UPPER LIMITATION OF KOLMOGOROV COMPLEXITY AND UNIVERSAL P. MARTIN-LÖF TESTS *

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In this paper we study the Kolmogorov complexity of initial strings in infinite sequences (being inspired by [9]), and we relate it with the theory of P.Martin-Löf random sequences. Working with partial recursive functions instead of recursive functions we can localize on an apriori given recursive set the points where the complexity is small. The relation between Kolmogorov's complexity and P. Martin-Löf's universal tests enables us to show that the randomness theories for finite strings and infinite sequences are not compatible (e.g.because no universal test is sequential).

we lay stress upon the fact that we work within the general framework of a non-necessarily binary alphabet.

Preliminaries

Throughout the paper $\mathbf{N} = \{0, 1, 2, \ldots\}$ will be the set of natural numbers. The integral part of a real number x will be denoted by $[x]$. If A is a finite set, then card A is the number of elements of A.

For every non-empty sets A and B we shall write $f : A \overset{o}{\to} B$ to denote a partial function, i.e. a function $f : A' \to B$, where A' is a subset of A. We shall consider that A' is the domain of f and we shall write A'=dom(f). We shall say that f is undefined at x and we shall write $f(x) = \infty$ in case x is not in dom(f). The graph of f is the set $\{(x, f(x)) | x \in \mathrm{dom}(f)\} \subset A \times B$. In case $f, g : A \overset{o}{\to} B$ are two partial functions such that dom$(f) \subset$ dom(g) and $g(x) = f(x)$ for every $x \in \mathrm{dom}(f)$, we say that g extends f.

We work with a finite alphabet $X = \{a_1, a_2, \ldots, a_p\}$, where $p \geq 2$ is a fixed natural (the binary case $p = 2$ is the most commonly used). The free monoid generated by X under concatenation is X^*. Its elements are called strings. The length of a string $x = x_1 x_2 \ldots x_n$ of X^* is $\rho(x) = n$. The empty string λ has length 0. If $x, y \in X^*$, then we write $x \subset y$ in case $y = xz$ for some string z. For every natural $n \geq 1$, put $X^n = \{x \in X^* | \rho(x) = n\}$. The set X^* is lexicographically ordered by $a_1 < a_2 < \ldots < a_p < a_1 a_1 < a_1 a_2 < \ldots a_1 a_p < \ldots$.

* Received April 4 , 1987.

We denote by X^∞ the set of all sequences $x = x_1x_2\dots x_n\dots$, where $x_n \in X$ for all natural $n \geq 1$. For such an $\underline{x}$ and for every natural $n \geq 1$, put $\underline{x}(n) = x_1x_2\dots x_n \in X^*$. If $y \in X^*, y = y_1y_2\dots y_m$ and $\underline{x} \in X^\infty, \underline{x} = x_1x_2\dots x_n\dots$, we shall write $y\underline{x}$ to denote the sequence $y_1y_2\dots y_mx_1x_2\dots x_n\dots$, and $\lambda\underline{x} = \underline{x}$. For every $V \subset X^*$, put $VX^\infty = \{y\underline{x} | y \in V, \underline{x} \in X^\infty\}$. In case V is a singleton, i.e. $V = \{y\}$, we write yX^∞ instead of VX^∞.

For the Recursive Function Theory see [10]. Dealing with computability, we do not distinguish between $\mathbf{N}, \mathbf{N}\backslash\{0\}$ and X^*. A recursively enumerable (r.e.) set is the domain of some partial recursive (p.r.) function.

A P. Martin-Löf test (M-L test in the sequel) is a (possibly empty) r.e. set $V \subset X^* \times (\mathbf{N}\backslash\{0\})$ having the following properties :

i) For every natural $m \geq 1$, one has $V_{m+1} \subset V_m$, where $V_m = \{x \in X^* | (x, m) \in V\}$.

ii) For all natural non-null m and n one has card $(X^n \cap V_m) < p^{n-m}/(p-1)$.

A M-L test having the following additional property :

iii) For every natural $m \geq 1$, and for all strings x, y in X^* with $x \subset y$ and $x \in V_m$ one has $y \in V_m$ is called a sequential M-L test (s. M-L test in the sequel).

A (sequential) M-L test U will be called a universal (universal sequential) M-L test if for every (sequential) M-L test V there exists a natural c (depending upon V and U) such that $V_{m+c} \subset U_m$ for all natural $m \geq 1$. For the existence of universal (universal sequential) M-L tests see [7], [8], [11], [1] and [3].

