| \leq \beta_0 \tag{3.17}$$

and

$$|x_{1h}^0| \leq K|\psi_h'(0)| \leq K\|D^2 g_h(0,0)\|^2 \leq K\Big\{\sum_{l=0}^{2} \|(T - T_h)D^l J(0,0)\|^2_{\mathcal{L}_l(R^2;V)}\Big\}. \tag{3.18}$$

Let $x_{2h}^0 = x_{2h}(x_{1h}^0)$; then, by (3.16), (3.17) and (3.18), we can choose h_0 small enough such that, for $0 < h \leq h_0$,

$$\Delta_h^0 \equiv \Delta_h(x_{1h}^0, X_{2h}^0) = 0 \quad \text{and} \quad \frac{\partial^3}{\partial x_1^3} g_h(x_{1h}^0, x_{2h}^0) \neq 0.$$

Finally, we obtain (3.15). The proof is completed.

Let $D^m g_h^0 = D^m g_h(x_{1h}^0, x_{2h}^0), m = 0, 1, 2, \cdots$, and

$$K_0(h) = g_h^0, \quad K_1(h) = \frac{\partial}{\partial x_1} g_h^0. \tag{3.19}$$

In general, we have

$$K_0(h) \neq 0, \quad K_1(h) \neq 0,$$

that is, the point (x_{1h}^0, x_{2h}^0) is not a critical point of g_h.

Let us consider equation

$$\tilde{g}_h(x_1, x_2) \equiv g_h(x_1, x_2) - K_0(h) - K_1(h)(x_1 - x_{1h}^0) = 0. \tag{3.20}$$

Then, it is easy to check that

$$\tilde{g}_h^0 = 0, \quad D\tilde{g}_h^0 = 0 \quad \text{and} \quad D^m \tilde{g}_h^0 = D^m g_h^0, \quad m \geq 2.$$

Using Lemma 6, we know

$$Q_h^T D^2 g_h^0 Q_h = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2^h \end{pmatrix}.$$

for some orthogonal matrix $Q_h \in \mathcal{L}(R^2; R^2)$. By Lemmas 4 and 6, we can choose Q_h such that (see [8])

$$\lambda_2^h \to \lambda_2, \quad \|Q_h - I\|_{\mathcal{L}(R^2;R^2)} \to 0, \quad \text{as } h \to 0$$

and

$$|\lambda_2^h - \lambda_2| + \|Q_h - I\|_{\mathcal{L}(R^2;R^2)} \leq K\|D^2 g_h^0 - D^2 g^0)\|_{\mathcal{L}(R^2;R^2)}. \tag{3.21}$$

Let

$$\begin{pmatrix} \tilde{x}_1(t,\sigma,a) \\ \tilde{x}_2(t,\sigma,a) \end{pmatrix} = Q_h \begin{pmatrix} t^2\sigma \\ t^3 a \end{pmatrix}, \tag{3.22}$$

$$\begin{aligned} \mathcal{F}_h(t,\sigma,a) &= \begin{pmatrix} t^{-6}\tilde{g}_h(x_{1h}^0 + \tilde{x}_1(t,\sigma,a), x_{2h}^0 + \tilde{x}_2(t,\sigma,a)) \\ \sigma^2 + a^2 - 1 \end{pmatrix}, \quad t \neq 0, \\ \mathcal{F}_h(0,\sigma,a) &= \lim_{t\to 0} \mathcal{F}_h(t,\sigma,a). \end{aligned} \tag{3.23}$$

Lemma 7. *Assume the hypotheses of all above lemmas. Then, there exist a positive constant t_0 and, for $0 < h \leq h_0$ small enough, a unique pair of C^{p-3} functions $t \in [-t_0, t_0] \to (\sigma_h(t), a_h(t)) \in R^2$, such that*

$$\mathcal{F}_h(t, \sigma_h(t), a_h(t)) = 0, \quad for |t| \leq t_0$$

and

$$\left|\frac{d^m}{dt^m}(\sigma_n(t)-\sigma(t))\right| + \left|\frac{d^m}{dt^m}(a_h(t)-a(t))\right|$$
$$\le K\Big\{\sum_{l=0}^{2}\|(T-T_h)D^l J(0,0)\|_{\mathcal{L}(R^2;V)} + \sup_{|t|\le t_0}\sum_{l=0}^{m+3}\|(T-T_h)D^l J(\xi(t),\alpha(t))\|_v\Big\} \tag{3.24}$$

where $0 \le m \le p-4$ and $K > 0$ is a constant independent of h.

