ON S–STABILITY *

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Abstract

We prove in this paper that no consistent and well–defined Runge–Kutta method is S–stable and point out the errors of the theorems on S–stability in [1].

1. Introduction

To further study the stability of a general R–K method

$$y_{n+1} = y_n + \sum_{i=1}^{r} b_i k_i, \quad k_i = hf(t_n + c_i h, y_n + \sum_{j=1}^{r} a_{ij} k_j), \quad i = 1(1)r, \tag{1.1}$$

which is used to solve a stiff initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0, \quad y_0, y, f \in R^N, t_0 < t \leq T, \tag{1.2}$$

A. Prothero and A. Robinson presented in [1] the concepts of S–stability and strong S–stability, and derived necessary and sufficient conditions for both stabilities (Theorems 2.1 and 2.2 in [1]). Then they discussed stabilities of several classes of well–defined and consistent R–K methods and concluded that these methods are S–stable or strongly S–stable.

Their work has a great influence on the research of numerical methods of stiff O. D. E.. The concepts and theorems of S–stability and strong S–stability have been adopted by many authors (see [2]–[7]).

Based on the definition of S–stability in [1], we now prove that consistent and well–defined R–K methods are not S–stable, and therefore not strongly S–stable. Then we point out the errors in Theorems 2.1 and 2.2 in [1].

For convenience, here we introduce briefly the definitions and some main conclusions of S–stability and strong S–stability in [1] and adopt the symbols of [1] as much as we can.

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2. Definition of S-Stability and Some Main Conclusions in [1]

Definition 2.1. *A R-K method (1.1) is said to be S-stable if it is applied to the test equation*

$$y' = \lambda(y - g(t)) + g'(t), \quad g \in G \tag{2.1}$$

(where λ is a complex constant with $\text{Re}(\lambda) < 0$, and G is the set of all functions defined in $[t_0, T]$, which have first bounded-derivative), and for any real positive constant λ_0 and any $g(t) \in G$, there exists a real positive constant h_0, such that

$$|\varepsilon_{n+1}| < |\varepsilon_n|, \ \forall h \in (0, h_0), \ \forall \lambda \text{ with } \text{Re}(-\lambda) \geq \lambda_0, \ t_n, t_{n+1} \in [t_0, T] \tag{2.2}$$

provided $y_n \neq g(t_n)$, where $\varepsilon_n = y_n - g(t_n)$.

Furthermore, (1.1) is said to be strongly S-stable if it is S-stable and

$$\varepsilon_{n+1}/\varepsilon_n \to 0, \ \forall h \in (0, h_0), \text{ as } \text{Re}(-\lambda) \to \infty, \ t_n, t_{n+1} \in [t_0, T]. \tag{2.3}$$

Since the solution of (2.1) is $y(t) = g(t) + (y_0 - g(t_0))e^{\lambda(t-t_0)}$ and $g(t)$ is quite arbitrary, the methods with S-stability and strong S-stability are very satisfactory. That is why many authors studied the construction of S-stable and strongly S-stable methods.

Correspondingly to [1], note $z = 1/(\lambda h)$. Applying (1.1) to (2.1), we obtain

$$\varepsilon_{n+1} = \alpha(z)\varepsilon_n + h\beta(z),$$

where

$$\begin{cases} \alpha(z) = 1 - b^T(A - zI)^{-1}e, \ A = (a_{ij}), \\ e = (1, 1, \cdots, 1)^T, \ b = (b_1, \cdots, b_r)^T, \\ \beta(z) = -G_0 + b^T(A - zI)^{-1}(\frac{1}{h}(\tilde{g} - g(t_n)e) - z\tilde{g}'), \\ G_0 = (g(t_{n+1}) - g(t_n))/h, \\ \tilde{g} = (g(t_n + c_1 h), \cdots, g(t_n + c_r h))^T, \\ \tilde{g}' = (g'(t_n + c_1 h), \cdots, g'(t_n + c_r h))^T. \end{cases} \tag{2.4}$$

Lemma 2.1. *Assume $R = \{z | 0 < \text{Re}(-z) \leq \bar{z}\}$ and $\bar{z}$ is a real positive number. Define*

$$\varepsilon(z, h, \varepsilon_0) = \alpha(z)\varepsilon_0 + h\beta(z), \ \forall \varepsilon_0 \in C, \ \forall h \in (0, \bar{h}), \ \forall z \in R, \tag{2.5}$$

where $\bar{h}$ is a real positive number . Then for any $g \in G$, there exists a real positive number $h_0 = h_0(\bar{z}, \varepsilon_0) \leq \bar{h}$, such that

$$|\varepsilon(z, h, \varepsilon_0)| < |\varepsilon_0|, \ \forall \varepsilon_0 \neq 0, \ \forall h \in (0, h_0), \ \forall z \in R$$

if and only if

i) $|\alpha(z)| < 1, \quad \forall z \in R,$ *and*

ii) $|\beta(z)|/(1 - |\alpha(z)|)$ *is bounded in* R.

