AN EXTRAPOLATION METHOD FOR BEM

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Abstract

This paper gives an asymptotic expansion of the error on the mesh point for Galerkin approximation of integral equations of the first kind. The extrapolation formula and some numerical results are given.

1. Introduction

It has been shown in [1]-[3] that the Richardson extrapolation can be applied to the elliptic Ritz projection with linear finite elements and increase the accuracy on mesh points z from

$$u_h(z) = u(z) + O(h^2|\ln h|)$$

to

$$\tilde{u}_h(z) = \frac{1}{3}\left[4u_{\frac{h}{2}}(z) - u_h(z)\right] = u(z) + O(h^3|\ln h|)$$

or

$$(O(h^4|\ln h|)),$$

where T_h is uniform triangulation and $T_{\frac{h}{2}}$ is generated from T_h by dividing each triangle into four congruent subtriangles.

In this paper, the above basic results are extended to boundary finite elements for integral equations of the first kind.

2. The Extrapolation for BEM

Let us consider the following boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = u_0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where Ω is a bounded domain with smooth boundary Γ. The equivalent integral equation is

$$\begin{aligned} \int_\Gamma q(x)E(x;y)ds_x = & \int_\Gamma u_0(x)\frac{\partial}{\partial n_x}E(x;y)ds_x + \frac{1}{2}u_0(y) \\ & - \int_\Omega f(x)E(x;y)dx \quad \forall\, y \in \Gamma, \end{aligned} \tag{2.2}$$

$$u(y)=\int_\Gamma q(x)E(x;y)ds_x-\int_\Gamma u_0(x)\frac{\partial}{\partial n_x}E(x;y)ds_x + \int_\Omega f(x)E(x;y)dx \quad \forall\, y\in\Omega, \tag{2.3}$$

where $E(x;y)$ is a fundamental solution of Laplace equation, i.e.

$$\triangle_x E(x:y)+\delta(x-y)=0$$

and $q=\left.\frac{\partial u}{\partial n}\right|_\Gamma$. For two-dimensional problems, it is known that

$$E(x;y)=-\frac{1}{2\pi}\ln|x-y|. \tag{2.4}$$

The integral equation (2.2) can be expressed by

$$(Aq)(y)=\int_\Gamma q(x)E(x;y)ds_x=F(y)\quad \forall y\in\Gamma \tag{2.5}$$

where

$$F(y)=\int_\Gamma u_0(x)\frac{\partial}{\partial n_x}E(x;y)ds_x+\frac{1}{2}u_0(y)-\int_\Omega f(x)E(x;y)dx.$$

The corresponding variational problem is

$$\begin{cases} \text{find } q\in H^{-\frac{1}{2}}(\Gamma)=V \quad \text{such that} \\ (Aq,q')=(F,q') \quad \forall q'\in V. \end{cases} \tag{2.6}$$

Let us consider the equation

$$\begin{cases} -\int_\Gamma \ln|t-s|e_\Gamma(s)dl_s=u_\Gamma, \\ \int_\Gamma e_\Gamma(s)dl_s=1, \end{cases} \tag{2.7}$$

where $e_\Gamma(s)$ and $c_\Gamma=\exp(-u_\Gamma)$ are defined as the equilibrium distribution and the transfinite diameter respectively [7]. It has been proved in [7] that when $c_\Gamma\neq 1$(i. e. $u_\Gamma\neq 0$), the solution of (2.6) exists and is unique.

Let L denote the arc length of Γ and let us identify functions on Γ with L-periodic functions on the real axis [9] :

$$x(s+L)=x(s),\quad q(s+L)=q(s)\quad \text{etc. for all } s\in R.$$

We consider the grid points x_j on Γ defined by

$$x_j=x(jh),\quad j=0,\pm1,\pm2,\ldots,\quad h:=\frac{L}{N+1}$$

and the periodic grid function q with values $q_h^j=q_h(x_j)$ such that

$$q_h^{j+(N+1)}=q_h^j.$$

Suppose $V_h(\Gamma_h)$ is a piecewise linear function space on $[0,L]$, where $\Gamma_h=\bigcup_i\Gamma_i$, $\Gamma_i=x_i\widehat{x_{i+1}}$ and c can represent different constants which are independent of h.