Proof. By (2.23) and (3.23), we have

$$\mathcal{F}(t,\sigma,a) = \begin{pmatrix} \lambda_2 a^2 + \dfrac{1}{2}\displaystyle\int_0^1 (1-s)^2 D^3 g(st^2\sigma, st^3 a)\cdot\begin{pmatrix}\sigma\\ ta\end{pmatrix}^3 ds \\ \sigma^2+a^2-1 \end{pmatrix}, \tag{3.25}$$

$$\mathcal{F}_h(t,\sigma,a) = \begin{pmatrix} \lambda_2^h a^2 + \dfrac{1}{2}\displaystyle\int_0^1 (1-s)^2 D^3 g_h(x_{1h}^0 + s\tilde{x}_1(t), x_{2h}^0 + s\tilde{x}_2(t))\left(Q_h\begin{pmatrix}\sigma\\ ta\end{pmatrix}\right)^3 ds \\ \sigma^2+a^2-1 \end{pmatrix}. \tag{3.26}$$

From the proof of Lemma 2, we can see that

$$\det\left(D_{(\sigma,a)}\mathcal{F}(0,\sigma(0),a(0))\right) \ne 0 \quad .$$

Therefore, for $t_0 > 0$ small enough, we have

$$\|D_{(\sigma,a)}\mathcal{F}(t,\sigma(t),a(t))^{-1}\|_{\mathcal{L}(R^2;R^2)} \le C, \quad \text{for} \quad |t| \le t_0 \quad .$$

By calculations, we obtain

$$\|D^m\mathcal{F}_h(t,\sigma,a) - D^m\mathcal{F}(t,\sigma,a)\|_{\mathcal{L}_m(R^2;R^2)}$$
$$\le K\{|\lambda_2^h - \lambda_2| + \|Q_h - I\|_{\mathcal{L}(R^2;R^2)} + \sup_{(x_1,x_2)\in U}\|D^{m+3}g - D^{m+3}g_h\|_{\mathcal{L}_{m+3}(R^2;R^2)}\}.$$

Hence, by Th.1 of [2], there exists a unique pair of C^{p-3} functions $t \to (\sigma_h(t), a_h(t))$, $|t| \le t_0$, satisfying

$$\mathcal{F}_h(t,\sigma_h(t),a_h(t)) = 0$$

and

$$\left|\frac{d^m}{dt^m}(\sigma_h(t)-\sigma(t))\right|+\left|\frac{d^m}{dt^m}(a_h(t)-a(t))\right|\le C\sum_{l=0}^{m}\left\|\frac{d^l}{dt^l}(\mathcal{F}(t,\sigma(t),a(t))-\mathcal{F}_h(t,\sigma(t),a(t)))\right\|_{R^2}$$

$$\le K\Big\{|\lambda_2^h-\lambda_2|+\|Q_h-I\|_{\mathcal{L}(R^2;R^2)}+\sum_{l=0}^{m+3}\sup_{|t|\le t_0}\|D^l g(t^2\sigma(t),t^3a(t))$$

$$-D^l g_h(x_{1h}^0+\tilde{x}_1(t),x_{2h}^0+\tilde{x}_2(t))\|_{\mathcal{L}_l(R^2;R)}\Big\}$$

$$\le K\Big\{\sum_{l=0}^{2}\|(T-T_h)D^lJ(0,0)\|_{\mathcal{L}_l(R^2;V)}+\sup_{|t|\le t_0}\sum_{l=0}^{m+3}\|(T-T_h)D^lJ(\xi(t),\alpha(t))\|_{\mathcal{L}_l(R^2;V)}\Big\}.$$

The proof is completed.

Furthermore, we have

Lemma 8. *Under the hypotheses of Lemma 7, there exist positive constants t_0, a and, for $0<h\le h_0$ small enough, a unique branch $\{(\tilde{x}_{1h}(t),\tilde{x}_{2h}(t)), |t|\le t_0\}$ of the solutions of (3.20) such that*

$$|\tilde{x}_{1h}(t)-x_1(t)|+|\tilde{x}_{2h}(t)-x_2(t)|\le a,$$

$$x_{1h}(0)=x_{1h}^0,\quad x_{2h}(0)=x_{2h}^0$$

and

$$\left|\frac{d^m}{dt^m}(\tilde{x}_{1h}(t)-x_1(t))\right|+\left|\frac{d^m}{dt^m}(\tilde{x}_{2h}(t)-x_2(t))\right|$$
$$\le K\sup_{|t|\le t_0}\sum_{l=0}^{m+3}\|(T-T_h)D^lJ(\xi(t),\alpha(t))\|_{\mathcal{L}_l(R^2;V)} \tag{3.27}$$

where K is a constant independent of h.

