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ITERATIVE METHODS FOR BOUNDARY INTEGRAL EQUATIONS

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Abstract

We review some iterative methods for solving boundary integral equations which arise in Dirichlet and Neumann problems for the Helmholtz and Laplace equations. In particular we show how these integral equations may be transformed so that they may be solved by Neumann-Poincare Picard iteration.

1. Introduction

We review some iterative methods for solving boundary integral equations which are simple, effective, and largely overlooked by engineers and scientists interested in numerical solutions of problems of practical interest. The boundary integral equations considered are representative of a wide class which may be solved by iteration, however here we will restrict attention to these arising in the Dirichlet and Neumann problems for the Laplace and Helmholtz equations.

The boundary value problems consist of finding solutions at the partial differential equation $(\nabla^2 + k^2)u = 0$ for points p either interior or exterior to a smooth closed bounded simply connected surface Γ in $I\!R^3$ with exterior Ω_+ and interior Ω_- which take on boundary values $u|_\Gamma = f$ (Dirichlet) or $\frac{\partial u}{\partial n}\Big|_\Gamma = f$ (Neumann). Included is the case when $k = 0$ where Laplace's equation is the governing field equation. In addition, considering the exterior problem, we impose a radiation condition when $k \neq 0, r(\frac{du}{\partial r} - iku) = o(1)$ uniformly in (θ, ϕ) as $r \to \infty$ where (r, θ, ϕ) are the spherical coordinates of a point p in Ω_+, and a regularity condition when $k = 0, u = O(\frac{1}{r})$ as $r \to \infty$.

As is well known these problems may be cast as boundary integral equations. Using the notation and results of Kleinman and Roach [11] we define single and double layer operators by

$$S\omega := \int_\Gamma \omega(q)\gamma(p,q)\, ds_q, \quad p \in I\!R^3, \tag{1}$$

$$D\omega := \int_\Gamma \omega(q) \frac{\partial}{\partial nq} \gamma(p,q)\, ds_q, \quad p \notin \Gamma, \tag{2}$$

where

$$\gamma(p,q) = -\frac{e^{ik|p-q|}}{2\pi|p-q|} \tag{3}$$

and boundary integral operators

$$K\omega = \int_\Gamma \omega(q) \frac{\partial}{\partial n_p} \gamma(p,q) ds_q, \quad p \in \Gamma, \tag{4}$$

$$\overline{K}^*_\omega = \int_\Gamma \omega(q) \frac{\partial}{\partial n_q} \gamma(p,q) ds_q, \quad p \in \Gamma, \tag{5}$$

$$\frac{\partial}{\partial n} D\omega = \frac{\partial}{\partial n_p} \int_\Gamma \omega(q) \frac{\partial}{\partial n_q} \gamma(p,q) ds_q, \quad p \in \Gamma. \tag{6}$$

Then the solution of the Dirichlet problem using the layer ansatz is

$$u = \mp D\omega, \quad \begin{matrix} + & p \in \Omega_+, \\ - & p \in \Omega_-, \end{matrix} \tag{7}$$

where ω is a solution of the boundary integral equation

$$\omega \mp \overline{K}^* \omega = f, \quad \begin{matrix} - \text{ exterior Dirichlet problem}, \\ + \text{ interior Dirichlet problem} \end{matrix} \tag{8}$$

whereas using Green's theorem, the so called direct method, the solution is given by

$$u = \mp \frac{1}{2}(Df - S\omega), \quad \begin{matrix} - & p \in \Omega_+, \\ + & p \in \Omega_-, \end{matrix} \tag{9}$$

where

$$\omega \mp K\omega = \mp D_n f, \quad \begin{matrix} - \text{ exterior Dirichlet problem}, \\ + \text{ interior Dirichlet problem}. \end{matrix} \tag{10}$$