The Galerkin method for (2.6) is

$$\begin{cases} \text{find } q_h \in V_h \quad \text{such that} \\ (Aq_h, q_h') = (F, q_h') \quad \forall q_h' \in V_h. \end{cases} \tag{2.8}$$

When $c_\Gamma \neq 1$ and h is small enough, the solution of (2.8) exists and is unique [7].

Lemma 1. Suppose $E(x;y)$ is the fundamental solution defined by (2.4). Then there exists an $\tilde{E}(x;y) \in V_h$ such that

$$\left\|\tilde{E}(\cdot;y) - E(\cdot;y)\right\|_{0,1,\Gamma} \leq ch^{\frac{3}{2}}|\ln h| \tag{2.9}$$

where c is a positive constant independent of h and y.

Proof. Suppose $M_y = \{x; \|x-y\| \leq 1\}, \Gamma_1 = \Gamma \cap M_y$, and $\Gamma_2 = \Gamma - \Gamma_1$.

From the definition of $E(x;y)$, we can know that $E(x;y) \in H_0^{1-\epsilon}(M_y)$ and

$$(\nabla E(\cdot;y), \nabla v) = v(y) \quad \forall v \in h_0^{1-\epsilon}(M_y).$$

It has been shown in [3] and [8] that there exists an $E_h \in W_h \subset H_0^1(M_y)$ (where W_h is a piecewise linear interpolation finite element subspace on M_y) such that

$$(\nabla E_h(\cdot;y), \nabla v) = v(y) \quad \forall v \in W_h$$

and

$$\left\|E_h(\cdot;y) - E(\cdot;y)\right\|_{1,1,M_y} \leq ch|\ln h|,$$

$$\left\|E_h(\cdot;y) - E(\cdot;y)\right\|_{0,1,M_y} \leq ch^2.$$

Therefore

$$\left\|E_h(\cdot;y) - E(\cdot;y)\right\|_{1,1,\Omega\cap M_y} \leq ch|\ln h|,$$

$$\left\|E_h(\cdot;y) - E(\cdot;y)\right\|_{0,1,\Omega\cap M_y} \leq ch^2.$$

By virtue of the trace theorem and $\Gamma_1 \subset \partial(\Omega \cap M_y)$,

$$\begin{aligned} &\left\|E_h(\cdot,y) - E(\cdot,y)\right\|_{0,1,\Gamma_1} \leq \left\|E_h(\cdot,y) - E(\cdot,y)\right\|_{0,1,\partial(\Omega\cap M_y)} \\ &\leq \left(h\left\|E_h(\cdot,y) - E(\cdot,y)\right\|^2_{1,1,\Omega\cap M_y} + h^{-1}\left\|E_h(\cdot,y) - E(\cdot,y)\right\|^2_{0,1,\Omega\cap M_y}\right)^{\frac{1}{2}} \\ &\leq \left(ch^3|\ln h|^2 + ch^3\right)^{\frac{1}{2}} = ch^{\frac{3}{2}}|\ln h|. \end{aligned}$$

Because $\|x-y\| > 1, \forall x \in \Gamma_2, E(x;y) \in C^\infty(\Gamma_2)$. Suppose $\pi : C(\Gamma) \to V_h(\Gamma_h)$ is a piecewise linear interpolation operator; then

$$\left\|\pi_x E(\cdot;y) - E(\cdot;y)\right\|_{0,1,\Gamma_2} \leq ch^2\left\|E(\cdot;y)\right\|_{2,1,\Gamma_2} \leq ch^2.$$

Setting

$$\tilde{E}(x;y)=\begin{cases} E_h(x;y), & x\in\Gamma_1,\\ \pi E(x;y), & x\in\Gamma_2,\end{cases}$$

it is clear that $\tilde{E}(\cdot;y)\in V_h$ and

$$\begin{aligned}&\left\|\tilde{E}(\cdot;y)-E(\cdot;y)\right\|_{0,1,\Gamma}\\ &\le \left\|E_h(\cdot;y)-E(\cdot;y)\right\|_{0,1,\Gamma_1}+\left\|\pi E(\cdot;y)-E(\cdot;y)\right\|_{0,1,\Gamma_2}\\ &\le ch^{\frac{3}{2}}|\ln h|+ch^2=ch^{\frac{3}{2}}|\ln h|.\end{aligned}$$