Corollary 2.1. A well–defined one–step method (1.1) is S–stable if and only if it is A–stable, and $\beta(z)/(1 - |\alpha(z)|)$ is bounded for all $z \in R$ and all $g(t) \in G$.

Theorem 2.1. *A well-defined A-stable one-step method (1.1) is S-stable if and only if*

i) $|\alpha_0| < 1$ *and* b_0^* *is finite, or*

ii) $|\alpha_0| = 1, \quad \alpha_1 \neq 0$ *and the method is stiffly accurate,*

where

$$\alpha_0 = Lt_{z\to 0}\alpha(z), \qquad \alpha_1 = Lt_{z\to 0}z^{-1}(1 - |\alpha(z)|),$$
$$b_0^* = Lt_{z\to 0}b^T(A - zI)^{-1}E(z),$$

$E(z)$ is an $r \times r^*$ matrix with elements

$$E_{ij} = \begin{cases} -z, & C_i = C_j^* = 0, \\ C_i, & C_i = C_j^* \neq 0, \\ 0, & otherwise, \end{cases}$$

where r^* is the number of different abscissae, and $\{C_j^*|_{j=1,2,\cdots,r^*}\}$ is the set of all different abscissae. We have, without loss of generality, an order $C_i^* < C_j^*$ if $i < j$.

Remark. In the next section , we are going to prove that Theorem 2.1 is wrong and Corollary 2.1 is right if R is replaced by the left half plane H (not including the imaginary axis) and "for all $g(t)$" by "for any $g(t)$". But , for convenience, we still call them "theorem" and "corollary" respectively.

Theorem 2.2. *A well-defined S-stable one-step method (1.1) is strongly S-stable if and only if the method is L-stable and stiffly accurate.*

3. Non-existence of S–Stable R–K Method

According to the lemma, corollary and theorem in Section 2, Prothero and Robinson discussed in [1] the S–stability of several classes of R–K methods and obtained corresponding results. For example, they concluded that an A–stable Euler method $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$ is strongly S–stable. In fact, as $\alpha(z) = z/(z-1)$, $\beta(z) = z/(z-1)\Big[g'(t_{n+1}) - \frac{g(t_{n+1}) - g(t_n)}{h}\Big], E(z) = 1, A = 1, b = 1, C_1 = 1, r = r^* = 1$, thus $\alpha_0 = 0, b_0^* = 1$, by Theorems 2.1 and 2.2, the method is S–stable and strongly S–stable. On the other hand, $|\beta(z)|/(1 - |\alpha(z)|) = |z|/(|z-1| - |z|)|$ $g'(t_{n+1}) - (g(t_{n+1}) - g(t_n))/h|$; as $z \to \infty (z \in R)$ along line $z = x + iy (i^2 = -1$ and x is a constant), $\beta(z)/(1 - |\alpha(z)|)$ is unbounded for any $h \in (0, h_0)$. By Corollary 2.1 and the remark, the method cannot be S–stable , or strongly S–stable. This contradicts the results in [1].

To explore this contradiction , we establish:

Lemma 3.1. *Suppose λ_0, h_0 are any fixed positive numbers.*

$$R_1 = \{z | z = 1/(\lambda h),\ \ \mathrm{Re}(-\lambda) \geq \lambda_0,\ \ 0 < h < h_0\},\quad H^- = \{z | \mathrm{Re}(z) < 0\}.$$

Then $R_1 = H^-$.

Proof. Note $D = \{z | z = \lambda h,\ \ \mathrm{Re}(-\lambda) \geq \lambda_0,\ \ 0 < h < h_0\}$. We first prove $D = H^-$. Clearly $D \subset H^-$; here we only prove $H^- \subset D$. Assume $z^* \in H^-$ and take $k \in (\frac{1}{h_0}, \infty)$ such that $\mathrm{Re}(-kz^*) \geq \lambda_0$. Then let $\lambda^* = kz^*$, $h^* = 1/k$. Clearly $\mathrm{Re}(-\lambda^*) \geq \lambda_0$, $0 < h^* < h_0$. and $z^* = \lambda^* h^*$, so we have $z^* \in D$. Therefore $D \supset H^-$. This indicates $D = H^-$. As the transformation $z = 1/\xi$ maps D into R_1 , and H^- into H^- , we get $R_1 = H^-$.