Proof. Let

$$\begin{pmatrix}\tilde{x}_{1h}(t)\\ \tilde{x}_{2h}(t)\end{pmatrix}=\begin{pmatrix}x_{1h}^0\\ x_{2h}^0\end{pmatrix}+Q_h\begin{pmatrix}t^2\sigma_h(t)\\ t^3a_h(t)\end{pmatrix}.$$

By Lemma 7 and (3.21), we have

$$\begin{pmatrix}x_{1h}(t)\\ x_{2h}(t)\end{pmatrix}-\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}x_{1h}^0\\ x_{2h}^0\end{pmatrix}+Q_h\begin{pmatrix}t^2(\sigma_h(t)-\sigma(t))\\ t^3(a_h(t)-a(t))\end{pmatrix}+(I-Q_h)\begin{pmatrix}t^2\sigma(t)\\ t^3a(t)\end{pmatrix}.$$

The proof is completed.

Moreover, we have also

Lemma 9. *Assume the hypotheses of the above lemmas. Then, there exist a neighborhood U of the origin in R^2 and, for $0<h\le h_0$ small enough, a diffeomorphism $j_h:(x_1,x_2)\in U\to(\bar{x}_1,\bar{x}_2)\in U$ suth that*

$$\begin{aligned}\tilde{g}_h(\bar{x}_1,\bar{x}_2)&=\tilde{g}_h(x_{1h}^0+x_1,x_{2h}^0+x_2)=\bar{x}_1^3-\bar{x}_2^2,\\ g_h(\bar{x}_1,\bar{x}_2)&=g_h(x_{1h}^0+x_1,x_{2h}^0+x_2)=\bar{x}_1^3-\bar{x}_2^2+K_1(h)\varphi_h(\bar{x}_1)+K_0(h)\end{aligned} \tag{3.28}$$

and

$$\max\Big\{\sup_{(x_1,x_2)\in U}\|Dj_n(x_1,x_2)\|_{\mathcal{L}(R^2;R,r^2)},\ \sup_{(x_1,x_2)\in U}\|Dj_h(x_1,x_2)^{-1}\|_{\mathcal{L}(R^2;R^2)}\Big\}\le C \tag{3.29}$$

where $|\varphi_h(x_1)|\le C$ *and* C *is a constant independent of* h.

Proof. By the proof of Lemma 6, we know that

$$\frac{\partial}{\partial x_2}\tilde g_h^0=0 \qquad \text{and} \qquad \Big|\frac{\partial^2}{\partial x_2^2}\tilde g_h^0\Big|\ge c_0>0$$

for $0<h\le h_0$ small enough. Hence there exists a smooth function $x_1\to i_h(x_1)$ such that

$$\frac{\partial}{\partial x_2}\tilde g_h(x_{1h}^0+x_1,x_{2h}^0+i_h(x_1))=0 \qquad \text{and} \qquad i_h(0)=0.$$

Let $\tilde x_1=x_1,\tilde x_2=x_2-i_h(\tilde x_1)$. Then

$$\tilde g_h(x_{1h}^0+x_1,x_{2h}^0+x_2)=\tilde g_h(x_{1h}^0+\tilde x_1,x_{2h}^0+\tilde x_2+i_h(\tilde x_1))$$

$$=\tilde g_h(x_{1h}^0+\tilde x_1,x_{2h}^0+i_h(\tilde x_1))+\tilde x_2^2\int_0^1(1-s)\frac{\partial^2}{\partial x_2^2}\tilde g_h(x_{1h}^0+\tilde x_1,x_{2h}^0+s\tilde x_2+i_h(\tilde x_1))ds.$$

Since

$$\tilde g_h^0=0,\quad \frac{d}{d\tilde x_1}\tilde g_h^0=\frac{\partial}{\partial x_1}\tilde g_h^0+\frac{\partial}{\partial x_2}\tilde g_h^0\cdot i_h'(0)=0,$$

$$\frac{d^2}{d\tilde x_1^2}\tilde g_h^0=\frac{\partial^2}{\partial x_1\partial x_2}\tilde g_h^0\cdot i_h'(0)+\frac{\partial^2}{\partial x_1^2}\tilde g_h^0=\Big(\frac{\partial^2}{\partial x_2^2}\tilde g_h^0\Big)^{-1}\cdot\Delta_h^0=0$$

and

$$D^m\tilde g_h\to D^m g$$

as $h\to 0$ uniformly in U which is some neighborhood of the origin in R^2 and

$$\frac{d^3}{d\tilde x_1^3}\tilde g_h^0\to\frac{\partial^3}{\partial x_1^3}g^0\ne 0,\qquad \text{as } h\to 0,$$

we have

$$\tilde g_h(x_{1h}^0+\tilde x_1,x_{2h}^0+i_h(\tilde x_1))=\tilde x_1^3\tilde\varphi_h(\tilde x_1),$$

where $|\tilde\varphi_h(\tilde x_1)|\ge\varepsilon_0>0$ for h and x small enough.