Similarly the solution of the Neumann problem using layers is given by

$$u = \pm S\omega, \quad \begin{matrix} +, & p \in \Omega_+, \\ -, & p \in \Omega_-, \end{matrix} \tag{11}$$

where

$$\omega \pm K\omega = f, \quad \begin{matrix} + \text{ exterior Neumann problem}, \\ - \text{ interior Neumann problem} \end{matrix} \tag{12}$$

while using the direct method the solution is

$$u = \pm\frac{1}{2}(Sf - D\omega), \qquad \begin{matrix} + & p \in \Omega_+, \\ - & p \in \Omega_-, \end{matrix} \tag{13}$$

where

$$\omega \pm \overline{K}^*\omega = \pm Sf, \qquad \begin{matrix} + \text{ exterior Neumann problem}, \\ - \text{ interior Neumann problem}. \end{matrix} \tag{14}$$

The formulation in terms of the Green theorem has the advantage of guaranteeing that the right hand side in each of the integral equations is in the range of the integral operator.

All of the equations are of the form

$$L\omega = (I - A)\omega = F \tag{15}$$

with F known, and A a compact operator on $L_2(\Gamma)$ if Γ is smooth (Lyapanov). Note that the integral equations for the Laplace case are obtained by setting $k = 0$ in the fundamental solution (3) which appears in the kernel of the layer and boundary integral operators. Note also that in general the boundary integral operators are non-selfadjoint hence L in (15) is generally non-selfadjoint (an exception occurs when $k = 0$ and Γ is a sphere).

The simplest iterative solution of (15) is in terms of the Neumann-Poincaré-Picard iterates

$$\begin{aligned} &\omega_0 : \text{ arbitrary}, \\ &\omega_{n+1} = A\omega_m + F. \end{aligned} \tag{16}$$

Unfortunately this sequence often does not converge for the problems under consideration and we discuss a few methods for restoring convergence.

2. Eigenvalue Shifting

Perhaps the oldest method for obtaining a convergent iterative process is the so called eigenvalue shifting technique first employed in potential theory ($k = 0$) for the operators described earlier, see Goursat [5, p. 195] and Kantorovich and Krylov [8, p. 118].

Essentially the method involves introducing an eigenvalue parameter in equation (15) obtaining

$$\lambda\omega - A\omega = F \tag{17}$$

then employing a Möbius transformation of the eigenvalue parameter retaining 1 and ∞ as fixed points, namely

$$\mu = \alpha\lambda + (1-\alpha) \quad \text{on} \quad \lambda = \frac{\mu}{\alpha} + \left(1 - \frac{1}{\alpha}\right), \tag{18}$$

where α is as yet undetermined, in which case (17) becomes

$$\mu\omega - (1-\alpha)\omega - \alpha A\omega = \alpha F \tag{19}$$

and on setting $\mu = 1$ and using $L = I - A$,

$$\omega - (I - \alpha L)\omega = \alpha F. \tag{20}$$

The strategy is to choose α so that the spectral radius of $I - \alpha L, r_\sigma(I - \alpha L)$ is less than one in which case the iteration

$$\begin{aligned} &\omega_0 \quad \text{arbitrary}, \\ &\omega_{n+1} = (I - \alpha L)\omega_n + \alpha F \end{aligned} \tag{21}$$

converges to the desired solution.

Situations when this strategy may be implemented are known. In [10] it is shown that, letting $\sigma(A)$ denote the spectrum of A, if $\sigma(A)$ is real,

$$\lambda^+ := \sup_{\substack{\lambda\in\sigma(A)\\ \lambda>0}} \{\lambda\}, \quad \lambda^- := \inf_{\substack{\lambda\in\sigma(A)\\ \lambda<0}} \{\lambda\}$$

then

$$\min_{\alpha\in R} r_\sigma(I - \alpha L) = \frac{\lambda^+ - \lambda^-}{2 - \lambda^+ - \lambda^-} \tag{22}$$

and this minimum is assumed when

$$\alpha = \frac{2}{2 - \lambda^+ - \lambda^-}. \tag{23}$$

In particular if $\lambda^+ > 1$ then $r_\sigma(I - \alpha L) > 1$ for any value of α. However when $\lambda^+ \leq 1$ and $\lambda^- = -1$ then

$$\min_{\alpha\in R} r_\sigma(I - \alpha L) = \frac{1 + \lambda^+}{3 - \lambda^+} \leq 1 \tag{24}$$