Let us consider the Green function g_z and the discrete Green function g_{zh}, which are defined as follows [11] :

$$\begin{aligned}&(f,A^*g_z)=f(z)\quad \forall f\in C(\Gamma), z\in\Gamma,\\ &(f_h,A^*g_{zh})=f_h(z)\quad \forall f_h\in V_h, z\in \text{ set of mesh points}\end{aligned}$$

where A^* is the conjugate operator of A. For the problem dealt with in this paper, $A^*=A$. It has been proved in [11] that

$$\left\|A^*g_z-Ag_{zh}\right\|_{0,1,\Gamma}\le c. \tag{2.10}$$

Lemma 2. Suppose g_z, g_{zh} are defined as above; then

$$\left\|A^*g_z-Ag_{zh}^*\right\|_{-1,1,\Gamma}\le ch, \tag{2.11}$$

$$\left\|g_{zh}\right\|_{0,1,\Gamma}\le ch^{-1}. \tag{2.12}$$

Proof. (2.11) can be easily derived by the duality argument and (2.10).

As to (2.12), it follows from the inverse estimation of finite elements and the regularities of the integral equation.

By virtue of inverse estimation of finite elements and regularity properties of integral equations,

$$\left\|g_{zh}\right\|_{0,1,\Gamma}\le ch^{-1}\left\|g_{zh}\right\|_{-1,1,\Gamma}\le ch^{-1}\left\|Ag_{zh}\right\|_{0,1,\Gamma}=ch^{-1}\|\delta\|_{0,1,\Gamma}=ch^{-1}$$

Lemma 3. Suppose Γ is smooth enough, $c_\Gamma\ne 1, q\in C^3(\Gamma)$ and $q_h\in V_h$ are solutions of (2.6) and (2.8) respectively; then

$$q(z)-q_h(z)=\frac{h^2}{12}q''(z)+O(h^{\frac{5}{2}}|\ln h|)\|q\|_{3,\infty,\Gamma} \tag{2.13}$$

where z are mesh points on $\Gamma, q''=\dfrac{d^2q}{ds^2}$.

Proof. Because $\pi q(z)=q(z)$ on every mesh point z and

$$(Aq,g_{zh})=(Aq_h,g_{zh})=(F,g_{zh}),$$

so

$$
\begin{aligned}
&q(z) - q_h(z) = \pi q(z) - q_h(z) = (\pi q - q_h, A^* g_{zh}) = (A\pi q - Aq_h, g_{zh}) \\
&= (A\pi q, g_{zh}) - (Aq_h, g_{zh}) = (A\pi q, g_{zh}) - (Aq, g_{zh}) \\
&= \int_\Gamma \int_\Gamma E(x;y)(\pi q(x) - q(x)) g_{zh}(y) ds_x ds_y \\
&= \int_\Gamma \int_\Gamma \tilde{E}(x;y)(\pi q(x) - q(x)) g_{zh}(y) ds_x ds_y \\
&+ \int_\Gamma \int_\Gamma (E(x;y) - \tilde{E}(x;y))(\pi q(x) - q(x)) g_{zh}(y) ds_x ds_y \\
&= I_1 + I_2.
\end{aligned}
\tag{2.14}
$$

By virtue of Lemme 1 and Lemme 2,

$$
\begin{aligned}
R_1 &= \int_\Gamma \left| \int_\Gamma (E(x;y) - \tilde{E}(x;y)) g_{zh}(y) ds_y \right| ds_x \\
&\leq \int_\Gamma \left| g_{zh}(y) \right| \int_\Gamma \left| E(x;y) - \tilde{E}(x;y) \right| ds_x ds_y \\
&\leq \max_{y \in \Gamma} \left\{ \int_\Gamma \left| E(x;y) - \tilde{E}(x;y) \right| ds_x \right\} \int_\Gamma \left| g_{zh}(y) \right| ds_y \\
&\leq \left\| E(x;y) - \tilde{E}(x;y) \right\|_{0,1,\Gamma} \left\| g_{zh} \right\|_{0,1,\Gamma} \leq ch^{\frac{3}{2}} |\ln h| ch^{-1} = ch^{\frac{1}{2}} |\ln h|.
\end{aligned}
\tag{2.15}
$$