Let

$$\tilde\psi_h(\tilde x_1,\tilde x_2)=\int_0^1(1-s)\frac{\partial^2}{\partial x_2^2}\tilde g_h(x_{1h}^0+\tilde x_1,x_{2h}^0+s\tilde x_2+i_h(\tilde x_1))ds.$$

Then, we have

$$|\tilde\psi_h(\tilde x_1,\tilde x_2)|\ge\varepsilon_0>0,\qquad \text{for } (\tilde x_1,\tilde x_2)\in U.$$

Setting

$$\bar x_1=\tilde x_1(\tilde\varphi_h(\tilde x_1))^{1/3},\quad \bar x_2=-(\operatorname{sgn}\frac{\partial^2}{\partial x_2^2}g^0)\tilde x_2|\tilde\varphi_h(\tilde x_1,\tilde x_2)|^{1/2},$$

we can obtain

$$\tilde{g}_h(x_{1h}^0 + x_1, x_{2h}^0 + x_2) = \bar{x}_1^3 - \bar{x}_2^2.$$

Since

$$\bar{x}_1 = \tilde{x}_1(\tilde{\varphi}_h(\tilde{x}_1))^{1/3} = x_1(\tilde{\varphi}_h(x_1))^{1/3},$$

we have

$$\begin{aligned} \left|\frac{d}{dx_1}\bar{x}_1(x_1)\right| &= |\tilde{\varphi}_h(x_1)^{1/3} + x_1\frac{1}{3}\tilde{\varphi}_h'(x_1)\tilde{\varphi}_h(x_1)^{-2/3}| \\ &\geq \varepsilon_0^{1/3} - |x_1|\frac{1}{3}\cdot\varepsilon_0^{-2/3}|\tilde{\varphi}_h'(x_1)| \geq \varepsilon_1 > 0 \end{aligned}$$

and

$$\bar{x}_1(0) = 0.$$

By the inverse function theorem, we get a unique function $x_1 = \tilde{\varphi}_h^{-1}(\bar{x}_1)$ such that

$$g_h(x_{1h}^0 + x_1, x_{2h}^0 + x_2) = \bar{x}_1^3 - \bar{x}_2^2 + K_1(h)\varphi_h(\bar{x}_1) + K_0(h)$$

where $\varphi_h(\bar{x}_1) = \tilde{\varphi}_h^{-1}(\bar{x}_1) - x_{1h}^0$.

Finally. let

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = j_h(x_1, x_2) = \begin{pmatrix} \tilde{x}_1(\tilde{\varphi}_h(\tilde{x}_1))^{1/3} \\ -(\operatorname{sgn}\dfrac{\partial^2}{\partial x_2^2}g^0)\tilde{x}_2|\tilde{\psi}_h(\tilde{x}_1, \tilde{x}_2)|^{1/2} \end{pmatrix}.$$

We have some $r > 0$ independent of h such that

$$|\det Dj_h(x_1, x_2)| = \left|\det\frac{\partial(\bar{x}_1, \bar{x}_2)}{\partial(\tilde{x}_1, \tilde{x}_2)}\right|\cdot\left|\det\frac{\partial(\tilde{x}_1, \tilde{x}_2)}{\partial(x_1, x_2)}\right| \geq r > 0, \quad \text{for } (x_1, x_2) \in U.$$

Because each element of $Dj_h(x_1, x_2)$ is bounded uniformly in U, the proof is completed.

Let U be a neighborhood of the origin in R^2. We denote by S, S_h and $\tilde{S}_h$ the sets of the solutions of equations

$$g(x_1, x_2) = 0, \quad g_h(x_{1h}^0 + x_1, x_{2h}^0 + x_2) = 0$$

and

$$\tilde{g}_h(x_{1h}^0 + x_1, x_{2h}^0 + x_2) = 0$$

in U respectively. Similarly, we denote by $\bar{S}_h$ and $\bar{\bar{S}}_h$ the sets of the solutions of equations

$$g_h(\bar{x}_1, \bar{x}_2) = 0 \quad \text{and} \quad \tilde{g}_h(\bar{x}_1, \bar{x}_2) = 0$$

in U respectively. Introducing the distance between two bounded closed sets defined by

$$d(A, B) = \max\left\{\sup_{y\in B}\inf_{x\in A}\|x - y\|, \sup_{y\in A}\inf_{x\in B}\|x - y\|\right\},$$

we have

Lemma 10. *Assume the hypotheses of all above lemmas. Then, we have*

$$d(S_h,\tilde{S}_h)\le C\Big\{|K_0(h)|+|K_1(h)\ |\Big\}^{\frac{1}{3}},$$

where C is a constant independent of h.