and is achieved when

$$\alpha = \frac{2}{3 - \lambda^+}. \tag{25}$$

Moreover if $\lambda^+ < 1$ then

$$\min_{\alpha\in R} r_\sigma(I - \alpha L) < 1.$$

Kress and Roach [7] treated the special case when $\lambda^+ = 0, \lambda^- = -1$ and showed, in agreement with the above, that when

$$\alpha = \frac{2}{3}, \quad r_\sigma(I - \alpha L) = \frac{1}{3}. \tag{26}$$

Of course the critical question is whether the requisite properties of the spectrum of A are satisfied for the operators of interest. The earliest answer was provided by Plemelj, see Jörgens [7, p. 134], who proved that when $k = 0$ and $A = -K$ and Γ was smooth then

$$\sigma(A) \subset [-1, 1) \text{ and } \lambda^- = -1. \tag{27}$$

In fact even if Γ is not smooth it was shown, [16], that this remains true.

However, except for the particular case cited above, Plemelj's theorem is not available and the eigenvalue shifting method is apparently not known to be applicable. Another iterative method available to us is described next which, while derived differently, is closely related to the method just discussed.

3. Bialy's Method

Originally derived for selfadjoint operators on a Hilbert space, e.g. Wiarda [23], the method we now describe has been discussed by many authors including Riesz-Nagy [10, p. 265], Bückner [4], Bialy [2] and its application to the boundary integral operators described previously in [13] and [14].

The essential features of the method are contained in the following theoram.

If L is a bounded selfadjoint operator mapping a Hilbert space H into itself and L is boundedly invertible then

$$\|I - \alpha L\| < 1 \quad \text{if} \quad \alpha \in \left(0, \frac{2}{\|L\|}\right).$$

This means that the equation $L\omega = F$ is solvable by first rewriting it as (20) and then employing the iteration method (21). In fact the optimal choice of α is given by Kantorovich and Akilov [9, p. 446] in terms of

$$m := \inf_{\|\omega\|=1} \|L\omega\| \text{ and } M := \sup_{\|\omega\|=1} \|L\omega\|$$

as follows:

$$\inf_{\alpha \in (0, \frac{2}{\|L\|})} \|I - \alpha L\| = \frac{M - m}{M + m} \tag{28}$$

and is achieved for

$$\alpha = \frac{2}{M + m}. \tag{29}$$

Of course, as noted earlier, the operators A and hence L treated here are in general non-selfadjoint however this drawback may be overcome by considering not equation (15) but rather the equation

$$L^* L\omega = L^* F \tag{30}$$

where L^* is the Hilbert space adjoint of L. Since L^*L is selfadjoint the previous results apply and we find the associated equation

$$\omega - (I - \alpha L^* L)\omega = \alpha L^* F$$

may be solved iteratively since

$$\|I - \alpha L^* L\| < 1 \quad \text{for all } \alpha \in \left(0, \frac{2}{\|L\|^2}\right).$$

Moreover

$$\inf_{\alpha \in (0, \frac{2}{\|L\|^2})} \|I - \alpha L^* L\| = \frac{M^2 - m^2}{M^2 + m^2} \tag{31}$$

and is achieved for

$$\alpha = \frac{2}{M^2 + m^2} \tag{32}$$

where m and M are defined exactly as before.

It is clear from equation (31) that $\|I - \alpha L^* L\| < 1$ only if $m > 0$, that is, L^*L is bounded below, which means that $L\omega = F$ must be uniquely solvable. Actually the iterates will converge albeit more slowly, even if $L\omega = 0$ has nontrivial solutions provided that F lies in the range of L and -1 is not an eigenvalue of $I - \alpha L^* L$. This stronger result was obtained by Browder and Petryshyn [3]. Additional references are given by Nashed [18] and applied to the integral equations of potential theory in [1].

