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THE CONVERGENCE OF CONTOUR DYNAMICS METHODS * 1)

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Abstract

In this paper the properties of contour dynamics methods of two-dimensional incompressible inviscid vortex flows are investigated. The error estimates and the convergence of the methods for piecewise constant vorticity patches using Euler's method are obained.

1. Introduction

The numerical methods of vortex flows have attracted much attention. Since 1970 the vortex methods for numerical simulation of incompressible vortex flows have been greatly developed, for example, the random vortex methods [1], vortex in cell or cloud in cell methods [1], the particle methods [2] and contour dynamics methods [3]. For two-dimensional incompressible inviscid flows with initial piecewise constant vorticity patches it is sufficient to track the boundaries of the patches for simulating the evolution of the vortex flows. Hence, N. Zabusky et al. [3] proposed the contour dynamics code for simulating the vortex motion of flows with piecewise constant vorticity blobs, and numerically revealed a number of phenomena of vortex flows. Today many scientists are studying the problems in physics and fluid mechanics by means of the contour dynamics methods.

Up to now, due to the complexity of the evolution of the vortex, the problems of stability, convergence and the error estimates of the contour dynamics methods have not been discussed yet. This paper is one of the series of our works on analysis of contour dynamics methods. In section 2, we give a preliminary investigation of the flow motion. In section 3, the physical models of our consideration are proposed, and several conditions on the behaviour of the contour are imposed. Sections 4 and 5 are devoted to the truncation error and the convergence problems of the contour dynamics methods. Finally a discussion on the results obtained is given.

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2. Preliminary considerations

In the conventional notations the inviscid incompressible vortex motions in two dimensions are described by the Helmholtz equations :

$$\begin{aligned}&\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=0,\\&\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega,\\&\omega=-\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\\&U=(u,v)^T=\left(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x}\right)^T.\end{aligned}\tag{2.1}$$

$u, v(x,y,t)$ are the velocity components of the fluid particles, and $\omega(x,y,t)$ is called the vorticity density of the flow.

From (2.1) we have the integrals for the stream function ψ

$$\psi(x,y,t)=-\frac{1}{2\pi}\iint\ln r\cdot\omega(\xi,\eta,t)d\xi d\eta$$

and the velocity U

$$U(x,y,t)=\frac{1}{2\pi}\iint\begin{pmatrix}-\frac{\partial}{\partial y}\\ \frac{\partial}{\partial x}\end{pmatrix}\ln r\cdot\omega(\xi,\eta,t)d\xi d\eta$$

where $r^2=(x-\xi)^2+(y-\eta)^2$.

Let

$$z=(x,y)^T, z'=(\xi,\eta)^T,$$

$$K=K(z-z')=\frac{1}{2\pi}\left(-\frac{\partial}{\partial y},\frac{\partial}{\partial x}\right)^T\ln r=\frac{1}{2\pi}\left(-\frac{y-\eta}{r^2},\frac{x-\xi}{r^2}\right)^T.$$

We express the velocity as a convolution of K and ω,

$$U(z)=K*\omega=\iint_{R^2}K(z-z')\omega(z')dz'.$$

Euler's equation can be written as

$$\dot{z}=U(z).$$

Let $\omega^t(x,y)=\omega(x,y,t)$ be the vorticity function and $z(s,t)$ the contour equation at time t and $z_0(s)=z(s,0)$, where s is an arc parameter of the contour.

In what follows we use the notation $|\cdot|$ to denote the absolute value when it operates on a sclar and to denote the Euclidian norm when it operates on a vector.

We have an obvious inequality

$$\iint_{R^2} \left| \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} \right| dz \geq \left| \begin{pmatrix} \iint_{R^2} a(z)dz \\ \iint_{R^2} b(z)dz \end{pmatrix} \right|.$$

2.1. The boundedness of the velocity

Let $B_D(z)$ be an open circle with radius $D \geq 1$ and centered at point $z, B_D = B_D(0)$.

Throughout the text, we make a basic but reasonable assumption on the physical models under consideration.

Assumption A. The vorticity function $\omega(z,t)$ is finite and has a compact support in B_D with sufficiently large D for $0 \leq t \leq T$ with T fixed.

Lemma 2.1. *Suppose $\omega(x,y)$ is a finite vorticity function, and its compact support is contained in a circle B_D. Then the velocity U is uniformly bounded as*

$$|U| \leq M_1 D.$$

Proof. Suppose $|\omega(x,y)| \leq C$ and let $\chi_D(z)$ be the characteristic function of domain B_D, *i.e.*

$$\chi_D(z) = \begin{cases} 1 & \text{if } z \in B_D, \\ 0 & \text{if } z \bar{\in} B_D. \end{cases}$$

Let P_{2D} denote a square containing B_D :

$$P_{2D} = \Big\{(x,y)\Big| |x| \leq D, |y| \leq D\Big\}.$$

We have

$$\begin{aligned} |K^*\omega| &= \left|\iint K(z-z')\omega(z')dz'\right| = \left|\iint_{B_D} K(z-z')\omega(z')dz'\right| \\ &\leq C\iint_{B_D} |K(z-z'))|\,\chi_D(z')dz' \leq C\iint_{B_D+z} |K(z')|dz'. \end{aligned}$$

If $z \in B_{2D}$, then $z + B_D \in B_{4D}$. Hence

$$|K^*\omega| \leq C\iint_{B_{4D}} \frac{dz'}{|z'|} = 8C\pi D.$$

If $z \bar{\in} B_{2D}$, then we have $\max(|x|, |y|) \geq \sqrt{2}D$, and

$$\begin{aligned} \iint_{B_D+z} \frac{dz'}{|z'|} &\leq \iint_{P_{2D}+z} \frac{dz'}{|z'|} = \int_{x-D}^{x+D}\int_{y-D}^{y+D} \frac{dx'dy'}{\sqrt{x'^2+y'^2}} \\ &\leq 2D\min\left(\ln\left|\frac{x+D}{x-D}\right|, \ln\left|\frac{y+D}{y-D}\right|\right) \\ &\leq 2D\ln(3+2\sqrt{2}). \end{aligned}$$

Therefore, we get the following estimate :

$$|K*\omega| \le \begin{cases} 8C\pi D, & \forall z \in B_{2D}, \\ 2CD\ln(3+2\sqrt{2}), & \forall z \bar{\in} B_{2D}. \end{cases}$$

Let $M_1 = 8C\pi$. The proof is completed.