Proof. First, we consider $d(\overline{S}_h,\overline{\overline{S}}_h)$. By the definition, we have

$$d(\overline{S}_h\overline{\overline{S}}_h)=\sup_{y\in\overline{\overline{S}}_h}\inf_{x\in S_h}\|x-y\|+\sup_{x\in S_h}\inf_{y\in\overline{\overline{S}}_h}\|x-y\|\le d_1+d_2$$

where

$$d_1=\sup_{|y_2|\le d_0^{1/3}}\Big(|\bar{x}_1(\bar{x}_2)-\bar{y}_1(\bar{y}_2)|\Big|_{x_2=y_2}\Big),$$

$$d_2=\sup_{0\le y_1\le d_0^{1/3}}\Big(|x_2(x_1)-y_2(y_1)|\Big|_{x_1=y_1}\Big),$$

$$\bar{S}_h=\{(\bar{x}_1(\bar{x}_2),\bar{x}_2)\}=\{(\bar{x}_1,\bar{x}_2(\bar{x}_1))\},$$

$$\overline{\overline{S}}_h=\{(\bar{y}_1(\bar{y}_2),\bar{y}_2)\}=\{(\bar{y}_1,\bar{y}_2(\bar{y}_1))\}$$

and d_0 is a positive constant which satisfies $\|x\|\le d_0$ for all $x\in U$.
Since $\bar{y}_1\ge 0$ for all $\bar{y}\in\overline{\overline{S}}_h$, we have (see Figure 2)

$$\begin{aligned}&|\bar{x}_1(\bar{x}_2)-\bar{y}_1(\bar{y}_2)|^3\\ &\le|(\bar{x}_1-\bar{y}_1)^2+3\bar{x}_1\bar{y}_1||\bar{x}_1-\bar{y}_1|\\ &\le|\bar{x}_1^3-\bar{y}_1^3|=|g_h(\bar{x}_1,\bar{x}_2)-\bar{g}_h(\bar{y}_1,\bar{y}_2)\\ &+K_0(h)\\ &+K_1(h)\varphi_h(\bar{x}_1)|\\ &\le C\{|K_0(h)|+K_1(h)|\}\end{aligned}$$

for $\bar{x}_1\ge 0$ and $\bar{x}_2=\bar{y}_2$, and

$$\begin{aligned}0>\bar{x}_1^3&=\bar{x}_2^2+K_0(h)+K_1(h)\varphi_h(\bar{x}_1)\\ &\ge -C\{|K_0(h)+|K_1(h)|\}\end{aligned}$$

