ON THE OPERATOR EQUATION $SX - XT = Q$

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Abstract

In this talk, we discuss the solvability of the operator equation $SX - XT = Q$, where S and $-T$ are generators of some semigroups of operators.

1. Introduction

Throughout my talk, let $\mathcal{X}$ and $\mathcal{Z}$ be two Banach spaces, and $B(\mathcal{X}, \mathcal{Z})$ be the Banach space of all bounded (linear) operators from $\mathcal{X}$ to $\mathcal{Z}$. Let S and $-T$ be the infinitesimal generators of (C_0)-semigroups $\{G(t) : t \geq 0\} \subset B(\mathcal{Z})$ and $\{H(t) : t \geq 0\} \subset B(\mathcal{X})$, respectively. For $Q \in B(\mathcal{X} \mathcal{Z})$, by a solution of

$$SX - XT = Q \tag{E}$$

we mean an operator X in $B(\mathcal{X}, \mathcal{Z})$ which maps the domain $D(T)$ of T into $D(S)$ such that

$$SXu - XTu = Qu$$

holds for all $u \in D(T)$.

Why are we interested in this equation? This goes back to early 1930s, when P.A.M. Dirac, H. Weyl, Von Neumann, and many of their comtemporaries were developing the mathematical theory of quantum mechanics. And they were very much concerned about stability of the spectrum of self-adjoint operators under small perturbations. In the simplest case, let S and P be bounded operators on a Banach space. If there exists a bounded invertible operator W such that

$$S = W^{-1}(S + P)W,$$

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then the spectrum of $S+P$ remains the same as that of S.

We observe that a sufficient condition for the similarity of $S+P$ and the unperturbed S is the simultaneous solvability of the following pair of equations:

$$SX - XS = Q \quad \text{and} \quad Q + PX = -P.$$

In this case, we have $(I+X)S = (S+P)(I+X)$. Thus $S+P$ is similar to S if $I+X$ is invertible. As a result of this observation, the first step towards a perturbation theory would seemed to be a thorough study of the commutator equation $SX - XS = Q$.

More generally, we consider the equation (E) for two (generally unbounded) operators S and T. We define a map Δ from $B(\mathcal{X}, \mathcal{Z})$ into itself which has as its domain $D(\Delta)$ the set of all those X in $B(\mathcal{X}, \mathcal{Z})$ for which $XD(T) \subset D(S)$ and $SX - XT$ is bounded on $D(T)$, and which sends each such X to the closure of $SX - XT$.

The solvability of (E) is equivalent to the question when is $Q \in R(\Delta)$ (the range of Δ)? This is closely related to Kato-Cook-Rosemblum approach to perturbation of self-adjoint operators, and the existence of "Wave Operator"

$$W_{\pm} = \lim_{t \to \pm\infty} e^{itT} e^{-itS}$$

(unitary or partial isometry) which impliments the similarity between $T = S + P$ and S.

2. Historical Remarks

The equation (E) has a long story starting from the space of matrices and ending up in complex Banach algebra.

(1) D.E. Rutherford (1932) studied the equation (E) with S, T and Q in M_n the space of n by n matrices. He showed that if the eigenvalues of S are distinct from those of $-T$, then (E) has a unique solution $X \in M_n$.

(2) E. Heinz; H. O. Cordes (1951, 1955) showed that when S and T are bounded operators on a Hilbert space , and if there exist constants $a > b$ such that $S + S^* \leq b$ and $T + T^* \geq a$ then Δ^{-1} exists as a bounded operator in $B(\mathcal{X})$ with the following representation

$$X = \Delta^{-1}(Q) = -\int_0^{\infty} e^{tS} Q e^{-tT} dt.$$

(3) M. Rosenblum (1956), by using dunford operational calculus, he was able to show that if there exists a Cauchy domain Ω such that $\sigma(T) \subset \Omega$ and $\sigma(S) \cap \overline{\Omega} = \emptyset$, then $X = \Delta^{-1}(Q)$ exists, and is given by

$$X = \frac{1}{2\pi i} \int_{\delta\Omega} (S-z)^{-1} Q (z-T)^{-1} dz$$

(contour integral around the spectrum of T). At about the same time D. C. Kleinecke showed that $\sigma(\Delta) = \sigma(S) - \sigma(T)$ in a seminar at U. C. Berkeley.