2.2. The continuity of the velocity field

lemma 2.2. *For the velocity field in Lemma 2.1, we have the estimate*

$$|U(z_1) - U(z_2)| \le k_1\delta + k_2\delta|\ln D/\delta|,$$

where $\delta = |z_1 - z_2|$, and k_1, k_2 are positive constants.

Proof. If $|z_1 - z_2| \ge D$, then by Lemma 2.1, we have

$$|U(z_1) - U(z_2)| \le |U(z_1)| + |U(z_2)| \le 16C\pi D \le 16C\pi\delta.$$

Now, consider the case $|z_1 - z_2| \le D$. At this time $z_2 \in z_1 + B_D$. Let

$$U(z_1) - U(z_2) = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{1}{2\pi}\iint_{B_{2\delta}(z_1)} \frac{1}{(x_1-\xi)^2 + (y_1-\eta)^2}\begin{pmatrix} y_1-\eta \\ -(x_1-\xi)\end{pmatrix}\omega(z')dz',$$

$$I_2 = \frac{1}{2\pi}\iint_{B_{2\delta}(z_1)} \frac{-1}{(x_2-\xi)^2 + (y_2-\eta)^2}\begin{pmatrix} y_2-\eta \\ -(x_2-\xi)\end{pmatrix}w(z')dz',$$

$$I_3 = \frac{1}{2\pi}\iint_{B_D\backslash B_{2\delta}(z_1)} \left[\frac{1}{(x_1-\xi)^2 + (y_1-\eta)^2}\begin{pmatrix} y_1-\eta \\ -(x_1-\xi)\end{pmatrix} - \frac{1}{(x_2-\xi)^2 + (y_2-\eta)^2}\begin{pmatrix} y_2-\eta \\ -(x_2-\xi)\end{pmatrix}\right]\omega(z')dz'.$$

We have the estimate

$$|I_1| \le \frac{C}{2\pi}\int_0^{2\pi}\left|\begin{pmatrix}\sin\theta \\ \cos\theta\end{pmatrix}\right| d\theta \int_0^{2\delta} d\rho = 2C\delta.$$

For each $z \in B_{2\delta}(z_1)$ it is obvious that

$$|z - z_2| \le |z - z_1| + |z_1 - z_2| \le 3\delta, \quad B_{2\delta}(z_1) \subset B_{3\delta}(z_2).$$

Therefore, we get

$$|I_2| \le \frac{C}{2\pi}\iint_{B_{3\delta}(z_2)} \frac{1}{(x_2-\xi) + (y_2-\eta)^2}\left|\begin{pmatrix} y_2-\eta \\ -(x_2-\xi)\end{pmatrix}\right| d\xi d\eta = 3\delta C.$$

Finally, we estimate I_3. Using the mean value theorem we have

$$\begin{aligned}&\left|\frac{1}{(x_1-\xi)^2+(y_1-\eta)^2}\begin{pmatrix} y_1-\eta \\ -(x_1-\xi)\end{pmatrix}-\frac{1}{(x_2-\xi)^2+(y_2-\eta)^2}\begin{pmatrix} y_2-\eta \\ -(x_2-\xi)\end{pmatrix}\right| \\ &\leq \frac{1}{(x^*-\xi)^2+(y^*-\eta)^2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}, \quad \forall(\xi,\eta)\bar{\in} B_{2\delta}(z_1)\end{aligned} \tag{2.2}$$

where (x^*,y^*) is on the segment $\overline{z_1z_2}$. Because

$$\sqrt{(x^*-x_1)^2+(y^*-y_1)^2}\leq\delta\leq\frac{1}{2}\sqrt{(x_1-\xi)^2+(y_1-\eta)^2}$$

we have

$$\begin{aligned}\sqrt{(x^*-\xi)^2+(y^*-\eta)^2} &\geq \sqrt{(x_1-\xi)^2+(y_1-\eta)^2}-\sqrt{(x^*-x_1)^2+(y^*-y_1)^2} \\ &\geq \tfrac{1}{2}\sqrt{(x_1-\xi)^2+(y_1-\eta)^2}.\end{aligned} \tag{2.3}$$

By (2.2) and (2.3) ,

$$\begin{aligned}|I_3| &\leq \frac{C}{2\pi}\iint_{B_D\setminus B_{2\delta}(z_1)}\frac{\delta}{(x^*-\xi)^2+(y^*-\eta)^2}d\xi d\eta \\ &\leq \frac{2\delta C}{\pi}\iint_{B_D\setminus B_{2\delta}(z_1)}\frac{d\xi d\delta}{(x_1-\xi)^2+(y_1-\eta)^2}.\end{aligned}$$

If $B_{2D}\cap B_{2\delta}(z_1)\neq\emptyset$, $z\in B_{2D}$ and $\tilde{z}\in B_{2D}\cap B_{2\delta}(z_1)$, we have

$$|z-z_1|\leq|z-\tilde{z}|+|\tilde{z}-z_1|\leq 4D+2\delta.$$

Hence, $B_{2D}\subset B_{2(2D+\delta)}(z_1)$ and

$$|I_3|\leq 2\pi^{-1}C\delta\iint_{B_{2(2D+\delta)}(z_1)\setminus B_{2\delta}(z_1)}\frac{d\xi d\eta}{(x_1-\xi)^2+(y_1-\eta)^2}\leq 4C\delta\ln\frac{3D}{\delta}.$$

If $B_{2D}\cap B_{2\delta}(z_1)=\emptyset$, then for any $z'\in B_D, z''\in B_{2\delta}(z_1)$ we have

$$|z''-z'|\geq|z''|-|z'|>2D-D=D.$$

Hence,

$$|I_3|\leq 2\pi^{-1}C\delta\iint_{B_D}\frac{dz'}{|z_1-z'|^2}\leq 2C\delta.$$

Thus, we get the estimate

$$|U(z_1)-U(z_2)|\leq\begin{cases}16C\pi\delta, & \text{if } |z_1-z_2|\geq D,\\ (7C+4C\ln 3)\delta+4C\delta\ln\frac{D}{\delta}, & \text{if } |z_1-z_2|\leq D.\end{cases}$$