The critical level induced by a M-L test V is the function $m_V : X^* \to \mathbf{N}$ given by $m_V(x) = \max\{m \in \mathbf{N} | x \in V_m\}$ in case $x \in V_1$, and $m_V(x) = 0$ otherwise.

Let $\varphi : X^* \times \mathbf{N} \xrightarrow{\circ} X^*$ be a p.r. function. According to A.N. Kolmogorov [6], we define the Kolmogorov complexity $K_\varphi : X^* \times \mathbf{N} \xrightarrow{\circ} \mathbf{N}$ as follows : $K_\varphi(x|n) = \min\{\rho(y) | \ y \in X^*, \varphi(y, n) = x\}$ if such y does exist, and $K_\varphi(x|n) = \infty$ in the opposite case. Now we can define, for every φ as adove, the set $V(\varphi) = \{(x, m) \in X^* \times \mathbf{N} | K_\varphi(x)\rho(x)) < \rho(x) - m\}$. It is readily seen that $V(\varphi)$ is a M-L test (see [1]). A p.r. function $\psi : X^* \times \mathbf{N} \xrightarrow{\circ} X^*$ is called a universal Kolmogorov algorithm in case for every p.r. function $\varphi : X^* \times \mathbf{N} \xrightarrow{\circ} X^*$ there exists a natural c (depending upon φ and ψ) such that $K_\psi(x|m) \leq K_\varphi(x|m) + c$ for all $x \in X^*$ and $m \in \mathbf{N}$. Universal Kolmogorov algorithms do exist (see [6], [1], [4]). Furthermore, for every universal Kolmogorov algorithm ψ, the M-L test. $V(\psi)$ is universal (see [2] and [4]).

Results

We begin with a slightly improved version of a result in [5].

Lemma 1. *Let $n(1), n(2), \dots, n(k)$ be natural numbers, $k \geq 1$. The following assertions are equivalent :*

A) *One has*

$$\sum_{i=1}^{k} p^{-n(i)} \geq 1. \tag{1}$$

B) *One can effectively find k strings $s(1), s(2), \ldots, s(k)$ in X^* with $\rho(s(i)) = n(i), i = 1, 2, \ldots, k$, such that*

$$\bigcup_{i=1}^{k} s(i)X^\infty = X^\infty. \quad (2)$$

Proof. A)$\Rightarrow$ B). We may assume that $n(1) \le n(2) \le \ldots \le n(k)$. In view of (1), the unmbers $n(1), n(2), \ldots, n(k)$ are not all distinct. So put $n(1) = n(2) = \ldots = n(t_1) = m_1 < n(t_1+1) = n(t_1+2) = \ldots = n(t_1+t_2) = m_2 < \ldots < n(t_1+t_2+\ldots+t_{u-1}+1) = n(t_1+t_2+\ldots+t_{u-1}+2)\ldots = n(t_1+t_2+\ldots+t_{u-1}+t_u) = m_u$. There are two distinct situations.

First situation. One has $t_1 \ge p^{m_1}$. In this case we shall take $\{s(1), s(2), \ldots, s(p^{m_1})\}$ to be X^{m_1}, in lexicographical order. The remaining strings $s(i)$ can be taken arbitrarily with $\rho(s(i)) = n(i)$, because one has

$$\bigcup_{i=1}^{p^{m_1}} s(i)X^\infty = X^\infty.$$

Second situation. There exists a natural $2 \le h \le u$ such that $t_1 \cdot p^{-m_1} + t_2 \cdot p^{-m_2} + \ldots + t_{h-1} \cdot p^{-m_{h-1}} < 1$ and $t_1 \cdot p^{-m_1} + t_2 \cdot p^{-m_2} + \ldots + t_{h-1} \cdot p^{-m_{h-1}} + t_h \cdot p^{-m_h} \ge 1$. Multiplying by p^{-m_h} one can effectively find a natural $1 \le t \le t_h$ such that $t_1 \cdot p^{-m_1} + t_2 \cdot p^{-m_2} + \ldots + t_{h-1} \cdot p^{-m_{h-1}} + t \cdot p^{-m_h} = 1$. The last equality can be written as follows :

$$t_1 \cdot p^{m_h - m_1} + t_2 \cdot p^{m_h - m_2} + \ldots + t_{h-1} \cdot p^{m_h - m_{h-1}} + t = p^{m_h}. \quad (3)$$

We choose $s(1), s(2), \ldots s(t_1)$ to be the first (in lexicographical order) strings of length m_1. We have $\bigcup_{i=1}^{t_1} s(i)X^\infty = \bigcup xX^\infty$, where x runs over the first $t_1 \cdot p^{m_h - m_1}$ strings of length m_h (in lexicographical order).