(4) M. Rosenblum (1969) generalized his results to the case S, T are unbounded self-adjoint operators in a separable Hilbert space $\mathcal{X} = \mathcal{Z} = \mathcal{H}$ with $\Delta X = \overline{SX - XT}$ by a perturbation method. Note that

$$\sigma(S + iyI) \cap \sigma(T - iyI) = \emptyset$$

for every $y > 0$, and by applying his previous results, he showed that

$$(S + iyI)X_y - X_y(T - iyI) = Q \qquad (E_y)$$

has a solution $X_y \in B(\mathcal{H})$. If the set $\{\|X_y\| : y > 0\}$ is bounded, then $\{X_y\}$ has a subsequence $\{X_{y_n}\}$ which converges weakly to some $X \in B(\mathcal{H})$, as $n \to \infty$. This limit X solves (E).

(5) J. M. Freeman (1970) considered the case where $\mathcal{X}, \mathcal{Z}$ are reflexive Banach spaces; and J. A. Goldstein (1978) considered the case where S and T are self-adjoint operators and $Q \in C_p$ (the Schatten-Von Neumann class of compact operators), $1 \le p \le \infty$.

3. Main Results

In the works of Rosenblum [4, 5], Freeman [1] and Goldstein [2], the solvability of (E) has been characterized basically by the uniform boundedness of $(\lambda - \Delta)^{-1}Q$ for λ near 0.

Recently Shaw and myself [6] give a different charaterization of the solvability of (E) using the existence of weak operator limit point X of

$$t^{-1} \int_0^t \int_0^s G(u)QH(u)du\, ds,$$

as $t \to \infty$. This X will be a solution of (E).

To describe our results, we need some definitions. Given the Banach space $\mathcal{X}$, let $\mathcal{Y}$ be a closed linear subspace of $\mathcal{X}^*$ (the dual space of $\mathcal{X}$). We use the weak topology $\sigma(\mathcal{X}, \mathcal{Y})$ on $\mathcal{X}$ introduced by W. Feller in 1953 [7]. A net $x_a \in \mathcal{X}$ is said to be $\sigma(\mathcal{X}, \mathcal{Y})$ convergent to $x \in \mathcal{X}$, if $< x_a, y > \to < x, y >$ $\forall y \in \mathcal{Y}$. The topology $\sigma(\mathcal{X}, \mathcal{Y})$ on $\mathcal{X}$ is weaker than the weak topology of $\mathcal{X}$ (of course if $\mathcal{Y} = \mathcal{X}^*$, they coincides).

Let $\{T(t)\} (t \ge 0)$ be a semigroup in $B(\mathcal{X})$. We assume

[H1]: $\mathcal{X}$ and $\mathcal{Y}$ are reciprocal, i.e.

$$\|x\| = \sup_{\substack{y \neq 0 \\ y \in \mathcal{Y}}} \frac{|< x, y >|}{\|y\|}, \quad \forall x \in \mathcal{X}.$$

[H2]: $\mathcal{Y}$ is invariant under $T^*(t)\forall t > 0$.

[H3]: For each $x \in \mathcal{X}, T(\cdot)x$ is $\sigma(\mathcal{X}, \mathcal{Y})$–continuous on $[0, \infty)$.

[H4]: For each $t > 0$ and $x \in \mathcal{X}, T(\cdot)x$ is $\mathcal{Y}$–Riemann integrable on $[0, t]$ in the sense that there exists $x_t \in \mathcal{X}$ such that

$$< x_t, y >= \int_0^t < T(s)x, y > ds, \quad \forall y \in \mathcal{Y}.$$

This x_t is unique due to (H1), and is called the $\mathcal{Y}$–Riemann integral of $T(\cdot)x$ on [0, t]. When $T(\cdot)x$ is strongly continuous, x_t coincides with the Bochner integral $\int_0^t T(s)x\,ds$. Such a semigroup $T(\cdot)$ is called a $\mathcal{Y}$–semigroup on $\mathcal{X}$. The $\mathcal{Y}$–generator A of $T(\cdot)$ is by definition the operator which maps $x \to \sigma(\mathcal{X}, \mathcal{Y})$–limit of $t^{-1}(T(t) - I)x$ as $t \to 0^+$ whenever this limit exists. A non-trivial example of a ($\mathcal{Y}$)–semigroup is the tensor product semigroup of two (C_0)–semigroups.