for $\bar{x}_1<0$ and $\bar{x}_2=\bar{y}_2$. At the same time, we have

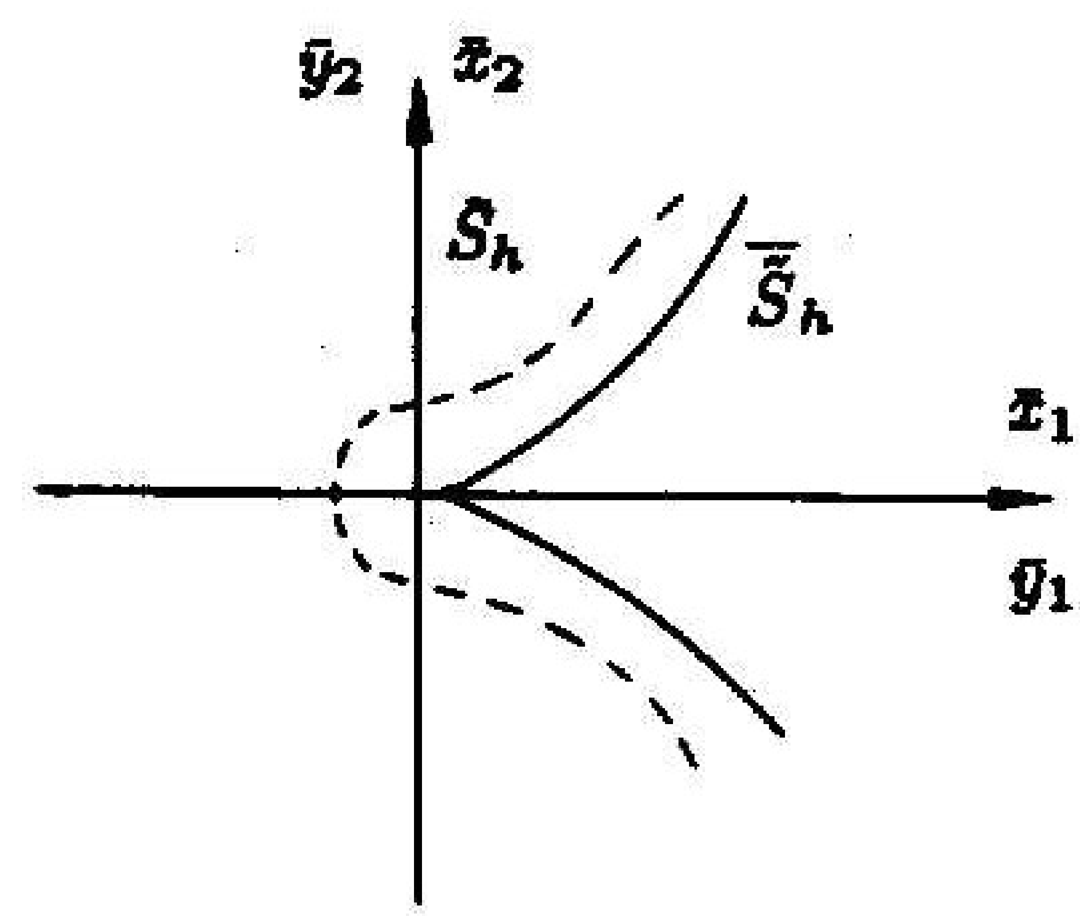

Fig. 2

$$\begin{aligned}\bar{y}_1^3=\bar{y}_2^2=\bar{x}_2^2&=\bar{x}_1^3-(K_0(h)+K_1(h)\varphi_h(\bar{x}_1))\\ &\le C\{|K_0(h)|+|K_1(h)|\}.\end{aligned}$$

To sum up, we have

$$d_1\le C\{|K_0(h)|+|K_1(h)|\}^{1/3}.$$

On the other hand, we can choose $\bar{x}_2(\bar{x}_1)$ so that the sign of $\bar{x}_2(\bar{x}_1)$ is the same as the sign of $\bar{y}_2(\bar{y}_1)$ for $\bar{x}_1=\bar{y}_1$. Thus, we have

$$|\bar{x}_2(\bar{x}_1)-\bar{y}_2(\bar{y}_1)|^2=|\bar{x}_2-\bar{y}_2|\cdot|\bar{x}_2-\bar{y}_2|\le|\bar{x}_2^2-\bar{y}_2^2|\le C\{|K_0(h)|+|K_1(h)|\},\qquad\text{for } \bar{x}_1=\bar{y}_1.$$

Hence, we can obtain

$$d_2 \le C\{|K_0(h)| + |K_1(h)|\}^{1/2}.$$

Finally, by $K_0(h), K_1(h) \to 0$ as $h \to 0$, we have

$$d(S_h, \bar{\tilde{S}}_h) \le C\{|K_0(h)| + |K_1(h)|\}^{1/3}.$$

Using (3.29), we complete the proof of the lemma.

With all lemmas above, we obtain

Theorem 1. *Assume that* (λ_0, u_0) *is a simple critical point of* F *satisfying* (2.16) *and* (2.18). *Assume in addition that* (3.4) $(p \ge 5)$ *and* (3.5) *hold. Then, there exist a neighborhood* O *of point* (λ_0, u_0) *in* $R \times V$ *and a positive constant* h_0 *such that, for* $0 < h \le h_0$, *the set* S_h *of solutions of equation* (3.2) *in* O *consists of only a branch of solutions. Furthermore, we have an estimate*

$$d(S_h, S) \le C\Big\{(|K_0(h)| + |K_1(h)|)^{1/3} \sup_{|t|\le t_0} \sum_{l=0}^{3} \|(T - T_h)D^l J(\xi(t), \alpha(t))\|_{\mathcal{L}_l(R^2;V)}\Big\} \quad (3.30)$$

where S *is the set of solutions of equation (2.3) in* O,

$$S = \{(\lambda(t), u(t)); |t| \le t_0\},$$
$$\lambda(t) = \lambda_0 + \xi(t), u(t) = u_0 + \alpha(t)\varphi_0 + v(\xi(t), \alpha(t))$$

and C *is a constant independent of* h.