Putting $k_1 = 16C\pi$ and $k_2 = 4C$, we have

$$|U(z_1) - U(z_2)| \le k_1\delta + k_2\delta \left|\ln \frac{D}{\delta}\right|.$$

This is what to be proved. Note that the above estimate can be replaced by the following inequality

$$|U(z_1) - U(z_2)| \le k_1\delta + k_2\delta \ln \frac{D+\delta}{\delta}.$$

2.3. The properties of finite vorticity patches.

Lemma 2.3. *Suppose $\omega^t(z)$ is finite and has a compact support for $0 \le t \le T$. Let $z(t)$ be the solution of equation*

$$\dot{z}(t) = k \times \omega^t, \tag{2.4}$$

$$z(t)|_{t=0} = z_0. \tag{2.5}$$

Then for any fixed T and any $0 \le t_1 < t_2 \le T$, the inequality

$$|z(t_1) - z(t_2)| \le M_1 \Delta t D \quad , \quad \Delta t = t_2 - t_1$$

holds, where M_1 is the constant in Lemma 2.1.

The proof is a direct consequence from Lemma 2.1.

Lemma 2.4. *Assume $z_1(0), z_2(0) \in$ supp $\omega^0(z) \subset B_D(z)$. Let $\delta(t) = |z_1(t) - z_2(t)|$. Then the following estimate is true*

$$D\exp\left(-\frac{k_1}{k_2}(e^{k_2 t} - 1)\right)\left(\frac{\delta(0)}{D}\right)^{\exp(k_2 t)} \le \delta(t) \le D\exp\left(\frac{k_1}{k_2}(1 - e^{-k_2 t})\right)\left(\frac{\delta(0)}{D}\right)^{\exp(-k_2 t)}.$$

Proof. First we note an elementary inequality

$$-|z|\left|\frac{dz}{dt}\right| \le |z|\frac{d|z|}{dt} \le |z|\left|\frac{dz}{dt}\right|.$$

By Lemma 2.2 and equation (2.4), we have

$$-2k_1\delta^2 - k_2\delta^2 \ln \frac{D^2}{\delta^2} \le \frac{d\delta^2}{dt} \le 2k_1\delta^2 + k_2\delta^2 \ln \frac{D^2}{\delta^2}.$$

The lemma is proved by solving this differential inequality.

From Lemma 2.4 it becomes clear that if the solution of (2.4)–(2.5) exists, then the solution is unique. Secondly, the initially uniformly continuous contours keep their unifrom continuities during the motion theoretically.

For finite piecewise constant vorticity distribution the particles of the fluid which are initially on the contour will remain on the contour during the motion. Therefore, we can track the trace of the fluid particles on the boundary of the vortex patches for simulating the vortex motion of these blobs. This is the main idea of the contour dynamics methods proposed by N. Zabusky et al. [3].

3. The physical models

In general, the properties of vortex flows are very complicated. Therefore, it is necessary to confine our consideration of the physical models. Let us consider the plane motions of a finite number of piecewise constant vorticity blobs. We will impose certain smoothness conditions on the boundaries of the vortex blobs a priori. We only consider the case of one contour because it is easy to extend the results to the multi–contour case.

The contour of the constant vorticity blob at time t is denoted by Γ^t. The circle $B_\varepsilon(z)$ is called an ε-neighbourhood of $z \in \Gamma^t$:

$$B_\varepsilon(z) = \{z' \| z - z'| \leq \varepsilon\}.$$

Consider an ε-neighbourhood $B_\varepsilon(z)$ of contour Γ^t with sufficiently small ε such that the part of the contour $\Gamma^t \cap B_\varepsilon(z)$ is divided by the point z into two branches $r_+^\varepsilon, r_-^\varepsilon$. Let $\rho_z(w) = |z - w|$ be the distance of z and w.

Assumption B. There are positive constants ε^* and β independent of t such that for any two points z_1, z_2 on either branches $r_+^{\varepsilon^*}, r_-^{\varepsilon^*}$, the follwing inequality holds :

$$|\rho_z(z_1) - \rho_z(z_2)| \geq \beta |z_1 - z_2|. \tag{3.1}$$

Remark. From Assumption B it is clear that

$$\min(| < zz_1z_2|, 2\pi - | < zz_1z_2|) \geq \frac{\pi}{2} \tag{3.2}$$

or, in other words,

$$\rho_z(z_2) > \rho_z(z_1). \tag{3.3}$$

Let the equation of the contour be

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq S$$

where s is the arc–length of the contour. We define

$$\rho_{s_0}(s)\sqrt{(x(s) - x(s_0))^2 + (y(s) - y(s_0))^2},$$
$$H_{s_1 s_2} = \rho_{s_1}(s_2) = \rho_{s_2}(s_1).$$

Lemma 3.1. *Suppose functions $x(s)$ and $y(s)$ are sufficiently smooth. Assume there exist positive constants $\varepsilon, \alpha_0, \alpha_1, k$ independent of t such that in the ε-neighbourhood of any s_0 the following inequalities are valid :*

$$\alpha_1^2 \geq (\dot{x}(s))^2 + (\dot{y}(s))^2 \geq \alpha_0^2 > 0, |\ddot{x}(s)| \leq k, |\ddot{y}(s)| \leq k.$$

Then the following inequality

$$|\rho_{s_0}(s_1) - \rho_{s_0}(s_2) \geq \beta H_{s_1 s_2}$$

holds for any $s_1, s_2 > s_0$ (or $< s_0$) and an absolute constant β.