According to the lemma, Definition 2.1 can be replaced by an equivalent definition as follows:

Definition 3.1. *A R–K method (1.1) for solving (1.2) is said to be S–stable if the sequence $\{\varepsilon_n\}$ obtained in applying (1.1) to the test equation (2.1) possesses the following properties:*

For any $g(t) \in G$, there exists $h_0 > 0$. Whenever $\varepsilon_n \neq 0$,

$$|\varepsilon_{n+1}| < |\varepsilon_n|,\quad \forall h \in (0, h_0),\quad \forall z \in H^-,\quad t_n, t_{n+1} \in [t_0, T].$$

In addition , if $\varepsilon_{n+1}/\varepsilon_n \to 0$ for $\forall h \in (0, h_0)$ as $\mathrm{Re}(-\lambda) \to \infty$, *(1.1) is said to be strongly S–stable.*

Also, we can establish a lemma corresponding to lemma 2.1:

Lemma 3.2. *Define $\varepsilon(z, h, \varepsilon_0) = \alpha(z)\varepsilon_0 + h\beta(z)$ for all complex ε_0, all real $h \in (0, \bar{h})$ and all $z \in H^-$, where $\bar{h}$ is some positive real number . Then for any $g \in G$, there exists a real positive number $h_0 = h_0(\varepsilon_0) \leq \bar{h}$ such that*

$$|\varepsilon(z, h, \varepsilon_0)| < |\varepsilon_0|,\quad \forall \varepsilon_0 \neq 0,\quad \forall h \in (0, h_0),\quad \forall z \in H^-,$$

if and only if

i) $\alpha(z)| < 1,\ \ \forall z \in H^-$, *and*

ii) $\beta(z)/(1 - |\alpha(z)|)$ *is bounded in* H^-.

Proof. The theorem can be demonstrated by using the method used in proving Lemma 2.1 in [1].

From Lemma 3.2, we can get at once

Corollary 3.1. A well–defined one-step method (1.1) is S–stable if and only if

i) $|\alpha(z)| < 1,\ \ \forall z \in H^-$, and

ii) for any $g \in G$, $\beta(z)/(1 - |\alpha(z)|)$ is bounded in H^-.

The first condition above is an A–stable condition , so S–stability is merely A–stability with condition ii). However, we have

Theorem 3.1. *Any well-defined and consistent method (1.1) cannot be S-stable; neither can it be strongly S-stable.*

Proof. From consistency, we conclude that $\alpha(z)$ is a rational approxiation of $\exp(\frac{1}{z})$; thus $Lt_{z\to\infty}\alpha(z) = 1$. As $Lt_{z\to\infty}\beta(z) = G_0 + b^T\tilde{g}$, there exists $g \in G$ such that $Lt_{z\to\infty}\beta(z) \neq 0$. This indicates that the second condition in Corollary 3.1 is never satisfied. This yields our theorem.

Now, we point out the errors in the proof of Theorem 2.1 in [1]. From the process of the proof we find that the authors of [1] ignored the equivalence of Definition 2.1 and Definition 3.1 and mistakenly substituted a subset R of H^- for H^-; moreover, they did not realize that R is a complex region including infinity whose upper and lower sides are infinite. For any bounded function $Q(z)$ in this region, the limit of $Q(z)$ as $z \to \infty$ must be bounded. However, according to consistence, we have $Lt_{z\to\infty}\alpha(z) = 1$. Thus, without any difficulty, under the conditions of A-stability and $|\alpha_0| < 1$ we infer that $(1 - |\alpha(z)|)^{-1}$ cannot be bounded in R. Similarly, $z(1 - |\alpha(z)|)^{-1}$ cannot be bounded in R if $|\alpha_0| = 1, \alpha_1 \neq 0$ and the A-stable condition is satisfied.

Finally, we'd like to point out that stability analysis of one-step methods by using (2.1) as a model equation is of certain significance. How to modify the definition of S-stability so that one-step methods possessing this property reflect well the error propagation behaviour in practical computation is still worth further research.

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