The procedure continues in the same manner. Assume that we have already constructed the strings $s(1), s(2), \ldots, s(t_1)$ (of length m_1), $s(t_1+1), s(t_1+2), \ldots, s(t_1+t_2)$ (of length m_2), $\ldots, s(t_1+t_2+\ldots+t_{i-1}+1), s(t_1+t_2+\ldots+t_{i-1}+2, \ldots, s(t_1+t_2+\ldots+t_{i-1}+t_i)$ (of length m_i), for $i < h$. Suppose also that $\bigcup_{j=1}^{T_i} s(j)X^\infty = \bigcup_{x \in A_i} xX^\infty$, where A_i consists of the first $t_1 \cdot p^{m_h - m_1} + t_2 \cdot p^{m_h - m_2} + \ldots + t_i \cdot p^{m_h - m_i}$ strings of length m_i (in lexicographical order), and $T_i = t_1 + t_2 + \ldots + t_i$.

In view of (3) it is seen that $X^{m_i} \backslash A_i$ is not empty . Let x be the first element (in lexicographical order) of the set $X^{m_i} \backslash A_i$. Then set y as the first (in lexicographical order) element of $X^{m_{i+1} - m_i}$, and $s(T_i + 1) = xy$. Construct then the next strings of length m_{i+1} (in lexicographical order), namely: $s(T_i+2), s(T_i+3), \ldots, s(T_i + t_{i+1}) = s(t_1 + t_2 + \ldots + t_{i+1})$ (if $i + 1 < h$), or $s(T_{h-1}+1), s(T_{h-1}+2), \ldots, s(T_{h-1}+t) =$

$s(t_1+t_2+\ldots+t_{h-1}+t)$ (if $i=h-1$). It is seen that $\bigcup_{j=1}^{T} s(j)X^{\infty}=X^{\infty}$, where $T=t_1+t_2+\ldots+t_{h-1}+t$ (again by (3)). So, if $k>T$, the remaining $s(i)$ (for $i>k$) can be taken arbitrarily with $\rho(s(i))=h(i)$ and condition (2) will be fulfilled.

B) $\Rightarrow$ A). Again assume that $n(1)\le n(2)\le\ldots\le n(k)$ and put $H_i=\{x\in X^{n(k)}|s(i)\subset x\}, 1\le i\le k$. Condition (2) implies that $\bigcup_{i+1}^{k} H_i\supset X^{n(k)}$ and this in turn implies that $\sum_{i=1}^{k}\text{card } H_i\ge p^{n(k)}$. This means that $\sum_{i=1}^{k} p^{n(k)-n(i)}\ge p^{n(k)}$, i.e. exactly (1). ∎

In order to avoid repetitions, we introduce

Definition 2. *A p.r. function* $F:\mathbf{N}\xrightarrow{\circ}\mathbf{N}$ *is said to be small if* $\sum_{n=0}^{\infty} p^{-F(n)}=\infty$.

Here we accept the convention that for every natural n which is not in dom(F) one has $p^{-F(n)}=0$. ∎

Lemma 3. *Let F be a small function and let k be an integer such that $F(n)+k\ge 0$ for all n in* dom(F). *Define the function $F+k$:* dom$(F)\to\mathbf{N}$ *by* $(F+k)(n)=F(n)+k$.

Then, $F+k$ is a small function.

Example 4. a) L. k be in $\mathbf{N}$. The constant function $F:\mathbf{N}\to\mathbf{N}$ given by $F(n)=k$ for all $n\in\mathbf{N}$ is a small function.

b) Take α to be a strictly positive rational, $\alpha<1$ or $\alpha\ge p$. The p.r. function given by $F(n)=[\log_\alpha n]$ for $n\ge 1$ is a small function. In particular, $F(n)=[\log_p n]$ is small.