So

$$
\begin{aligned}
I_2 &= \|\pi q - q\|_{0,\infty\Gamma} \int_\Gamma \left| \int_\Gamma (E(x;y) - \tilde{E}(x;y)) g_{zh}(y) ds_y \right| ds_x \\
&\leq ch^2 \|q\|_{2,\infty,\Gamma} R_1 = ch^{\frac{5}{2}} |\ln h| \ \|q\|_{2,\infty,\Gamma}.
\end{aligned}
\tag{2.16}
$$

By Taylor expansion, on $\Gamma_i = \widehat{x_i x_{i+1}}$

$$
\pi q(x(s)) = q(x(s) + \frac{1}{2}(s_{i+1} - s)(s - s_i) q''(x(s)) + O(h^3) \|q\|_{3,\infty,\Gamma}.
$$

Hence

$$
\begin{aligned}
I_1 &= \int_\Gamma \int_\Gamma \tilde{E}(x;y)(\pi q(x) - q(x)) g_{zh}(y) dl_x dl_y \\
&= \frac{1}{2} \int_\Gamma \int_\Gamma \tilde{E}(x;y)(s_{i+1} - s)(s - s_i) q''(x(s)) g_{zh}(y) ds dl_y \\
&+ O(h^3) \|q\|_{3,\infty,\Gamma} \int_\Gamma \left| \int_\Gamma \tilde{E}(x;y) g_{zh}(y) dl_x \right| dl_x = J_1 + J_2.
\end{aligned}
\tag{2.17}
$$

By (2.15),

$$\begin{aligned} R_2 &= \int_\Gamma \left| \int_\Gamma \tilde{E}(x;y) g_{zh}(y) ds_y \right| ds_x \le \int_\Gamma \left| \int_\Gamma E(x;y) g_{zh}(y) ds_y \right| ds_x \\ &+ \int_\Gamma \left| \int_\Gamma (\tilde{E}(x;y) - E(x;y)) g_{zh}(y) ds_y \right| ds_x \\ &= \int_\Gamma \left| (A g_{zh})(x) \right| ds_x + R_1 \le \left\| A g_{zh} \right\|_{0,1,\Gamma} + c h^{\frac{1}{2}} |\ln h|^{\frac{1}{2}} \\ &\le \|\delta\|_{0,1,\Gamma} + c h^{\frac{1}{2}} |\ln h|^{\frac{1}{2}} \le c, \end{aligned} \tag{2.18}$$

$$|J_2| \le O(h^3) \|q\|_{3,\infty,\Gamma} R_2 = O(h^3) \|q\|_{3,\infty,\Gamma}. \tag{2.19}$$

On Γ_i ,

$$\tilde{E}(x;y) = \frac{s_{i+1} - s}{h} \tilde{E}(x_i;y) + \frac{s - s_i}{h} \tilde{E}(x_{i+1};y), \tag{2.20}$$

$$\begin{aligned} R_3 &= \frac{1}{2} \sum_i \int_{\Gamma_i} (s_{i+1} - s)(s - s_i) q''(x(s)) \tilde{E}(x(s);y) ds \\ &= \frac{1}{2} \sum_i \int_{\Gamma_i} (s_{i+1} - s)(s - s_i) q''(x(s_i)) \tilde{E}(x(s);y) ds \\ &+ \frac{1}{2} \sum_i \int_{\Gamma_i} (s_{i+1} - s)(s - s_i)(q''(x(s)) - q''(x(s_i))) \tilde{E}(x(s);y) ds \\ &= R_4 + R_5, \end{aligned}$$

