EXTRAPOLATION OF FINITE ELEMENT APPROXIMATION IN A RECTANGULAR DOMAIN*

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Abstract

Recently, the Richardson extrapolation for the elliptic Ritz projection with linear triangular elements on a general convex polygonal domain was discussed by Lin and Lu. We go back in this note to the simplest case, i.e. the bilinear rectangular elements on a rectangular domain which is a parallel case of the one-triangle model in the early work of Lin and Liu. We find that the finite element argument for the Richardson extrapolation with an accuracy of $O(h^4)$ needs only the regularity of $H^{4,\infty}$ for the solution u but the finite difference argument for extrapolation with $O(h^{3+\alpha})$ accuracy needs $u \in C^{5+\alpha}(0 < \alpha < 1)$. Moreover, a formula is suggested to guarantee the extrapolation of $O(h^4)$ accuracy at fine gridpoints as well as at coarse gridpoints.

1. Error expansion for rectangular elements

Let S be a square domain with a square mesh T^h of size $h, u \in H_0^1$ the solution of the Poisson equation with zero boundary condition, $S^h \subset H_0^1$ the piecewise bilinear finite element space over T^h, and $u^h \in S^h$ the Ritz projection defined by

$$(\nabla u^h, \nabla v) = (\nabla u, \nabla v), \quad \forall v \in S^h.$$

Let $u^I \in S^h$ be the interpolant of u.

If $u \in H^{4,\infty}$, we have

$$u^h - u^I = h^2\Phi + O(h^4|\log h|) \tag{1}$$

where the coefficient Φ is independent of h.

* Received June 9 , 1984.

Proof of (1) . By Lin and Liu [2], for a function E satisfying $E(\alpha) = E(\beta) = 0$, we have

$$\begin{aligned}\int_\alpha^\beta E dx &= -\frac{1}{12}(\beta-\alpha)^2\int_\alpha^\beta \partial_x^2 E dx - \frac{1}{6}\int_\alpha^\beta Q(x)\partial_x^3 E dx\\ &= -\frac{1}{12}(\beta-\alpha)^2\int_\alpha^\beta \partial_x^2 E dx + \frac{1}{24}\int_\alpha^\beta P(x)\partial_x^4 E dx\end{aligned}$$

with

$$Q(x) = (x-\alpha)(x-\beta)(x-\frac{\alpha+\beta}{2}),\quad P(x) = (x-\alpha)^2(x-\beta)^2,$$

and $P' = 4Q$.

Let $K \in T^h$ with sides 1 and 2 coinciding with x-direction. Then, for $v \in S^h$,

$$\begin{aligned}(\nabla(u^h - u^I), \nabla v) &= (\nabla(u-u^I), \nabla v)\\ &= \sum_K \int_K \partial_y(u-u^I)\partial_y v + \sum_K \int_K \partial_x(u-u^I)\partial_x v = \sum_K I(K) + \sum_K II(K),\\ I(K) &= (\int_1 - \int_2)(u-u^I)\partial_y v dx = -\frac{h^2}{12}(\int_1 - \int_2)\partial_x^2 u \partial_y v dx\\ &-\frac{1}{6}(\int_1 - \int_2)Q(x)(\partial_x^3 u\partial_y v + 3\partial_x^2 u \partial_{xy} v)dx = -\frac{h^2}{12}\int_K \partial_y\partial_x^2 u\partial_y v\\ &-\frac{1}{6}\int_K Q(x)(\partial_y\partial_x^3 u\partial_y v + 3\partial_y\partial_x^2 u\partial_{xy} v),\\ \sum I(K) &= \frac{h^2}{12}\int_S \partial_y^2\partial_x^2 u\cdot v - \frac{1}{6}\int_S Q(x)(\partial_y\partial_x^3 u\partial_y v - 3\partial_y^2\partial_x^2 u\partial_x v),\end{aligned}$$

and, similarly,

$$\sum II(K) = \frac{h^2}{12}\int_S \partial_y^2\partial_x^2 u\cdot v - \frac{1}{6}\int_S Q(y)(\partial_x\partial_y^3 u\partial_x v - 3\partial_x^2\partial_y^2 u\partial_y v).$$