Lemma 5. *Let g be a small function with recursive graph. Then one can effectively find another small function G with recursive domain such that*

a) *The function g extends G.*

b) *For every $n\in$* dom(G) *one has $G(n)\le n$.*

c) *For every natural k there exists at most one natural n with $G(n)=n-k$.*

Proof. Define the p.r. function $G:\mathbf{N}\xrightarrow{\circ}\mathbf{N}$ as follows: $G(n)=g(n)$ if $g(n)\le n$ and $m-g(m)\ne n-g(n)$ for every natural $m<n$; $G(n)=\infty$ otherwise. Since g has a recursive graph, it follows that the conditions in the above definition are recursive and G satisfies A), b), c). It remains to prove that G has a recursive domain and

$$\sum_{n=0}^{\infty} p^{-G(n)}=\infty. \tag{4}$$

To this aim define the sets

$$A = \left\{ n \in \mathbf{N} \mid \; g(n) \le n \right\},$$
$$A_k = \left\{ n \in \mathbf{N} \mid \; g(n) = n - k \right\}, \quad k \in \mathbf{N}.$$

Notice that $A = \bigcup_{k=0}^{\infty} A_k$ and that the sets A_k are pairwise disjoint. Because g is small and $\sum_{n \in \mathbf{N} \setminus A} p^{-g(n)} = \infty$, one has $\sum_{n \in A} p^{-g(n)} = \infty$, which means that

$$\sum_{k \in B} \sum_{n \in A_k} p^{-g(n)} = \infty, \tag{5}$$

where $B = \{k \in \mathbf{N} | A_k \neq \emptyset\}$.

For every k in B denote by n_k the smallest element of A_k. Then $\mathrm{dom}(G) = \{n_k | k \in B\}$. So, $G(n) < \infty$ iff $G(n) \le m$ for some $m \le n$. Accordingly , $\mathrm{dom}(G)$ is recursive. We can write (5) in the form

$$a + b = \infty, \tag{6}$$

where $a = \sum_{k \in B} \sum_{n \in A_k \setminus \{n_k\}} p^{-g(n)}$ and $b = \sum_{k \in B} p^{-g(n_k)} = \sum_{n=0}^{\infty} p^{-G(n)}$.

Caution ! The sum over the empty set is null.

For every k in B one has

$$\begin{aligned} \sum_{n \in A_k \setminus \{n_k\}} p^{-g(n)} &= \sum_{n \in A_k \setminus \{n_k\}} p^{-(n-k)} \le \sum_{n = n_k + 1}^{\infty} p^{k-n} \\ &= p^{k-n_k}/(p-1) = p^{-g(n_k)}/(p-1). \end{aligned}$$

It follows that

$$a \le \sum_{k \in B} p^{-g(n_k)}/(p-1) = b/(p-1).$$

From (6) we deduce that $\infty = a + b \le b + b/(p-1)$; hence $b = \infty$, which is precisely (4). ∎

Proposition 6. Let g be a small function with recursive graph. Then we can effectively construct a recursive function $f : \mathbf{N} \to X^*$ such that for every sequence $\underline{x}$ in X^∞ the set $A(\underline{x}) = \{n \in \mathbf{N} | f(n) = \underline{x}(\rho(f(n)))$ and $\rho(f(n)) = g(n)\}$ is infinite.

Proof. Given g, construct G according to Lemma 5. Because $\mathrm{dom}(G)$ is recursive, we can define f by the following procedure:

Stage 0.

1. Compute $n_0 = \min \left\{ n \in \mathbf{N} \middle| \sum_{j=0}^{n} p^{-G(j)} \ge 1 \right\}$.

2. Extract from the vector $(G(0), G(1), \ldots, G(n_0))$ all finite components and call them $(G(i(0)), G(i(1)), \ldots, G(i(k_0))), i(0) < i(1) < \ldots < i(k_0)$.

3. Construct k_0 strings $s(0), s(1), \ldots, s(k_0)$ in X^* such that $\rho(s(j)) = G(i(j)), 0 \le j \le k_o$, and for every $\underline{x}$ in X^∞ there exists a natural $0 \le j \le k_o$ satisfying $s(j) = \underline{x}(\rho(s(j)))$. This is done by Lemma 1, because of the choice of n_0 : $\sum_{j=0}^{k_0} p^{-G(i(j))} \ge 1$.