Let $T(t)$ be a $\mathcal{Y}$–semigroup on $\mathcal{X}$ with $\mathcal{Y}$–generator A. We define $S(t)$ to be the map which takes x into x_t (its $\mathcal{Y}$–Riemann integral on [0, t]). Then $S(t)x \in D(A)$ and

$$AS(t)x = T(t)x - x, \quad \forall x \in \mathcal{X},$$

$$S(t)Ax = T(t)x - x, \quad \forall x \in D(A).$$

For each $q \in \mathcal{X}$ we define the function:

$$q(t) = -t^{-1} \int_0^t S(s)q\,ds = -t^{-1} \int_0^t \Big[\int_0^s T(u)q\,du\Big]\,ds,$$

where the integration with respect to u is a $\mathcal{Y}$–Riemann integral and the one with respect to s a Bochner integral.

Let $R(A)$ and $N(A)$ denote the range and null spaces of A, respectively. We need the following mean ergodic theorem.

Theorem. *Let $T(\cdot)$ be a $\mathcal{Y}$–semigroup on $\mathcal{X}$ with the $\mathcal{Y}$–generator A. Suppose there exists a constant $M > 0$ such that*

(1) $t^{-1}\|S(t)\| \leq M$ *for all* $t > 0$*;*

(2) $\|T(t)x\| = o(t)(t \to \infty)\forall x \in D(A)$*;*

(3) *there exists a sequence* $t_n \to \infty$ *such that*

$$x = \mathcal{Y} - \lim_{n \to \infty} q(t_n) \ \textit{exist}.$$

Then $q \in R(A)$ and x is a solution of the equation

$$Ax = q.$$

Proof (sketch). Let P maps $x \to s - \lim_{t \to \infty} t^{-1}S(t)x$ whenever exists (Cesàro Ergodic Theory). It is known that P is a bounded linear projection with $R(P) = N(A), N(P) = \overline{R(A)}$, and domain $D(P) = N(A) \oplus \overline{R(A)}$ which is equal to

$$\{x \in \mathcal{X} : \exists t_n \to \infty \text{ such that } w\text{-}\lim t_n^{-1}S(t_n)x \text{ exists, as } n \to \infty\}.$$

Since $\{q(t_n)\}$ is $\sigma(\mathcal{X}, \mathcal{Y})$ convergent and $\mathcal{Y}$ is a Banach space, by using the uniform boundedness principle and (H1) we show that

$$\begin{aligned} q \in^{\perp} [\cap_{t>0} N(T^*(t) - I^*)] &= \overline{\text{span}}\{R(T(t) - I) : t > 0\} \\ &= \overline{\text{span}}\{R(AS(t)) : t > 0\} \subset \overline{R(A)}. \end{aligned}$$

Taking $\mathcal{Y}$–Riemann integrals on both sides of the identity $AS(s)q = T(s)q - q$, we obtain

$$Aq(t_n) = q - t_n^{-1} S(t_n) q,$$

which converges strongly to $q - Pq = q$ as $n \to \infty$. Consequently the $\sigma(\mathcal{X}, \mathcal{Y})$–closedness of A ensures that $x \in D(A)$ and $Ax = q$.

Corollary. If in addition $T(\cdot)$ is mean-ergodic (i.e. $D(P) = \mathcal{X}$) then $q \in R(A)$ if and only if $x = \mathcal{Y} - \lim_{n \to \infty} q(t_n)$ exists.

Remark. Our results improve those of U. Krengel and M. Lin (1984) [3] in two respects:

(1) The class of $\mathcal{Y}$–semigroups is substantially larger than the class of (C_0)–semigroups.