$$\begin{aligned} R_4 &= \frac{1}{2} \sum_i q''(x(s_i)) \int_{\Gamma_i} (s_{i+1} - s)(s - s_i)(\frac{s_{i+1} - s}{h} \tilde{E}(x_i;y) \\ &+ \frac{s - s_i}{h} \tilde{E}(x_{i+1};y)) ds = \frac{1}{2} \sum_i q''(x(s_i))(\tilde{E}(x_i;y) \\ &\int_{\Gamma_i} (s_{i+1} - s)(s - s_i) \frac{s_{i+1} - s}{h} ds + \tilde{E}(x_{i+1};y) \int_{\Gamma_i} (s_{i+1} - s)(s - s_i) \frac{s - s_i}{h} ds) \\ &= \frac{h^2}{12} \sum_i q''(x(s_i))(\frac{h}{2} \tilde{E}(x_i;y) + \frac{h}{2}(x_{i+1};y)). \end{aligned}$$

By (2.20) and

$$\int_{\Gamma_i} \frac{s - s_i}{h} ds = \frac{h}{2} \quad , \quad \int_{\Gamma_i} \frac{s_{i+1} - s}{h} ds = \frac{h}{2},$$

$$\begin{aligned} R_4 &= \frac{h^2}{12} \sum_i q''(x(s_i)) \int_{\Gamma_i} \left(\frac{s_{i+1} - s}{h} \tilde{E}(x_i;y) + \frac{s - s_i}{h} \tilde{E}(x_{i+1};y) \right) ds \\ &= \frac{h^2}{12} \sum_i q''(x(s_i)) \int_{\Gamma_i} \tilde{E}(x(s);y) ds = \frac{h^2}{12} \sum_i \int_{\Gamma_i} q(x(s)) \tilde{E}(x(s);y) ds \\ &+ \frac{h^2}{12} \sum_i \int_{\Gamma_i} (q''(x(s_i)) - q''(x(s))) \tilde{E}(x(s);y) ds, \end{aligned}$$

$$
\begin{aligned}
R_3 &= R_4 + R_5 = \frac{h^2}{12}\sum_i \int_{\Gamma_i} q''(x(s))\tilde{E}(x(s);y)ds \\
&+\frac{h^2}{12}\sum_i \int_{\Gamma_i} (q''(x(s_i)) - q''(x(s)))\tilde{E}(x(s);y)ds \\
&+\frac{1}{2}\sum_i \int_{\Gamma_i} (s_{i+1}-s)(s-s_i)(q''(x(s)) - q''(x(s_i)))\tilde{E}(x(s);y)ds,
\end{aligned}
$$

$$
\begin{aligned}
J_1 &= \frac{1}{2}\int_\Gamma \int_\Gamma \tilde{E}(x(s);y)(s_{i+1}-s)(s-s_i)q''(x(s))g_{zh}(y)dsdl_y \\
&= \int_\Gamma R_3(y)g_{zh}dl_y = \frac{h^2}{12}\int_\Gamma\int_\Gamma \tilde{E}(x;y)q''(x)g_{zh}(y)dl_xdl_y \\
&+\frac{h^2}{12}\int_\Gamma \sum_i \int_{\Gamma_i} \tilde{E}(x(s);y)(q''(x(s_i)) - q''(x(s)))g_{zh}(y)dsdl_y \qquad (2.21) \\
&+\frac{1}{2}\int_\Gamma \sum_i \int_{\Gamma_i} (s_{i+1}-s)(s-s_i)(q''(x(s_i)))\tilde{E}(x(s);y) \\
&\quad g_{zh}(y)dsdl_y = K_1 + K_2 + K_3,
\end{aligned}
$$

$$
\begin{aligned}
|K_2| &\le \frac{h^2}{12}\max_{\substack{0\le i\le N\\ x(s)\in\Gamma_i}}\left|q''(x(s_i)) - q''(x(s))\right|\int_\Gamma\left|\int_\Gamma \tilde{E}(x;y)g_{zh}(y)dl_y\right|dl_x \\
&\le \frac{h^2}{12}O(h)\|q\|_{3,\infty,\Gamma}R_2 = O(h^3)\|q\|_{3,\infty,\Gamma},
\end{aligned} \qquad (2.22)
$$