4. Put $f(i(j)) = s(j)$ for all natural $0 \le j \le k_0$, and $f(m) = \lambda$ for every m in $\{0, 1, \ldots, n_0\} \backslash \{i(0), i(1), \ldots, i(k_0)\}$.

Stage q+1.

1. Compute $n_{q+1} = \min\{n \in \mathbf{N} | n > n_q$ and $\sum_{j=n_q+1}^{n} p^{-G(j)} \ge 1\}$.

2. Extract from the vector $(G(n_q+1), G(n_q+2), \ldots, G(n_{q+1}))$ the finite components, thus obtaining the vector $(G(i(k_q+1)), G(i(k_q+2)), \ldots, G(i(k_{q+1}))), i(k_q+1) < i(k_q+2) < \ldots < i(k_{q+1})$.

3. Find the strings $s(k_q+1), s(k_q+2), \ldots, s(k_{q+1})$ in X^* having $\rho(s(j)) = G(i(j))$ for $j = k_1+1, k_q+2, \ldots, k_{q+1}$, such that for each $\underline{x}$ in X^∞ there exists a natural $k_q+1 \le j \le k_{q+1}$ with $s(j) = \underline{x}(\rho(s(j)))$.

4. Define $f(i(j)) = s(j)$ for all j in $\{k_q+1, k_q+2, \ldots, k_{q+1}\}$, and $f(m) = \lambda$ for m in $\{n_q+1, n_q+2, \ldots, n_{q+1}\} \backslash \{i(k_q+1), i(k_q+2), \ldots, i(k_{q+1})\}$.

The procedure above defines a recursive function f. For every $\underline{x}$ in X^∞ the set $A(\underline{x})$ is infinite because dom(G) is infinite and G is small. ∎

Proposition 7. Let g be a small function with recursive graph and let $\psi : X^* \mathbf{N} \overset{\circ}{\to} X^*$ be a universal Kolmogorov algorithm. Then we can find a natural c (depending upon g and ψ) such that for every $\underline{x}$ in X^∞ there exist infinitely many n in dom(g) having the property

$$K_\psi(\underline{x}(n)|n) \le n - g(n) + c. \tag{7}$$

Proof. Given g we construct G like in Lemma 5. With the aid of G we construct the recursive function $f : \mathbf{N} \to X^*$ having the property that for every sequence $\underline{x}$ in X^∞ the set $A(\underline{x}) = \{n \in \mathbf{N} | f(n) = \underline{x}(\rho(f(n)))$ and $\rho(f(n)) = G(n)\}$ is infinite (we have made use of proposition 6). Now we can define the p.r. function $\varphi : X^* \times \mathbf{N} \overset{\circ}{\to} X^*$ as follows : $\varphi(y, n) = f(n)y$ if $G(n) = n - \rho(y)$, and $\varphi(y, n) = \infty$ otherwise.

Take a sequence $\underline{x} = x_1 x_2 \ldots x_n \ldots$ in X^∞. Notice that the set $A'(\underline{x}) = \{n \in \mathbf{N} | \underline{x}(\rho(f(n)) = f(n)$ and $\rho(f(n)) = G(n) < n\}$ is infinite because $A(\underline{x})$ defined above is infinite and the set $\{n \in \mathbf{N} | f(n) = \underline{x}(\rho(f(n)))$ and $\rho(f(n)) = G(n) = n\}$ has at most one element according to Lemma 5. For every n in $A'(\underline{x})$ we construct the string $y(f, n) = x_{u+1} x_{u+2} \cdots x_{u+n-G(n)}$, where $u = \rho(f(n)) = G(n)$. We have $\varphi(y(f, n), n) = f(n)y(f, n) = \underline{x}(n)$, which shows that $K_\varphi(\underline{x}(n)|n) \le \rho(y(f, n)) = n - G(n)$. Kolmogorov's theorem furnishes a natural c such that $K_\psi(\underline{x}(n)|n) \le K_\varphi(\underline{x}(n)|n) + c \le n - G(n) + c = n - g(n) + c$ for every n in $A'(\underline{x}) \subset$ dom(g). ∎

Our next purpose is to get rid of the constant c which appears in (7). To this aim we prove

Lemma 8. *Let F be a small function with recursive domain. Then, we can effectively find a small function F^* with the same domain as F and having the following supplementary property: for every natural c there exists a natural N_c such that $F^*(n) \geq F(n) + c$ for all $n \in \mathrm{dom}(F)$ and $n \geq N_c$.*