Proof. According to the assumption of the lemma, in the ε-neighbourhood of s_0 we have the expansion

$$\rho_{s_0}(s_1) = \rho_{s_0}(s_2) + \rho'_{s_0}(s^*)(s_1 - s_2)$$

where s^* is between s_1 and s_2,

$$\rho'_{s_0}(s^*) = \frac{[(x(s^*) - x(s_0))\dot{x}(s^*) + (y(s^*) - y(s_0))\dot{y}(s^*)]}{[x(s^* - x(s_0))^2 + (y(s^*) - y(s_0))^2]^{1/2}}.$$

Expand $x(s_0)$ and $y(s_0)$ at s^* :

$$x(s_0) = x(s^*) + \dot{x}(s^*)(s_0 - s^*) + \tfrac{1}{2}(s_0 - s^*)^2\ddot{x}(s_1^{**}),$$
$$y(s_0) = y(s^*) + \dot{y}(s^*)(s_0 - s^*)\tfrac{1}{2}(s_0 - s^*)^2\ddot{y}(s_2^{**})$$

where s_1^{**}, s_2^{**} are between s_0 and s^*. Then we have

$$|[\dot{x}(s^*)(x(s^*) - x(s_0)) + \dot{y}(s^*)(y(s^*) - y(s_0))]|$$
$$\geq [\dot{x}(s^*)^2 + \dot{y}(s^*)^2]|s_0 - s^*| - \frac{1}{2}(s_0 - s^*)^2\sqrt{2}k\sqrt{\dot{x}(s^*)^2 + \dot{y}(s^*)^2}.$$

By the Schwarz inequality we get

$$[(x(s^*) - x(s_0))^2 + (y(s^*) - y(s_0))^2]^{1/2}$$
$$\leq \sqrt{\dot{x}(s^*)^2 + \dot{y}(s^*)^2}|s_0 - s^*|\frac{\sqrt{2}}{2}(s_0 - s^*)^2k.$$

Hence, when ε is sufficiently small, we have

$$|\rho_{s_0}(s_1) - \rho_{s_0}(s_2)| \geq \{[\alpha_0^2 - \frac{\sqrt{2}}{2}\varepsilon k\alpha_1]/[\alpha_1 + \frac{\sqrt{2}\varepsilon}{2}k]\} \cdot |s_1 - s_2| \equiv \beta|s_1 - s_2| \geq \beta H_{s_1 s_2},$$

where s_1, s_2 are in ε-neighbourhood of s_0, and the proof is completed.

According to Lemma 3.1, Assumption B is actually true if the contour is quite smooth.

By Assumption B it is not difficult to prove the following

Contour Property 3.1. Consifer the contour Γ^t. Let s_t be the arc length of Γ^t. Γ^t_ε denotes the ε-neighbourhood of contour Γ^t :

$$\Gamma^t_\varepsilon = \bigcup_{z \in \Gamma^t} B_\varepsilon(z)$$

$\varepsilon \le \varepsilon^*, 0 \le t \le T$. Then there exist constants M, M^*, α, independent of ε, t and N_ε^t points $Q_i, i = 1, \ldots, N_\varepsilon^t$, such that

$$\begin{aligned} \rho_{Q_i}(Q_j) = \ & |Q_i - Q_j| \ge \alpha\varepsilon, \qquad \forall i \ne j, \\ & N_\varepsilon^t \cdot M\varepsilon \le M^* S_t, \\ & \textstyle\bigcup_{i=1}^{N_\varepsilon^t} B_{M\varepsilon}(Q_i) \supset \Gamma_\varepsilon^t. \end{aligned}$$

and the maximal number of the circles $B_{M\varepsilon}(Q_i)$ intersecting $B_{M\varepsilon}(z)$ for an arbitrary point $z \in \Gamma^t$ does not exceed d which is independent of t and ε.

Assumption C. There exist constants $M_2, M_3 (1 \le M_2 \le M_3)$ independent of t such that if the distances $H_1^t = |z_1^t - z_2^t|$ and $H_2^t = |w_1^t - w_2^t|$ of two arbitrary pair of points z_1^t, z_2^t and w_1^t, w_2^t on the contour Γ^t satisfy the condition

$$1/M_2 \le H_1^0 / H_2^0 \le M_2,$$

then for any $0 \le t \le T$ the following inequalities hold :

$$1/M_3 \le H_1^t / H_2^t \le M_3.$$

Assumption C means that if N points z_i^0 are chosen on the contour Γ^0, then there exists a constant M_4 such that

$$H^t / h^t \le M_4,$$

where M_4 is independent of t, and

$$\begin{aligned} H^t &= \max_j \left| z_{j+1}^t - z_j^t \right|, \\ h^t &= \min_j \left| z_{j+1}^t - z_j^t \right| \end{aligned}$$

and $z_{N+1}^t = z_1^t$.

Let us approximate the initial contour Γ^0 by an N-polygon with vertices $z_1^0, \ldots, z_N^0 (z_{N+1}^0 = z_1^0)$. Let z_j^t be the corresponding point on the contour Γ^t of z_j^0. The segment $\overline{z_j^t z_{j+1}^t}$ of the polygon and the arc $\widehat{z_j^t z_{j+1}^t}$ of the contour form an area A_j^t whosę diameter is D_j^t. We have the follwing.

Contour Property 3.2. Under Assumprions B and C, there exist constants M_5, M_6 such that for any $0 \le t \le T$ the following inequalities hold :

$$H^t / h^t \le M_5, \quad D_j^t / h_j^t \le M_6, \quad j = 1, \ldots, N.$$

4. The truncation error of velocity calculation

Lemma 4.1. *Suppose $\omega(z)$ and $\omega_n(z)$ have the same vorticity density inside the contours Γ^t and Γ_n respectively, and Γ_n belongs to the ε-neighbourhood of Γ^t. Let*

$$I = U - U_n = \iint k(z_j - z')(\omega(z') - \omega_n(z'))dz'$$

be the difference of the velocities U and U_n induced by $\omega(z)$ and $\omega_n(z)$ respectively. Then the estimate

$$|I| \le M\varepsilon\Big[c_1 + c_2\big|\ln\frac{1}{M\varepsilon}\big|\Big]$$

is valid for some constants C_1, C_2 independent of ε and t.