$$
|K_3| \le \frac{h^2}{2}\max_{\substack{0\le i\le N\\ x(s)\in\Gamma_i}}\left|q''(x(s)) - q''(x(s_i))\right|\int_\Gamma\left|\int_\Gamma \tilde{E}(x;y)g_{zh}(y)dl_y\right|dl_x = O(h^3)\|q\|_{3,\infty,\Gamma}
$$

$$
\begin{aligned}
K_1 &= \frac{h^2}{12}\int_\Gamma\int_\Gamma E(x;y)q''(x)g_{zh}(y)ds_xds_y \\
&+\frac{h^2}{12}\int_\Gamma\int_\Gamma (\tilde{E}(x;y) - E(x;y))q''(x)g_{zh}(y)ds_xds_y \\
&= \frac{h^2}{12}\int_\Gamma\int_\Gamma E(x;y)q''(x)g_z(y)ds_xds_y \\
&+\frac{h^2}{12}\int_\Gamma\int_\Gamma E(x;y)q''(x)(g_{zh}(y) - g_z(y))ds_xds_y \\
&+\frac{h^2}{12}\int_\Gamma\int_\Gamma (\tilde{E}(x;y) - E(x;y))q''(x)g_{zh}(y)ds_xds_y \\
&= L_1 + L_2 + L_3
\end{aligned} \qquad (2.23)
$$

$$
L_1 = \frac{h^2}{12}\int_\Gamma\int_\Gamma E(x;y)q''(x)g_z(y)ds_xds_y = \frac{h^2}{12}(q'', A^*g_z) = \frac{h^2}{12}q''(z). \qquad (2.24)
$$

According to Lemma 2,

$$\begin{aligned}|L_2| &= \frac{h^2}{12}\left|\int_\Gamma q''(x)(A^*g_{xh} - A^*g_x)(x)ds_x\right| \\ &\le \frac{h^2}{12}\|q''\|_{1,\infty,\Gamma}\|A^*g_{xh} - A^*g_x\|_{-1,1,\Gamma} \\ &\le \frac{h^2}{12}\|q\|_{3,\infty,\Gamma}ch = O(h^3)\|q\|_{3,\infty,\Gamma}.\end{aligned} \tag{2.25}$$

By (2.15),

$$L_3 \le \frac{h^2}{12}\|q\|_{2,\infty,\Gamma}R_1 \le \frac{h^2}{12}\|q\|_{2,\infty,\Gamma}ch^{\frac{1}{2}}|\ln h| = 0(h^{\frac{5}{2}}|\ln h|)\|q\|_{2,\infty,\Gamma}. \tag{2.26}$$

Form (2.14), (2.16), (2.17), (2.19), (2.21)–(2.26), we obtain

$$\begin{aligned}&q(z) - q_h(z) = I_1 + I_2 = J_1 + J_2 + I_2 = K_1 + K_2 + K_3 + J_2 + I_2 \\ &= L_1 + L_2 + L_3 + K_2 + K_3 + J_2 + I_2 = \frac{h^2}{12}q''(z) + O(h^{\frac{5}{2}}|\ln h|)\|q\|_{3,\infty,\Gamma}.\end{aligned}$$

From Lemma 3, it is easy to deduce the extrapolation formula for Galerkin method for integral equations of the first kind. We have

Theorem 1. Suppose q, q_h, Γ, V_h are defined such that all conditions in Lemma 3 are satisfied; then

$$\tilde{q}_h(z) = \frac{1}{3}(4q_{\frac{h}{2}}(z) - q_h(z)) = q(z) + O(h^{\frac{5}{2}}|\ln h|)\|q\|_{3,\infty,\Gamma} \tag{2.27}$$

where $z \in \Gamma$ are mesh points.

3. Numerical Test

Let us consider

$$\begin{cases} -\Delta u = o & x^2 + y^2 < \frac{1}{4} \\ u = x + y & x^2 + y^2 = \frac{1}{4} \end{cases}$$

It is easy to show that $c_\Gamma = \frac{1}{2} \neq 1$ and all conditions in Theorem 1 are satisfied.

The numerical results with Γ divided into 32 ($h \doteq 0.1$) and 100 ($h = 0.0314$) elements are shown in Table 1 and Table 2 respectively. For simplicity, we only show the results in the first quardrand.