Proof. Let $r : \mathbf{N} \to \mathbf{N}$ be a recursive strictly increasing function such that $\mathrm{dom}(F) = \{r(i) | i \in \mathbf{N}\}$. Put $u(n) = F(r(n))$ for every natural n. Then

$$\sum_{n=0}^{\infty} p^{-u(n)} = \infty. \tag{8}$$

In view of (8) we can effectively find a recursive strictly increasing function $A : \mathbf{N} \to$

$$\sum_{n=A(i)+1}^{A(i+1)} p^{-u(n)} \geq p^{i+1} \tag{9}$$

for all natural i. Now we can define the recursive function $v : \mathbf{N} \to \mathbf{N}$ by $v(n) = u(n) + i + 1$, if $A(i) + 1 \leq n < A(i+1)$. From (9) it follows that

$$\sum_{n=A(i)+1}^{A(i+1)} p^{-v(n)} \geq 1. \tag{10}$$

From (10) we get

$$\sum_{n=0}^{\infty} p^{-v(n)} = \infty.$$

The function F^* is defined by $F^*(r(n)) = v(n)$ for all natural n. ∎

The following theorem is the basic result of the present paper.

Theorem 9. *Let F be a small function with recursive domain and let $\psi : X^* \times \mathbf{N} \overset{\circ}{\to} X^*$ be a universal Kolmogorov algorithm. Then, for each sequence $\underline{x}$ in X^∞, the inequality*

$$K_\psi(\underline{x}(n)|n) \leq n - F(n)$$

holds for infinitely many n in $\mathrm{dom}(F)$.

Proof. With the aid of F we construct the small function F^* given by Lemma 8. Applying Proposition 7 to F^* instead of g we get a natural c (depending ultimately upon F and ψ) such that the set $H(\underline{x}) = \{n \in \mathbf{N} | K_\psi(\underline{x}(n)|n) \leq n - F^*(n) + c\}$ is infinite for every $\underline{x}$ in X^∞. For this c we can find, using Lemma 8, a natural N_c such that $F^*(n) \geq F(n) + c$ for all $n \geq N_c$ in $\mathrm{dom}(F)$. It follows that for every $\underline{x}$ in X^∞ the set $T(\underline{x}) = H(\underline{x}) \cap \{n \in \mathrm{dom}(F) | F^*(n) \geq F(n) + c\}$is still infinite and for every n in $T(\underline{x})$ one has $K_\psi(\underline{x}(n)|n) \leq n - F^*(n) + c \leq n - F(n)$. ∎

In the sequel we shall use Theorem 9 to derive a series of results concerning M-L tests.

Corollary 10. Let $\psi : X^* \times \mathbf{N} \overset{\circ}{\to} X^*$ be a universal Kolmogorov algorithm. Then ψ has the following properties:

(P) Assume F is a small function taking at most finitely many zero values and having a recursive domain . Then, for every $\underline{x}$ in X^∞, one has $(\underline{x}(n), F(n)) \in V(\psi)$ for infinitely many natural n.

(PP) Assume $k \geq 1$ is natural. Then, for every $\underline{x}$ in X^∞, one has $(\underline{x}(n), k) \in V(\psi)$ for infinitely many natural n.

Proof. We must prove only (P), because (PP) follows from (P) taking in particular $F : \mathbf{N} \to \mathbf{N}, F(n) = k$ for all natural n.

Define the small function $G = F + 1$. According to Theorem 9, for every $\underline{x}$ in X^∞one has $K_\psi(\underline{x}(n)|n) \leq n - G(n) < n - F(n)$ for infinitely many natural n. This means that for these n one has $(\underline{x}(n), F(n)) \in V(\psi)$. ∎

Theorem 11. *Let U be a universal M-L test. Then U has the following properties:*

(PU) *Assume F is a small function with recursive domain taking at most finitely many zero values. Then, for every $\underline{x}$ in X^∞ one has $(\underline{x}(n), F(n)) \in U$ for infinitely many natural n.*

(PPU) *Assume $k \geq 1$ is natural. Then, for every $\underline{x}$ in X^∞ one has $\underline{x}(n) \in U_k$ for infinitely many natural n.*

Proof. Property (PPU) follows immediately from (PU), as in the proof of Corollary 10.