Let $G_z^h \in S^h$ be the discrete Green function of G_z and $g_z \in H_0^1 \cap H^2$ the regular Green function defined by Frehse and Rannacher [1]. One has

$$\int_S |G_z^h - G_z| \le ch^2|\log h|,$$

$$\int_S |D(G_z^h - g_z)| \le ch|\log h|,$$

$$\int_S |D^2 g_z| \le c|\log h|.$$

Thus,

$$\begin{aligned}(u^h - u^I)(z) &= (\nabla(u^h - u^I), \nabla G_z^h) = \frac{h^2}{6}\int_S \partial_y^2\partial_x^2 u G_z \\ &-\frac{1}{6}\int_S Q(x)(\partial_y\partial_x^3 u\partial_y g_z - 3\partial_y^2\partial_x^2 u\partial_x g_z) \\ &-\frac{1}{6}\int_S Q(y)(\partial_x\partial_y^3 u\partial_x g_z - 3\partial_x^2\partial_y^2 u\partial_y g_z) + O(h^4|\log h|).\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}4\int_S Q(x)\partial_y\partial_x^3 u\partial_y g_z &= -\int_S P(x)(\partial_y\partial_x^4 u\partial_y g_z + \partial_y\partial_x^3 u\partial_{yx} g_z) \\ &= \int_S P(x)(\partial_x^4 u\partial_{yy} g_z - \partial_y\partial_x^3 u\partial_{yx} g_z), \\ 4\int_S Q(x)\partial_y^2\partial_x^2 u\partial_x g_z &= \int_S P(x)(\partial_y\partial_x^3 u\partial_{xy} g_z - \partial_y^2\partial_x^2 u\partial_{xx} g_z). \\ &\cdots\end{aligned}$$

Finally, we obtain (1) with

$$\Phi(z) = \frac{1}{6}\int_S \partial_y^2\partial_x^2 u G_z dxdy.$$

2. Extrapolation at fine gridpoints

Let z be the angle points of K. Then, by (1),

$$(u^h - u)(z) = h^2\Phi(z) + O(h^4|\log h|). \tag{2}$$

Let z_{12} be the middle point of two adjacent angle points z_1 and z_2. Define

$$\bar{u}^h(z_{12}) = u^{h/2}(z_{12}) + \frac{1}{6}(u^{h/2} - u^h)(z_1) + \frac{1}{6}(u^{h/2} - u^h)(z_2).$$

We have

$$(\bar{u}^h - u)(z_{12}) = O(h^{4-\epsilon}). \tag{3}$$

Proof of (3). By (2),

$$\Phi(z_i) = \frac{4}{3h^2}(u^h - u^{h/2})(z_i) + O(h^2|\log h|), \quad i = 1, 2.$$

Note from the error estimate for the linear interpolation of Φ that

$$\Phi(z_{12}) = \frac{1}{2}\Phi(z_1) + \frac{1}{2}\Phi(z_2) + O(h^{2-\epsilon}),$$

we have

$$\Phi(z_{12})=\frac{2}{3h^2}(u^h-u^{h/2})(z_1)+\frac{2}{3h^2}(u^h-u^{h/2})(z_2)+O(h^{2-\epsilon}). \tag{4}$$

But

$$u^{h/2}(z_{12})=u(z_{12})+\frac{h^2}{4}\Phi(z_{12})+O(h^4|\log h|).$$

Thus, (3) follows from (4).

Appendix

Let us recall the early work [3] where the error expansion for linear triangular elements on a right triangular domain was considered. The proof in [3] and the proof in Section 1 of the present paper are similar. We now explain it as follows.

Let K and K' be two adjacent triangular elements with common side coinciding with x-direction and other sides 1, 2, 3, 4 coinciding with

$$y=h-x,\qquad y=-x,\qquad x=h,\qquad x=0$$

respectively. Noting that

$$\partial_x v=\ \text{const in } K\cup K'$$

we have

$$\int_{K\cup K'}\partial_x(u-u^I)\partial_x v=\partial_x v\int_{K\cup K'}\partial_x(u-u^I).$$

By Green's formula

$$\int_{K\cup K'}\partial_x(u-u^I)=(\int_1+\int_2+\int_3+\int_4)(u-u^I)dy$$

and

$$\begin{aligned}\Big(\int_3+\int_4\Big)(u-u^I)dy &=\int_{-h}^{0}\Big(-\frac{h^2}{12}\partial_y^2u(h,y)+\frac{1}{24}P(y+h)\partial_y^4u(h,y)\Big)dy\\ &\quad+\int_h^0\Big(-\frac{h^2}{12}\partial_y^2u(0,y)+\frac{1}{24}P(y)\partial_y^4u(0,y)\Big)dy\\ &=\int_{K\cup K'}\Big(-\frac{h^2}{12}(\partial_x-\partial_y)\partial_y^2u+\frac{1}{24}P(x+y)(\partial_x-\partial_y)\partial_y^4u\Big)\end{aligned}$$

with $P(t)=t^2(h-t)^2$. Similarly

$$\begin{aligned}\Big(\int_1+\int_2\Big)\quad(u-u^I)dy&=-\Big(\int_1+\int_2\Big)(u-u^I)dx\\ &=\int_h^0\Big(\frac{h^2}{12}(\partial_x-\partial_y)^2u(x,h-x)-\frac{1}{24}P(x)(\partial_x-\partial_y)^4u(x,h-x)\Big)dx\\ &\quad+\int_0^h\Big(\frac{h^2}{12}(\partial_x-\partial_y)^2u(x,-x)-\frac{1}{24}P(x)(\partial_x-\partial_y)^4u(x,-x)\Big)dx\\ &=\int_{K\cup K'}\Big(-\frac{h^2}{12}\partial_y(\partial_x-\partial_y)^2u+\frac{1}{24}P(x)\partial_y(\partial_x-\partial_y)^4u\Big).\end{aligned}$$