Proof. The area where $\omega(z) - \omega_n(z) \ne 0$ belongs to the ε-neighbourhood Γ_ε of Γ. By Assumption B, there exists a constant M, and a corresponding family of circles $F = \{B_{M\varepsilon}(Q_i)\}_{i=1}^{N_\varepsilon}$ which cover Γ_ε. According to the Contour Property 3.1, for any point z_j on the contour Γ there exist at most d circles of the family F intersecting $B_{M\varepsilon}(z_j)$. Hence, the circle $B_{3M\varepsilon}(z_j)$ will completely include these d circles. By I_d we denote the set of the subscripts Q_i of the circles intersecting $B_{M\varepsilon}(z_j)$.

If $i \in I_d$ we have

$$\iint_{B_{M\varepsilon}(Q_i)} \frac{dz'}{|z_j - z'|} \le \iint_{B_{3M\varepsilon}(z_j)} \frac{dz'}{|z_j - z'|} = 6\pi M\varepsilon,$$

hence ,

$$\sum_{i\in I_d} \iint_{B_{M\varepsilon}(Q_i)} \frac{dz'}{|z_j - z'|} \le 12\pi d M\varepsilon.$$

Now, consider the case when $i\bar{\in} I_d$. Assume $Q_i = (s_i, r_i)$ and $z' = (x', y') \in B_{M\varepsilon}(Q_i)$. According to the mean value theorem we have

$$\begin{aligned}
&\frac{1}{|z_j - z'|} - \frac{1}{|z_j - Q_i|} \\
&= \frac{x_j - s_i - \theta(x' - s_i)}{|z_j - Q_i - \theta(z' - Q_i)|^3}(x' - s_i) + \frac{y_j - r_i - \theta(y' - r_i)}{|z_j - Q_i - \theta(z' - Q_i)|^3}(y' - r_i) \\
&\le \frac{1}{|z_j - Q_i - \theta(z' - Q_i)|^2}\sqrt{(x' - s_i)^2 + (y' - r_i)^2} \\
&\le \frac{M\varepsilon}{(|z_j - Q_i| - \theta M\varepsilon)^2}, \qquad \theta \in (0,1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i\bar{\in} I_d} \iint_{B_{M\varepsilon}(Q_i)} \frac{dz'}{|z_j - z'|} &\le \sum_{i\bar{\in} I_d}\Big[\iint_{B_{M\varepsilon}(Q_i)} \frac{M\varepsilon dz'}{(|z_j - Q_i| - \theta M\varepsilon)^2} + \frac{\pi M^2\varepsilon^2}{|z_j - Q_i|}\Big] \\
&= \sum_{i\bar{\in} I_d}\Big[\frac{\pi(M\varepsilon)^3}{(|z_j - Q_i| - \theta M\varepsilon)^2} + \frac{\pi(M\varepsilon)^2}{|z_j - Q_i|}\Big].
\end{aligned}$$

Let ε^* be the one in Assumption B. Separate the above sum into two parts : $\sum_{i\bar{\in} I_d} = \sum_{i\in I_1} + \sum_{i\in I_2}$, where I_1 is the index set for $Q_i \in B_{\varepsilon^*}(z_j)$ and I_2 the index set i for $Q_i \bar{\in} B_{\varepsilon^*}(z_j), i\bar{\in} I_d$.

Consider the sum over I_1 first.

When $3M\varepsilon \geq \varepsilon^*$, the set I_1 is empty. So, we assume $3M\varepsilon < \varepsilon^*$. In this case,

$$B_{3M\varepsilon}(z_j) \subseteq B_{\varepsilon^*}(z_j).$$

The part of the contour $\Gamma \cap B_{\varepsilon^*}(z_j)$ is divided into two branches by z_j. We consider the sum over I_1 spread on one of these two branches . For convenience, we order z_j and all the points Q_i on this branch as $z_j, Q_{i_1}, \ldots, Q_{i_L}$. For the indices $i_k \in I_1, i_k \bar{\in} I_d$ we have

$$\text{and} \quad \left. \begin{array}{l} |z_j - Q_{i_k}| > 2M\varepsilon, \rho_{z_j}(Q_{i_{k+1}}) > \rho_{z_j}(Q_{i_k}), k = 1, \ldots, L-1 (\text{ see}(3.3)) \\ |Q_{i_{k+1}} - Q_{i_k}| \geq \alpha\varepsilon > 0, k = 1, \ldots, L-1 \ \ (\text{ see } (3.4)) \end{array} \right\} \quad (4.3)$$

Now , from (4.2), (4.3) we have the estimates

$$\begin{aligned} &\sum_{k=1}^{L} \left[\frac{\pi(M\varepsilon)^3}{(|z_j - Q_{i_k}| - \theta M\varepsilon)^2} + \frac{\pi(M\varepsilon)^2}{|z_j - Q_{i_k}|} \right] \\ &\leq \frac{1}{\alpha\beta\varepsilon} \sum_{k=2}^{L} \left(\frac{\pi(M\varepsilon)^3}{|z_j - Q_{i_k}| - \theta M\varepsilon)^2} + \frac{\pi(M\varepsilon)^2}{|z_j - Q_{i_k}|} \right) \left(\rho_{z_j}(Q_{i_k}) - \rho_{z_j}(Q_{i_{k-1}}) \right) \\ &+ \frac{\pi(M\varepsilon)^3}{(|z_j - Q_{i_1}| - M\varepsilon)^2} + \frac{\pi(M\varepsilon)^2}{|z_j - Q_{i_1}|} \\ &\leq \frac{1}{\alpha\beta\varepsilon} \int_{2M\varepsilon}^{\varepsilon^*} \left(\frac{\pi(M\varepsilon)^3}{(\rho - \theta M\varepsilon)^2} + \frac{\pi M^2\varepsilon^2}{\rho} \right) d\rho + \frac{\pi(M\varepsilon)^3}{(M\varepsilon)^2} + \frac{\pi(M\varepsilon)^2}{2M\varepsilon} \\ &\leq \frac{\pi}{\alpha\beta} \left[\frac{1}{2(2-\theta)} + \left| \ln \frac{\varepsilon^*}{2M\varepsilon} \right| \right] M^2\varepsilon + \frac{3}{2}\pi M\varepsilon. \end{aligned}$$

Similarly we can estimate the sum spread on the other branch. So we have

$$\sum_{i\in I_1} \iint_{B_{M\varepsilon}(Q_i)} \frac{dz'}{|z_j - z'|} \leq \tilde{C}_1 M\varepsilon + \tilde{C}_2 M\varepsilon \left| \ln \frac{1}{M\varepsilon} \right|. \quad (4.4)$$

Next, consider the sum over I_2.