Proof. Since

$$d(S_h, S) \le d(S_h, \tilde{S}_h) + d(\tilde{S}_h, S)$$

where S is the set of solutions of equation (2.9) in some neighborhood U of the origin in R^2, we can draw the conclusion from Lemmas 7–10. The proof is completed.

Remark 2. By Lemma 4 and Lemma 6, we can give the estimate for $K_0(h)$ and $K_1(h)$ as follows:

$$|K_0(h)| + |K_1(h)| \le C\Big\{\sum_{l=0}^{2} \|(T - T_h)D^l J(0,0)\|_{\mathcal{L}_l(R^2;V)}\Big\}.$$

Remark 3. If $K_0(h) = K_1(h) = 0$, the point (λ_{h0}, u_{h0}) is a cusp point of F_h, where

$$\lambda_{h0} = \lambda_0 + \xi_{h0}, \quad u_{h0} = u_0 + \alpha_{h0}\varphi_0 + v_h(\xi_{h0}, \alpha_{h0})$$

and

$$\begin{pmatrix} \xi_{h0} \\ \alpha_{h0} \end{pmatrix} = H \begin{pmatrix} x_{1h}^0 \\ x_{2h}^0 \end{pmatrix}.$$

Moreover, we have the estimate

$$\begin{aligned} |\lambda_0 - \lambda_{h0}| + \|u_{h0} - u_0\|_v &\le C\{|x_{1h}^0| + |x_{2h}^0| + \|v_h(\xi_{h0}, \alpha_{h0})\|_v\} \\ &\le C\Big\{\sum_{l=0}^{2} \|(T - T_h)D^l J(0,0)\|_{\mathcal{L}_l(R^2;V)}\Big\}. \end{aligned}$$

4. Application: Galerkin Approximation of A Nonlinear Problem

We consider a two-point boundary value problem of ODE

$$\begin{aligned} -u'' - u &= u^3 - c_0\lambda^2 \sin x, \quad 0 < x < \pi, \\ u(0) &= u(\pi) = 0 \end{aligned} \tag{4.1}$$

where $c_0 = \sqrt{2/\pi}$.

Let $V = H_0^1(0,\pi)$ and $W = L^2(0,\pi)$. We define a linear operator $L : H_0^1(0,\pi) \cap H^2(0,\pi) \subset V \to W$ by $Lu = -u''$. It is known that the inverse $T = L^{-1}$ of L is well defined and compact from W into V. Let $G(\lambda, u) = -u - u^3 + c_0\lambda^2 \sin x$. Then, the problem (4.1) is equivalent to equation

$$F(\lambda, u) = u + TG(\lambda, u) = 0 \tag{4.2}$$

and $G \in C^p(V; W), p \geq 5$.

Clearly, the point $(\lambda_0, u_0) = (0,0) \in R \times V$ is a solution of equation (4.2). Moreover, -1 is an eigenvalue of the compact operator TD_uG^0 with algebraic multiplicity 1 and $\varphi_0 = c_0$ $\sin x$ is a normalised orthogonal eigenfunction corresponding to -1. Because D_uF^0 is a self-adjoint operator, we have $\varphi_0^* = \varphi_0$ and $(D_uF^0)^*\varphi_0^* = 0$,

$$\langle \varphi_0, \varphi_0^* \rangle = \int_0^\pi c_0^2 \sin^2 x dx = 1.$$

We define the projection operator $Q : V \to V_2 = \{v \in V; < v, \varphi_0^* >= 0\}$ by

$$Qv = v - < v, \varphi_0^* > \varphi_0. \tag{4.3}$$

Then, equation (4.2) is equivalent to the system

$$QF(\lambda, u) = 0,$$

$$(I - Q)F(\lambda, u) = 0.$$

Let $\lambda = \lambda_0 + \xi = \xi, u = u_0 + \alpha\varphi_0 + v, v \in V_2$. By the implicit function theorem, there exist positive constants ξ_0, α_0 and a unique smooth mapping $v : [-\xi_0, \xi_0] \times [-\alpha_0, \alpha_0] \to v(\xi, \alpha) \in V_2$ such that

$$QF(\xi, \alpha\varphi_0 + v(\xi, \alpha)) = 0 \quad \text{and} \quad v(0,0) = 0. \tag{4.5}$$

By the system (4.4), we have the bifurcation equation

$$\begin{aligned} f(\xi, \alpha) &\equiv \langle F(\xi, \alpha\varphi_0 + v(\xi, \alpha)), \varphi_0^* \rangle \\ &= \xi^2 - c_1\alpha^3 + \text{ h.o.t.} \end{aligned} \tag{4.6}$$

where $c_1 = \int_0^\pi \varphi_0^4 dx$ and h.o.t. $= O(\xi^3 + \xi^2\alpha + \xi\alpha^2 + \alpha^4)$.