4. Comparison of Bialy and Eigenvalue Shifting

We now have two methods of solving equation (15); eigenvalue shifting where

$$\omega_{n+1} = (I - \alpha L)\omega_n + \alpha F, \tag{33}$$

and the Bialy method where

$$\omega_{n+1} = (I - \alpha L^* L)\omega_n + \alpha L^* F. \tag{34}$$

Some measure of comparison of the two methods is available in the case when Plemelj's theorem (27) holds. Then as noted earlier

$$\alpha = \frac{2}{3 - \lambda^+} \Rightarrow r_\sigma(I - \alpha L) = \frac{1 + \lambda^+}{3 - \lambda^+}. \tag{35}$$

Moreover since $\sigma(I - \alpha L^* L) = \{1 - \alpha(1-\lambda)^2 | \lambda \in \sigma(A)\}$ it is straightforward to establish that

$$\alpha = \frac{2}{4 + (1-\lambda^+)^2} \Rightarrow r_\sigma(I - \alpha L^* L) = \frac{4 - (1-\lambda^+)^2}{4 + (1-\lambda^+)^2} \tag{36}$$

and this is the smallest value of the spectral radius over all real α.

In the case that $\lambda^+ = 0$ we find the following optimal results,

$$\alpha = \frac{2}{3} \Rightarrow r_\sigma(I - \alpha L) = \frac{1}{3} \tag{37}$$

while

$$\alpha = \frac{2}{5} \Rightarrow r_\sigma(I - \alpha L^* L) = \frac{3}{5}. \tag{38}$$

We note in passing that C. Neumann's original solution of the potential problem corresponds to a choice of $\alpha = 1/2$ in which case $r_\sigma(I - \alpha L) = \frac{1}{2}$ which, though not optimal for eigenvalue shifting, is clearly superior to the optimal result using the Bialy method.

This discussion indicates that the eigenvalue shifting method is preferable but the question arises as to its applicability when Plemelj's theorem is not available.

5. Generalized Over–Relaxation

In general the operator A will be non-selfadjoint and its spectrum will be complex. Nevertheless we wish to employ the same iterative algorithm used in the eigenvalue shifting method, (21), even though we don't have Plemelj's theorem on which to base a proof of convergence. This algorithm is the simplest form of the generalized over relaxation method of Petryshyn [19] which is the operator equivalent of a stationary one step Richardson method in matrix theory (e.g., Varga [22, p. 141]).

Of interest is whether it is possible to choose α, now possibly complex valued, for which $r_\sigma(I - \alpha L) < 1$. One answer is provided in [15], phrased in terms of $\sigma(L)$ rather than $\sigma(A)$ where $\sigma(L) = \{1 - \lambda | \lambda \in \sigma(A)\}$. Choose $0 \leq \arg \mu < 2\pi$ and define

$$\theta_m = \inf_{\mu \in \sigma(L)} \arg \mu \tag{39}$$

and

$$\theta_M = \sup_{\mu \in \sigma(L)} \arg \mu. \tag{40}$$

Then the crucial result may be stated in the following.

Theorem. *if $m > 0$ and $\theta_M = \theta_m < \pi$ then there exists $\alpha \in \mathbb{C}$ such that*

$$r_\sigma(I - \alpha L) < I. \tag{41}$$

This result considerably enlarges the class of operators, beyond those for which Plemelj's theorem is available, for which the iteration method (21) can be shown to converge. In addition the range of values of α for which (41) holds is characterized in [15] in terms of $\sigma(L)$. The conditions for (41) to hold are

$$0 < |\alpha| < \frac{\sin \theta/2}{M} \tag{42}$$

and

$$-\frac{\pi}{2} + \frac{\theta}{2} - \inf_{\mu \in \sigma(L)} \arg \mu < \arg \alpha < \frac{\pi}{2} - \frac{\theta}{2} - \sup_{\mu \in \sigma(L)} \arg \mu \tag{43}$$

where θ is any value in the interval $(0, \pi - \theta_M + \theta_m)$.

Not withstanding these results, the task of actually choosing an appropriate value of α is still a problem. It depends, in this analysis, on having detailed knowledge of $\sigma(L)$. Unfortunately, in particular cases such information is generally unavailable or at best may be found only at the expense of considerable computational effort. Nevertheless, as discussed in the next section, this method may be employed to good advantage in numerical calculation.