For any $i \in I_2, Q_i$ is outside the circle $B_{\varepsilon^*}(z_j)$. Hence,

$$|z_j - Q_i| \geq \varepsilon^*.$$

There are two cases. When $3M\varepsilon < \varepsilon^*$, for every $z' \in B_{M\varepsilon}(Q_i)$ we have

$$\begin{aligned} &|z_j - z'| \geq |z_j - Q_i| - |Q_i - z'| \geq \varepsilon^* - M\varepsilon \geq (1 - \tfrac{1}{3})\varepsilon^*, \\ &\sum_{i\in I_2} \iint_{B_{M\varepsilon}(Q_i)} \frac{dz'}{|z_j - z'|} \leq \sum_{i\in I_2} (1 - \frac{1}{3})^{-1} \frac{1}{\varepsilon^*} \pi(M\varepsilon)^2 \leq \frac{3}{2}\pi M^* S_t \varepsilon. \end{aligned} \quad (4.5)$$

When $3M\varepsilon \geq \varepsilon^*, Q_i$ is outside the circle $B_{\varepsilon^*}(z_j)$. The total number of these circles centered at $\{Q_i\}$ is less than

$$M^* S_t / M\varepsilon \leq 3\frac{M^*}{\varepsilon^*} S_t.$$

By Lemma 2.1 we have

$$\iint_{B_{M\varepsilon}(Q_i)} \frac{dz'}{|z_j - z'|} \leq 8\pi M\varepsilon,$$

hence,

$$\sum_{i\in I_2} \iint_{B_{M\varepsilon}(Q_i)} \frac{dz'}{|z_j - z'|} \leq \frac{M^*}{\varepsilon^*} S_t 8\pi M\varepsilon \cdot 3.$$

The proof is completed by synthesizing (4.1), (4.4), (4.5) and (4.6).

Corollary 4.1. Replace the support of $w^t(z)$ by an N-polygonal region with N vertices on the boundary of the support. The vorticity function $\bar{\omega}^t(z)$ with this N-polygon as its support has the same vorticity density as $\omega^t(z)$ has. Let H be the maximal length of the N sides of the polygon, and

$$I = \iint_{R^2} k(z_j - z')[\omega^t(z') - \bar{\omega}^t(z')]dz'.$$

Then the estimate

$$|I| \leq (C_3 - C_4|\ln \frac{1}{H}|)H$$

holds under Assumptions B and C for some constants C_3, C_4 independent of t.

5. The convergence of Euler's explicit method

Let $\bar{\omega}^t(z)$ be the N-polygonal approximation of $\omega^t(z)$. We solve the following initial value problem

$$\begin{aligned} &\dot{z}_j = K * \omega^t, \\ &z_j\Big|_{t=0} = z_j^0 \ , \quad j = 1, 2, \ldots, N \end{aligned} \tag{5.1}$$

by using explicit Euler's method

$$\begin{aligned} &\hat{z}_j^{n+1} = \hat{z}_j^n + \Delta t K(\hat{z}_j^n) * \hat{\omega}^n, \\ &\hat{z}_j^0 = z_j^0, \quad j = 1, 2, \ldots, N, \end{aligned} \tag{5.2}$$

where $\hat{\omega}^n$ is a vorticity distribution with the support of an N-polygon with verices $\hat{z}_j^n, j = 1, \ldots, N$.

Introduce the intermediate contour at time $t + \Delta t$ with vertices $\bar{z}_j^n, j = 1, \ldots, N$, where $\bar{z}_j^n$ are calculated as follows :

$$\begin{aligned} \bar{z}_j^{n+1} &= z_j^n + \Delta t K(z_j^n) * \bar{\omega}^n \\ z_j^n &= z_j(n\Delta t), \qquad j = 1, 2, \ldots, N \end{aligned} \tag{5.3}$$

where $\bar{\omega}^n$ is a vorticity function with the support of an N-polygon vertexed at $z_j^n, j1, \ldots, N$.

Lemma 5.1. *Assume* supp $\omega^0(z) \subset B_D(z)$. *Then the solution of (5.2) has the following properties :*

$$|\hat{z}_j^{k+1} - \hat{z}_j^k| \le M_1 \Delta t D_k,$$
$$|\hat{z}_j^k - z_j^0| \le e^{M_1 T} D$$

where

$$D_{k+1} = (1 + M_1 \Delta t) D_k, \qquad D_0 = D,$$
$$k = 0, 1, 2, \ldots, N_T - 1, \qquad N_T = [T/\Delta t].$$

Using Lemmas 2.1 and 2.3 the proof is completed by induction.

Next, we estimate the computational error. First, estimate the error at the first time step. We have

$$\bar{z}_j^1 = z_j^0 + \Delta t \iint_{R^2} K(z_j^0 - z') * \bar{\omega}^0(z') dz'.$$

The exact solution of the physical problem shoud be

$$z_j^1 = z_j^0 + \int_0^{\Delta t} \iint_{R^2} K(z_j - z') * \omega^t(z') dt.$$

Therefore, we get

$$|\bar{z}_j^1 - z_j^1| \le \int_0^{\Delta t} (I_1 + I_2 + I_3) dt$$

where

$$\begin{aligned} I_1 &= \iint_{R^2} |K(z_j - z') * [\omega^0(z') - \omega^t(z')]| dz', \\ I_2 &= \iint_{R^2} |K(z_j^0 - z') * [\bar{\omega}^0(z') - \omega^0(z')]| dz', \\ I_3 &- \iint_{R^2} |[K(z_j^0 - z') - K(z_j - z')] * \omega^0(z')| dz'. \end{aligned}$$