(2) The condition $\|T(t)x\| = o(t)$ as $t \to \infty$ is weaker than their condition $\|T(t)\| = o(t)$ as $t \to \infty$.

Application to $\Delta X = Q$

To each pair $x \in \mathcal{X}$ and $z^* \in \mathcal{Z}^*$, let f_{x,z^*} denote the linear functional on $B(\mathcal{X}, \mathcal{Z})$ defined by

$$< X, f_{x,z^*} > = < Xx, z^* > \ (X \in B(\mathcal{X}, \mathcal{Z})).$$

Let $\mathcal{Y}$ be the closed linear span of the set

$$\{f_{x,z^*}; x \in \mathcal{X}, z^* \in \mathcal{Z}^*\}$$

in $B(\mathcal{X}, \mathcal{Z}))^*$, and let $\mathcal{T}(\cdot)$ be the tensor product semigroup of (C_0)–semigroups $G(\cdot)$ and $H(\cdot)$ in $B(\mathcal{Z})$ and $B(\mathcal{X})$, respectively. It is known that $\mathcal{T}(\cdot)$ is a ($\mathcal{Y}$)–semigroup of operators on $B(\mathcal{X}, \mathcal{Z})$, and the topology $\sigma(B(\mathcal{X}, \mathcal{Z}), \mathcal{Y})$ coincides with the weak operator topology of $B(\mathcal{X}, \mathcal{Z})$.

Theorem 1. *The following statements are equivalent:*

(1) $X \in D(\Delta)$;

(2) $\mathcal{Y} - \lim t^{-1}(\mathcal{T}(t)X - X)$ *exists as* $t \to 0^+$.

Moreover, if $X \in D(\Delta)$, *then* $\Delta X = \mathcal{Y} - \lim t^{-1}(\mathcal{T}(t)X - X)$.

We can now consider the equation $\Delta X = Q$. We define for each $Q \in B(\mathcal{X}, \mathcal{Z})$, the function $Q(t)$ by

$$\begin{aligned} Q(t)x &= -t^{-1} \int_0^t \int_0^s (\mathcal{T}(u)Q)x \, du \, ds \\ &= -t^{-1} \int_0^t \int_0^s (G(u)QH(u)x) du \, ds, \end{aligned}$$

for $x \in \mathcal{X}$.

Theorem (Main). *Suppose there exists a constant $M > 0$ such that*

(1) $t^{-1}\|\int_0^t \mathcal{T}(s)E\| \le M\|E\| \forall E \in B(\mathcal{X}, \mathcal{Z})$ *and* $t > 0$,

(2) $\|\mathcal{T}(t)E\| = o(t)(t \to \infty) \forall E \in D(\Delta)$.

If there exists a sequence $t_n \to \infty$ such that $X = wo - \lim Q(t_n)$ (weak operator limit) exists as $n \to \infty$, then $Q \in R(\Delta)$ and $\Delta X = Q$.

Results discussed here are treated in full in [6].

Remarks. (1) The assumptions in the sbove theorem are fulfilled if $\mathcal{T}(\cdot)$ is uniformly bounded. This is the case when the sum of the types of $G(\cdot)$ and $H(\cdot)$ is less than 0 (cf. [1, proposition 7]).

(2) If $\mathcal{Z}$ is reflexive, then the unit ball of $B(\mathcal{X}, \mathcal{Z})$ is compact relative to the weak operator topology. In this case $\Delta X = Q$ has a solution X if and only if $\{\|Q(t)\| : t > 0\}$ is bounded.

(3) When $\mathcal{X} = \mathcal{Z} = \mathcal{H}$ is a separable Hilbert space, and S, T are skew-adjoint operators on $\mathcal{H}$, and $Q \in C_p$ (the Schatten–Von Neumann class , $1 \le p \le \infty$), we obtained necessary and sufficient condition for $\Delta X = Q$ to have a unique solution X in C_p.

(4) When S, T are bounded operators, $\mathcal{T}(\cdot)$ is (C_0) and the result coincides with that of Krengel and Lin [3]. However, if S, T are unbounded, $\mathcal{T}(\cdot)$ is no longer (C_0) and their result does not apply.

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