By calculations, we have

$$f(0,0) = f'_\xi(0,0) = f'_\alpha(0,0) = f''_{\xi\alpha}(0,0) = f''_{\alpha\alpha}(0,0) = 0,$$
$$f''_{\xi\xi}(0,0) = 2 \neq 0, \quad f'''_{\alpha\alpha\alpha}(0,0) = -6c_1 \neq 0. \tag{4.7}$$

Therefore, we obtain

$$\Delta = \det D^2 f^0 = 0 \quad \text{and} \quad \text{rank} D^2 f^0 = 1.$$

Furthermore, taking $h_1 = (h_{11}, h_{12})^T = (0,1)^T$, we can check that the conditions of Lemmas 1 and 2 hold. Thus, we know that $(\lambda_0, u_0) = (0,0)$ is a cusp point of F and the set of solutions of equation (4.2) in some neighborhood O of (λ_0, u_0) is diffeomorphic to a part of a semi-cubical parabola.

Now, let us consider Galerkin approximation of problem (4.1).

We introduce a continuous bilinear form $a : (u,v) \in V \times V \to a(u,v) \in R$ by

$$a(u,v) = (u', v') = \int_0^\pi u'v'dx.$$

As we know, a is V-elliptic in the sense that there exists a positive constant γ such that

$$a(u,u) \geq \gamma \|u\|_v^2 \quad \forall u \in V. \tag{4.8}$$

By [9], we also have

$$a(Tf, v) = a(v, T^* f) = (f, v), \quad \forall v \in V, \forall f \in V'. \tag{4.9}$$

The weak form of problem (4.1) is

Find $(\lambda, u) \in R \times V$ such that

$$a(u,v) + (G(\lambda, u), v) = 0, \quad \forall v \in V. \tag{4.10}$$

Let $\{V_h\}_{h>0}$ be a family of finite dimensional subspaces of V and

$$\lim_{h\to 0} \inf_{v_h \in V_h} \|v - v_h\|_v = 0, \quad \forall v \in V. \tag{4.11}$$

Galerkin approximation of problem (4.1) consists in finding solutions $(\lambda, u_h) \in R \times V_h$ of

$$a(u_h, v_h) + (G(\lambda, u_h), v_h) = 0, \quad \forall v_h \in V_h. \tag{4.12}$$

Let us define the operator $\Pi_h \in \mathcal{L}(V; V_h)$ and $T_h \in \mathcal{L}(W; V_h)$ by

$$a(\Pi_h u - u, v_h) = 0, \quad \forall v_h \in V_h, \forall u \in V$$

and

$$a(T_h f, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad \forall f \in V'$$

respectively. Clearly, we have

$$T_h = \Pi_h T$$

and equation (4.12) is equivalent to

$$F_h(\lambda, u_h) \equiv u_h + T_h G(\lambda, u_h) = 0. \tag{4.13}$$

From [9], we have

$$\lim_{h\to 0} \|v - \Pi_h v\|_v = 0, \quad \forall v \in V \tag{4.14}$$

and

$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(W;V)} = 0. \tag{4.15}$$

Using Thoerem 1, we obtain immediately

Theorem 2. *Assume that condition (4.11) holds. Then, there exist a neighborhood O of (λ_0, u_0) in $R \times V$ and a positive constant h_0 such that, for $0 < h \le h_0$, the set S_h of solutions of problem (4.12) in O consists of only a C^{p-3} branch of solutions and the set S_h is diffeomorphic to the curve defined by*

$$x^3 - y^2 + \alpha\varphi(x) + \beta = 0, \quad x^2 + y^2 \le c_0,$$

where α, β, c_0 are constants, $c_0 > 0$ and $\varphi(x)$ is a smooth function. Furthermore, we have the estimate

$$\begin{aligned} d(S_h, S) &\le C\Big\{(\|(I - \Pi_h)TDG^0\|_{\mathcal{L}(V;V)})^{1/3} \\ &\quad + \sup_{|t|\le t_0} \sum_{l=0}^{3} \|(I - \Pi_h)T\frac{d^l}{dt^l}G(\lambda(t), u(t))\|_{\mathcal{L}_l(R^2;V)}\Big\} \end{aligned}$$

where S is the set of solutions of problem (4.1) in O,

$$\begin{aligned} &S = \{(\lambda(t), u(t)); |t| \le t_0\}, \\ &\lambda(t) = \xi(t), \quad u(t) = \lambda(t)\varphi_0 + v(\xi(t), \alpha(t)) \end{aligned}$$

and C and t_0 are constants independent of h.

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