For $0 \le t \le \Delta t$ by Lemma 2.3 we have

$$|z(0) - z(t)| \le M_1 \Delta t D \equiv \bar{M} \Delta t.$$

Hence, for $0 \le t \le \Delta t$ the contour Γ^t belongs to the $\bar{M}\Delta t$-neighbourhood of Γ^0. Then by Lemma 4.1 we get

$$|I_1| \le (C_1 + C_2 |\ln \frac{1}{M\bar{M}\Delta t}|) M\bar{M}\Delta t = (C_5 + C_6 |\ln \frac{1}{\Delta t}|) \Delta t.$$

For I_2, according to Corollary 4.1, we have

$$|I_2| \le (C_3 + C_4|\ln \frac{1}{H^0}|)H_0,$$

where

$$H_0 = \max_j |z_j^0 - z_{j+1}^0|.$$

Similarly, for I_3 we get the estimate

$$|I_2| \le (C_{10} \ln \frac{1}{\Delta t} + C_9)\Delta t.$$

Finally, we have the error estimate

$$|\bar{z}_j^1 - z_j^1| \le [(C_{11} + C_{12} \ln \frac{1}{\Delta t})\Delta t + (C_3 + C_4 \ln \frac{1}{H^0})H_0]\Delta t \equiv g_0 \Delta t, \tag{5.5}$$

where g_0 denotes the terms in brackets, and $g_0 \ll 1$ if $\Delta t, H_0$ are sufficiently small.

In the same way we can extend the estimate (5.5) for any n :

$$|\bar{z}_j^{n+1} - z_j^{n+1}| \le \Delta t g_n,$$

where

$$g_n = (C_{11} + C_{12} \ln \frac{1}{\Delta t})\Delta t + (C_3 + C_4 \ln \frac{1}{H_n})H_n$$

and

$$H_n = \max_j |z_j^n - z_{j+1}^n| \quad , \qquad n\Delta t \le T.$$

Suppose $H_0 \le 1$. By Lemma 2.4 we have

$$H_n \le C_{13} H_0^{\exp(-k_2 n \Delta t)} \le C_{13}' H_0^{\exp(-k_2 T)} \equiv H_T,$$
$$H_n \ll 1, \quad H_T \ll 1, \quad g_n \ll 1.$$

Denote the bound for $\max_j |\hat{z}_j^n - z_j^n|$ by $f_n (n > 1)$ and $f_1 = \Delta t \cdot g_0$, and let

$$g_T = (C_{11} + C_{12} \ln \frac{1}{\Delta t})\Delta t + (C_{14} + C_{15} \ln \frac{1}{H_0})H_T \quad , \quad g_n < g_T, g_n \ll 1, \tag{5.6}$$

$$C_{19} = \left[MC_1 + C_2 \left|\ln \frac{1}{M}\right|\right] + \left[k_1 + k_2 \ln(De^{M_1 T})\right],$$
$$C_{20} = k_2 + MC_2,$$
$$\Omega = \max\left\{1, \left(\frac{1}{T}\right)^{C_{20}T}\right\} e^{C_{20}T + C_{19}T}.$$

Theorem 5.1. *Suppose $C_{20}T < 1, \Delta t, H_0$ are sufficiently small, such that*

$$\Omega T g_T^{1 - C_{20}T} \le \frac{1}{e}.$$

Then the following estimate

$$f_n \le \Omega T g_T^{1-C_{20}T} \tag{5.7}$$

holds for $n = 1, \dots N_T (= T/\Delta t)$.

Proof. We prove the theorem by induction. For $n = 1$, because $\Omega > 1$ and by (5.6), the conclusion is true . Assume (5.7) is true for all $n \le k$. Consider the case of $n = k+1 \le \left[\frac{T}{\Delta t}\right]$.

From (5.2) , (5.3) we get

$$|\hat{z}_j^{k+1} - \bar{z}_j^{k+1}| \le |\hat{z}_j^k - z_j^k| + \Delta t \iint_{R^2} |K(\hat{z}_j^k - z') - K(z_j^k - z')| \cdot |\hat{\omega}^k(z')| dz'$$
$$+ \Delta t \iint_{R^2} |K(z_j^k - z')||\hat{\omega}^k(z') - \bar{\omega}^k(z')| dz' \equiv |\hat{z}_j^k - z_j^k| + \Delta t R_1 + \Delta t R_2.$$

where

$$R_1 = \iint_{R^2} |K(\hat{z}_j^k - z') - K(z_j^k - z')||\hat{\omega}^k(z')| dz',$$
$$R_2 = \iint_{R^2} |K(z_j^k - z')||\hat{\omega}^k(z') - \bar{\omega}^k(z')| dz'.$$

By Lemma 5.1, the supprot of $\hat{\omega}^k(z')$ is included in the circle $B_{D\exp(M_1T)}$. Therfore, by Lemma 2.2, we have

$$|R_1| \le k_1 |\hat{z}_j^k - z_j^k| + k_2 |\hat{z}_j^k - z_j^k| \ln \left(\frac{De^{M_1T}}{|\hat{z}_j^k - z_j^k|} \right).$$

By the assumption of induction

$$f_n \le \frac{1}{e} \le \frac{D\exp(M_1T)}{e} \quad , \quad \forall n \le k.$$

So we have

$$\begin{aligned} \Delta t |R_1| &\le k_1 \Delta t f_k + k_2 \Delta t f_k \ln \left[\frac{D\exp(M_1T)}{f_k} \right] \\ &\le \Delta t [k_1 + k_2 \ln(D\exp(M_1T))] f_k + k_2 \Delta t f_k \ln \frac{1}{f_k}. \end{aligned}$$

Similarly, by Lemma 4.1, we can get the estimate for R_2

$$|R_2| \le M f_k \left[C_1 + C_2 |\ln \frac{1}{M}| + C_2 \ln \frac{1}{f_k} \right].$$