It can be shown that the accuracy of the usual linear finite element method with 200 mesh points can be achieved by our extrapolation method with only 32 mesh points.

Table 1

x	y	q	q_h	$\tilde{q}_h$	$q - q_h$	$q - \tilde{q}_h$
0.50000000	0.00000000	1.00000000	1.00138247	1.00003779	−0.00138247	−0.00003779
0.49039264	0.09754516	1.17587560	1.17750211	1.17591949	−0.00162661	−0.00004388
0.46193977	0.19134172	1.30656296	1.30836962	1.30661276	−0.00180666	−0.00004980
0.41573481	0.27778516	1.38703984	1.38895789	1.38709186	−0.00191805	−0.00005202
0.35355399	0.35355399	1.41421357	1.41616960	1.41426596	−0.00195603	−0.00005239
0.27778516	0.41573481	1.38703986	1.38895799	1.38709113	−0.00191814	−0.00005127
0.19134172	0.46193977	1.30656298	1.30837025	1.30661161	−0.00180727	−0.00004864
0.09754516	0.49039264	1.17587562	1.17750137	1.17592078	−0.00162575	−0.00004517

Table 2

x	y	q	$q_{\frac{h}{2}}$	$\tilde{q}_h$	$q-q_{\frac{h}{2}}$	$q-\tilde{q}_h$
0.50000000	0.00000000	1.00000000	1.00003995	1.00000052	−0.00003995	−0.00000052
0.49901337	0.03139526	1.06081724	1.06086139	1.06082045	−0.00004415	−0.00000320
0.49605736	0.06266662	1.11744793	1.11749054	1.11744602	−0.00004262	0.00000191
0.49114363	0.09369066	1.16966856	1.16971673	1.16967191	−0.00004818	0.00000335
0.48429158	0.12434495	1.21727304	1.21732573	1.21727960	−0.00005269	−0.00000656
0.47552826	0.15450850	1.26007351	1.26011929	1.26006761	−0.00004578	−0.00000590
0.46488825	0.18406228	1.29790103	1.29795383	1.29790397	−0.00005279	0.00000294
0.45241353	0.21288965	1.33060634	1.33065867	1.33060609	−0.00005233	−0.00000025
0.43815334	0.24087684	1.35806035	1.35811315	1.35806082	−0.00005281	0.00000048
0.42216397	0.26791340	1.38015471	1.30821211	1.38015789	−0.00005740	−0.00000317
0.40450850	0.29389263	1.39680225	1.39685106	1.39679390	−0.00004882	−0.00000834
0.38525662	0.31871200	1.40793723	1.40799828	1.40794490	−0.00006105	0.00000767
0.36448432	0.34227356	1.41351573	1.41357018	1.41351458	−0.00005445	−0.00000115
0.34227356	0.36448432	1.41351573	1.41357526	1.41352145	−0.00005953	0.00000572
0.31871200	0.38525662	1.40793724	1.40799157	1.40793565	−0.00005433	−0.00000159
0.29389263	0.40450850	1.39680225	1.39685863	1,39680375	−0.00005639	0.00000151
0.26791340	0.42216397	1.38015473	1.38021202	1.38015835	−0.00005729	−0.00000362
0.24087684	0.43815334	1.35806037	1.35811549	1.35806370	−0.00005512	−0.00000332
0.21288905	0.45241353	1.33060636	1.33066139	1.33060962	−0.00005503	−0.00000326
0.18406228	0.46488825	1.29790105	1.29795587	1.29790600	−0.00005482	−0.00000495
0.15450850	0.47552826	1.26007353	1.26012515	1.26007646	−0.00005162	−0.00000294
0.12434495	0.48429158	1.21727307	1.21732610	1.21727996	−0.00005303	−0.00000688
0.09369006	0.49114363	1.16966859	1.16971333	1.16966681	−0.00004475	−0.00000177
0.06266662	0.49605736	1.11744796	1.11749502	1.11745211	−0.00004706	0.00000416
0.03139526	0.49901337	1.06081729	1.06081724	1.06081724	−0.00004154	−0.00000004

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