Hence,

$$\int_S \partial_x(u-u^I)\partial_x v = -\frac{h^2}{12}\int_S \partial_x\partial_y(\partial_x-\partial_y)u\cdot\partial_x v + O(h^4)\|v\|_{1,2}$$
$$= \frac{h^2}{12}\int_S \partial_x^2\partial_y(\partial_x-\partial_y)u\cdot v + O(h^4)\|v\|_{1,2}.$$

A similar expansion holds for

$$\int_S \partial_y(u-u^I)\partial_y v.$$

Finally, we have, for $v \in S^h$,

$$(\nabla(u^h-u^I),\nabla v) = \frac{h^2}{12}\int_S \partial_x\partial_y(\partial_x-\partial_y)^2 u\cdot v + O(h^4)\|v\|_{1,2}$$
$$= h^2(\nabla\Phi^h,\nabla v) + O(h^4)\|v\|_{1,2}$$

with $\Phi^h \in S^h$. Hence, we have

$$(\nabla(u^h-u^I-h^2\Phi^h),\nabla v) = O(h^4)\|v\|_{1,2},$$
$$\|u^h-u^I-h^2\Phi^h\|_{1,2} \le ch^4,$$
$$\|u^h-u^I-h^2\Phi^h\|_{0,\infty} \le ch^4|\log h|^{1/2},$$

i.e. the error expansion (1) holds. This is however, not new at the finite difference angle since the linear elements over a uniform mesh on a triangular domain is almost equivalent to the finite difference approximation where error expansion has been given by Marchuk and Shaidurov [6].

The proof in [3] mentioned above is based on the finite element argument, but, unfortunately, it is limited to a one-triangular domain and cannot be generalized even to the quadrilateral domain. This problem has been solved recently in [4] and [5] where the error expansion was proved for the quadrilateral domain and the general convex polygonal domain, respectively.

Final Remark on the Nonuniform Partition

The error expansion for rectangular elements has not been further developed in the last three years (1985–1987) and our attention was focussed on the triangular elements with the efforts to weeken the uniform condition imposed on the partition. We cannot, however, free partition completely from uniform condition if persist in using the triangular elements. In order to free partition from uniform condition we have to go back the rectangular elements with the following extensions:

(1) Extend the error expansion in Section 1 to the nonuniform rectangular mesh;

(2) Extend the Poisson equation in Section 1 to the variable coefficient equation corresponding to the functional

$$\int_S p_{11}u_x v_x + p_{22}u_y v_y + p_{12}(u_y v_x + u_x v_y)$$

with $v \neq 0$ even on ∂S;

(3)Extend the square domain S, by a bilinear mapping Ψ_S, to an arbitrary convex quadrilateral Ω_1. Under the mapping Ψ_S, the line parallet to x or y axis in S is transformed to the line linking the two equi-proportionate points of the two opposite edges in Ω_1 and a nonuniform rectangular mesh in S is transformed to a nonuniform quadrilateral mesh in Ω_1;

(4) A general polygonal domain Ω can be decomposed into several fixed convex macro-quadrilaterals $\Omega_1, \Omega_2, \cdots$. Each of Ω_i is transformed back to S. For a function V defined on Ω_i, let v be the function defined on $S : v = V \cdot \Psi_S$. On the other hand, a function v defined on S determines a function V on Ω_i. Define the piecewise isoparametric bilinear finite element space:

$$\Omega^h = \left\{ V \in H_0^1 : V \cdot \Psi_S \text{ piecewise bilinear on } S, \cdots \right\}$$

and, for $U = u\Psi_S^{-1}$,

$$U^I = u^I \Psi_S^{-1}, \quad u^I \in S^h.$$

Then, a typical integral is transformed by

$$\int_{\Omega_i} (U - U^I)_\xi V_\xi d\xi d\eta \to \int_S \Big(p_{11}(u - u^I)_x v_x \\ + p_{22}(u - u^I))_y v_y + p_{12}\Big((u - u^I)_x v_y + (u - u^I)_y v_x \Big) \Big) dx dy.$$

Thus, we obtain an error expansion for isoparametric bilinear finite element approximation defined on a nonuniform (regular) quadrilateral mesh constructed in linking some equi-proportionate points of the opposite edges in each macro-quadrilateral $\Omega_i (i = 1, 2, \cdots) \subset \Omega$. The interested reader is referenced to a forthcoming paper: Lin Qun, Finite element error expansion for nonuniform quadrilateral meshes, Systems Science and Mathematical Sciences, 2:3 (1989).

References

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