At last, we get

$$\begin{aligned} |\hat{z}_j^{k+1} - z_j^{k+1}| &\le |\hat{z}_j^{k+1} - \bar{z}_j^{k+1}| + |\bar{z}_j^{k+1} - z_j^{k+1}| \\ &\le \Delta t g_k + f_k + \Delta t [C_{19} + C_{20} \ln \frac{1}{f_k}] f_k. \end{aligned}$$

Hence, we obtain a fundamental relation for the error estimation f_{k+1}

$$f_{k+1}=\Delta t g_k+f_k+\Delta t[C_{19}+C_{20}\ln\frac{1}{f_k}]f_k. \tag{5.8}$$

Now we are going to check inequality (5.6) for $n=k+1$. From (5.8) ,

$$f_{k+1}=\sum_{i=1}^{k}\Delta t g_i+\sum_{i=1}^{k}\Delta t\left[C_{19}+C_{20}\ln\frac{1}{f_i}\right]f_i\geq(k+1)\Delta t g_0.$$

In addition, from (5.7) we can also get an inequality

$$f_{k+1}\geq(k+1-j)^{\Delta t g_j}\ ,\ \ 0\leq j\leq k. \tag{5.9}$$

By (5.8), (5.9) we have

$$\begin{aligned} f_{k+1}&=\Delta t g_k+\sum_{i=0}^{k-1}\prod_{j=i}^{k-1}\left[1+\Delta t\left(C_{19}+C_{20}\ln\frac{1}{f_{j+1}}\right)\right]\Delta t g_i\\ &\leq\sum_{i=0}^{k-1}\prod_{j=i}^{k-1}\left[1+\Delta t\left(C_{19}+C_{20}\ln\frac{1}{(j+1-i)\Delta t g_i}\right)\right]\Delta t g_i+\Delta t g_k. \end{aligned} \tag{5.10}$$

Let

$$P_m=\prod_{i=1}^{m}\left[1+\Delta t\left(C_{19}+C_{20}\ln\frac{1}{i\Delta t g_j}\right)\right].$$

Because $\ln(1+x)\leq x$ for $x\geq 0$, we get

$$\begin{aligned} \ln P_m&\leq\Delta t\sum_{i=1}^{m}[C_{19}+C_{20}\ln\frac{1}{i\Delta t g_j}],\\ P_m&\leq e^{C_{19}m\Delta t}\left(\prod_{i=1}^{m}\frac{1}{i\Delta t g_j}\right)^{C_{20}\Delta t}\\ &=e^{C_{19}m\Delta t}\left(\frac{1}{g_j}\right)^{C_{20}m\Delta t}\left(\frac{1}{m!\Delta t^m}\right)^{C_{20}\Delta t}. \end{aligned}$$

The time increment Δt equals T/N_T. Hence we have

$$m!\Delta t^m=\prod_{i=1}^{m}\left(\frac{i}{N_T}\right)T^m\geq\prod_{i=1}^{N_T}\left(\frac{i}{N_T}\right)T^m$$

and

$$\left(\frac{1}{m!\Delta t^m}\right)^{C_{20}\Delta t}\leq\left(\frac{1}{\prod_{i=1}^{N_T}\frac{i}{N_T}}\right)^{C_{20}\frac{T}{N_T}}\left(\frac{1}{T}\right)^{C_{20}m\Delta t}.$$

Note that

$$\frac{T}{N_T}\sum_{i=1}^{N_T}\ln\frac{1}{(\frac{i}{N_T})}\le T\int_0^1\ln\frac{1}{x}dx=T,$$
$$\left(\frac{1}{m!\Delta t^m}\right)^{C_{20}\Delta t}\le\left(\frac{1}{T}\right)^{C_{20}m\Delta t}e^{C_{20}T}.$$

Therefore,

$$P_m\le\Omega\left(\frac{1}{g_j}\right)^{C_{20}m\Delta t}.$$

Applying (5.10) and (5.11) we get

$$\begin{aligned}f_{k+1}&\le\Delta tg_k+\Omega\Delta t\sum_{i=1}^{k-1}\Delta tg_i^{1-C_{20}(k-i)\Delta t}\\&\le\Omega\sum_{i=0}^{k}\Delta tg_k^{1-C_{20}(k-i)\Delta t}\le(k+1)\Delta t\Omega g_k^{1-C_{20}k\Delta t}\le T\Omega g_k^{1-C_{20}T}\\&\le T\Omega g_T^{1-C_{20}T}\le\frac{1}{e}.\end{aligned}$$

The induction is comleted.

Remark. Theorem 5.1 shows the following error estimate of the approximate solution by explicit Euler's method :

$$\max_j|\hat{z}_j^n-z_j^n|\le\Omega Tg_T^{1-C_{20}T}$$

where $g_T=O(\Delta t\ln\Delta t+H_0\ln H_0)$ is defined in (5.6).

6. Discussion

we have proved the convergence of the discrete contour dynamics methods for a certain class of vortex flows with explicit Euler's method for sufficiently small $T(C_{20}T<1)$. But , one can always regularize the distribution of the points z_j on the contour at each time step in such a way that the conclusion will be true for any finite time interval T. In [7], it is proved that if the contour at each time step is regularized to meet certain conditions, the approximate solution converges to the solution of the equation

$$\dot{z}=K(z)*\omega,$$
$$z|_{t=0}=z_0.$$

So far we have obtained the convergence results of the explicit Euller's method which, as well known is of first-order accuracy. It can be expected that similar results hold for predictor-corrector methods used in [5].

Finally we make a remark concerning the canonical properties of the method used in contour dynamics codes.

According to [8], the contour dynamics system

$$\dot{z} = K(z) * \omega \tag{*}$$

is a Hamiltonian system of infinite degrees of freedom. The approximation of (*) yields a system

$$\dot{z}_j + K(z_j) * \omega \quad , \quad j = 1, \ldots, N \tag{**}$$

which is a Hamiltonian system of N degrees of freedom. But, the contour dynamics methods in [3–6] do not preserve the Hamiltonian and other canonical properties of the system. How to construct new contour schemes to preserve these good properties of the system is an important problem. The canonical difference schemes of [9] will certainly resolve the above problem. The new methods will preserve the symplectic properties of the physical problem. The detailed numerical algorithms and computational results obtained will be reported elsewhere.

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