6. Numerical Implementation

In attempting to actually solve equation (15) iteratively the equation is first discretized, either using a Galerkin approximation or collocation, leading to an approximate, matrix, equation

$$L^{(N)}\omega^{(N)} = F^{(N)} \tag{44}$$

where $\omega^{(N)}$ and $F^{(N)}$ are N dimensional vectors and $L^{(N)}$ is an $N \times N$ matrix. Then these matrix equations are solved iteratively using the algorithms

$$\omega_{n+1}^{(N)} = (I - \alpha L^{(N)})\omega_n^{(N)} + \alpha F^{(N)} \tag{45}$$

and

$$\omega_{n+1}^{(N)} = I - \alpha L^{(N)*} L^{(N)} \omega_n^{(N)} + \alpha L^{(N)*} F^{(N)} \tag{46}$$

where $L^{(N)*}$ is the Hermitian transpose of the matrix $L^{(N)}$. An operation count shows that each iteration in the eigenvalue shifting or generalized over relaxation method, (45), requires one matrix-vactor multiplication or N^2 operations while the Bialy scheme, (46), requiress two sequential matrix vector multiplications (which is more economic than forming the matrix product $L^{(N)*}L^{(N)}$) or $2N^2$ operations per

iteration. Comparing the operation count in solving (44) by Gaussian elimination, $\frac{N^3}{3}$, with nN^2 and $2nN^2$, the number of operators needed for n iterations in each of the above methods we see that if the desired accuracy is achieved in less than $\frac{N}{3}$ or $\frac{N}{6}$ iterations, for the generalized over relaxation and Bialy methods respectively, then these solution methods compete favorably with Gaussian elimination.

Critical to the rate of convergence of the sequence of iterates is the choice at the relaxation parameter α as was evident in the previous analysis. While an optimal choice of α is possible only with the benefit of detailed knowledge of $\sigma(L)$, it is still possible to use these methods to good advantage even in the absence of such knowledge. In [14] the Bialy method, (46), was employed to solve the problem of acoustic scattering by a rigid obstacle Γ. In this case $A = -\overline{K}^*$ and a modified Green's function technique was employed to eliminate the occurrence of "irregular frequencies," values of k for which the homogeneous equation has nontrivial solutions which occur at eigenvalues of the interior Dirichlet problem. The modified Green's function method, proposed by Jones [6] and further developed by Ursell [21] as well as [12], essentially involves adding a sum of radiating multipoles to the fundamental solution (3). It has the effect of ensuring that L is bounded below, i.e., $m > 0$. Numerical results were found for a variety of simple configurations including spheres, spheroids, and spherically capped cylinders however the relaxation parameter α was chosen in all cases to be the same, namely the optimal value for a sphere which can be determined because the equation may be solved exactly in this case. Despite the fact that this choice is clearly not optimal for nonspherical shapes, it was nontheless found that the Bialy method produced good results, residual errors of order 10^{-3}, for a modest number of iterates, sufficiently small so that the Bialy method competes favorably with Gaussian elimination in all cases treated.

Even more dramatic results are presented in [15] for the generalized over-relaxation method (45). Since $\sigma(L)$ is complex, the optimal α (that value which minimizes $r_\sigma(I - \alpha L)$) will also be complex and a means of making a reasonable choice without determining $\sigma(L)$ precisely is essential to the successful implementation of the iteration method. One method of choosing α proposed by van den Berg, as described in [15], consists of finding that α which minimizes $\|(I - \alpha L)F\|$. A necessary condition for this function to be a minimum is found to determine α to be

$$\alpha = \frac{(F, LF)}{\|LF\|^2}. \tag{47}$$

As reported in [15], this choice makes the generalized overrelaxation method (45) remarkably effective in a variety of examples including the boundary integral equations considered here.

The iterative methods described here will never successfully compete against more sophisticated techniques such as multistep nonstationary Richardson or conjugate gradient methods if storage is not limited or extreme accuracy is needed.

However if only modest error criteria need be met and storage is limited then the extremely simple algorithms discussed in this paper can yield useful numerical results with a minimum of computer size, computation time, and programming complexity.

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