To prove (PU), consider a universal Kolmogorov algorithm and put $V = V(\psi)$. The universality of U yields a natural c such that $V_{F(n)+c} \subset U_{F(n)}$ for every n in $\mathrm{dom}(F)$, $F(n) > 0$. Since $F + c$ is small, we can use Corollary 10 to show that $(\underline{x}(n), F(n) + c) \in V$ for infinitely many natural n in $\mathrm{dom}(F)$, which implies $(\underline{x}(n), F(n)) \in U$, for the same n. ∎

Theorem 11 can be reformulated as follows:

Theorem 12. *Let U be a universal Kolmogorov algorithm and let F be a small function with recursive domain. Then, for every $\underline{x}$ in X^∞ one has*

$$m_U(\underline{x}(n)) \geq F(n)$$

for infinitely many n in $\mathrm{dom}(F)$.

In particular, for all $\underline{x}$ in X^∞ and all natural k one has $m_U(\underline{x}(n)) \geq k$ for infinitely many natural n. ∎

Comments. 1) Theroem 11 contains Corollary 10 because $V(\psi)$ is universal whenever ψ is a universal Kolmogorov algorithm (see [2], [4]).

2) Every p.r. function with recursive domain has a recursive graph, but the converse is false. Working with p.r. functions instead of recursive functions as in the seminal paper [9] (see also [5]) we can obtain more precise results. For instance, using the small function $F(n) = [\log_p n]$ defined only on the set P of primes, one can deduce from Theorem 9 that $K_\psi(\underline{x}(n)|n) \leq n - [\log_p n]$ for an infinity of primes n. Here we have made use of the fact that $\sum_{n \in P} n^{-1} = \infty$. ∎

Theorem 13. *Let V be a s.M-L test. Then, for every natural $m \geq 1$, one has $V_m X^\infty \neq X^\infty$.*

Proof. A sub-tree is a non-empty set $S \subset X^*$ such that for every $x \in S$ one has $A(x) = \{y \in X^* | y \subset x\} \subset S$. Every $\underline{x} \in X^\infty$ puts into evidence the set $B(\underline{x}) = \{\underline{x}(n) | n \in \mathrm{N}, n \geq 1\} \bigcup \{\lambda\}$. Notice that $B(\underline{x})$ is an infinite sub-tree which is linearly order with the order relation given by $\subset$. According to [10] we have the following result, called König's Lemma : For every infinite sub-tree $S \subset X^*$,there exists an $\underline{x}$ in X^∞ such that $B(\underline{x}) \subset S$.

Passing to our proof, it will be sufficient to show that $V_1 X^\infty \neq X^\infty$. To this aim, put $S = X^* \backslash V_1, S_0 = \{\lambda\}$ and, for every natural $n \geq 1$, put $S_n = X^n \cap S$. We shall see that S is an infinite sub-tree.

Indeed, for every natural $n \geq 1$, one has card $(X^n \cap V_1) < p^{n-1}/(p-1)$, which implies that card $(S_n) = p^n-$ card $(X^n \cap V_1) > p^n - p^{n-1}/(p-1) \geq p^{n-1}$. So S is infinite.

In order to see that S is a sub-tree, pick some $x \in S$ and show that $A(x) \subset S$. Assuming the contrary, let y be in $A(x)$ such that y is not in S and put $n = \rho(x) \geq 1$. On the other hand $y \notin S$, i.e. $y \in V_1$, which contradicts the fact that $x \notin V_1$ because $x \supset y$.

Applying König's Lemma to the infinite sub-tree S, we can find a sequence $\underline{x}$ in X^∞ with $B(\underline{x}) \subset S$. This implies that $\underline{x} \notin V_1 X^\infty$. ▮

Corollary 14. No universal M-L test is a s.M-L test. In particular, for every universal Kolmogorov algorithm ψ, the M-L test $V(\psi)$ is not a s.M-L test.

Proof. Let $\underline{x}$ be in X^∞. According to Theorem 11, $\underline{x}(n) \in U_1$ for at least one natural $n \geq 1$ (in fact, for infinitely many n), which means that $\underline{x} \in U_1 X^\infty$. So, $U_1 X^\infty = X^\infty$, which shows that U cannot be sequential (see Theorem 13). ▮

Remark. Corollary 14 points out that Martin-Löf theories of randomness for finite strings ([7],[8],[1]) and infinite sequences ([7],[8],[12],[5]) are not compatible.

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