THE FINITE ELEMENT METHOD FOR NONLINEAR ELASTICITY*

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Abstract

In this paper we consider the finite element method for nonlinear elasticity in the case when body force is small. The incremental method and the improved incremental method are investigated, their convergence are proved and the error estimates are obtained.

§1. Introduction

In the present paper we discuss the system of nonlinear elasticity :

$$\begin{cases} -\partial_j(\sigma_{ij}+\sigma_{kj}\partial_k u_i) = f_i, & \Omega, \\ u_i = 0, & \partial\Omega, \end{cases} \quad i=1,2,3 \tag{1.1}$$

where $u=(u_1,u_2,u_3,)^T$ is the displacement vector, $f=(f_1,f_2,f_3)^T$ is the exterior body force, and

$$\sigma_{ij}=\lambda E_{kk}(u)\delta_{ij}+2\mu E_{ij}(u)+o(E),$$

while

$$E(u)=\frac{1}{2}(\nabla u^T+\nabla u+\nabla u^T\nabla u). \tag{1.2}$$

We confine ourselves to the case when body force is sufficiently small. Hence we may suppose (see §5 for details) :

$$\sigma_{ij}=\lambda E_{kk}(u)\delta_{ij}+2\mu E_{ij}(u). \tag{1.3}$$

There are some mathematical results about the system (1.1). Especially, Ciarlet and Destuynder [1] proved the existence and uniqueness of solutions for (1.1) when f is sufficiently small. Bernadou, Ciarlet and Hu [2] proved the convergence of

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semi-discrete incremental methods for (1.1) in the case when f is sufficiently small. Ciarlet [3] summed up these results. These theoretical results and the wide application of finite element methods have promoted the investigation of the finite element methods for solving (1.1).

The present paper proves the existence, the uniqueness and the convergence of the finite element solutions and the convergence of the incremental methods for (1.1) under the condition that $\partial\Omega$ has certain smoothness and $f \in (W^{1,p}(\Omega))^3$ is sufficiently small. Error estimates of the incremental methods are also given.

In the present paper we assume that $\Omega \subset \mathcal{R}^3$ is a bounded connected open set with $\partial\Omega$ sufficiently smooth and consisting of finite disjoint simple closed surfaces.

Let $\Omega_h \subset \Omega, h > 0$, be regions composed of a finite number of polyhedrons. To each Ω_h there corresponds a regular triangulation (cf. Ciarlet [7]). We assume that Ω_h satisfy $\max_{x\in\partial\Omega_h}$ dist $(x, \partial\Omega) \leq kh^2$, where $k > 0$ is a constant which only depends on Ω.

For convenience, let $\mathcal{W}^{1,p}(\Omega)$ denote $(W^{1,p}(\Omega))^3, \mathcal{C}(\Omega)$ denote $(C(\Omega))^3, \mathcal{H}_0^1(\Omega)$ denote $(H_0^1(\Omega))^3$, etc.

Denote

$$V_h = \left\{u \in \mathcal{C}(\Omega) | u|_{\Omega\setminus\Omega_h} = 0, u|_K \in \mathcal{P}_2(K), \forall K \in \mathcal{J}_h\right\}$$

where $P_2(K)$ is the set of polynomials of second degree defined on K.

In §2 we will review some theoretical results about the system (1.1). The proofs of these results may be found in Chapter 2 of [3]. Some of the results are given in a more generalized form, but the original proofs remain valid if we note the results of Agmon, Douglis and Nirenberg [4].

In §3 we will investigate such properties as the existence, the uniqueness and the error estimates, etc. of the solutions of the corresponding variational problem of (1.1), when the problem is confined to the finite element space V_h .

In §4 we will investigate incremental methods, and their convergence and the error estimates.

In §5 some additional notes on σ_{ij} are given.

§2. Some theoretical results

We write (1.1) in the form of an operator

$$\begin{aligned} &A \ : \ \mathcal{W}^{m+2,p}(\Omega) \cap \mathcal{W}_0^{1,p}(\Omega) \to \mathcal{W}^{m,p}(\Omega), \\ &Au = f, \end{aligned} \tag{2.1}$$

$A'(0)$ denotes the Fréchet derivative of A at 0. Let

$$\begin{aligned} &\varepsilon(u) = \tfrac{1}{2}(\nabla u^T + \nabla u), \\ &\tilde{\sigma}_{ij}(u) = \lambda\varepsilon_{kk}(u)\delta_{ij} + 2\mu\varepsilon_{ij}(u). \end{aligned}$$

Then $A'(0)v = f$ can be expressed in component form as

$$\begin{cases} -\partial_j \bar{\sigma}_{ij}(v) = f_i \quad , \quad \Omega, \\ \qquad v_i = o \quad , \quad \partial\Omega, \end{cases} \quad i = 1,2,3. \tag{2.2}$$

In order to change the problem (2.2) into a variational form , we define

$$\begin{aligned} a(u,v) &= \int_\Omega (\lambda \varepsilon_{kk}(u)\varepsilon_{ll}(v) + 2\mu\varepsilon_{ij}(u)\varepsilon_{ij}(v))dx, \\ L(v) &= \int_\Omega f \cdot v dx \end{aligned}$$

where $u, v \in \mathcal{H}_0^1(\Omega)$. The variational problem of (2.2) can be expressed as

$$\begin{cases} \text{Find } u \in \mathcal{H}_0^1(\Omega), \text{ such that} \\ a(u,v) = L(v), \quad \forall v \in \mathcal{H}_0^1(\Omega). \end{cases} \tag{2.3}$$

Theorem 2.1 (Korn's inequality [5]). *Suppose $\partial\Omega$ is sufficiently smooth. Then*

$$\left\{ v \in \mathcal{L}^2(\Omega) | \varepsilon_{ij}(v) \in L^2(\Omega), 1 \le i,j \le 3 \right\} = \mathcal{H}^1(\Omega),$$

and there are constants $C_1 > 0$ and $C_2 > 0$, such that

$$C_1 \|v\|_{1,\Omega} \le (\|v\|_{0,\Omega}^2 + |\varepsilon(v)|_{0,\Omega}^2)^{1/2} \le C_2 \|v\|_{1,\Omega}. \tag{2.4}$$

Corollary (cf. [3]). The semi-norm $|\varepsilon(v)|_{0,\Omega}$ is equivalent to the norm $\|v\|_{1,\Omega}$ in $\mathcal{H}_0^1(\Omega)$.

It follows easily from Theorem 2.1 that $a(u,v)$ is a continuous positive definite bilinear form defined on $\mathcal{H}_0^1(\Omega) \times \mathcal{H}_0^1(\Omega)$ provided that $\lambda > 0$ and $\mu > 0$.

Theorem 2.2 (cf. [3]). *The variational problem (2.3) has a unique solution $u \in \mathcal{H}_0^1(\Omega)$, if $\lambda > 0$ and $\mu > 0$.*

Theorem 2.3. *Suppose $\partial\Omega$ is sufficiently smooth, and $f \in \mathcal{W}^{m,p}(\Omega), p \ge 2$. Then the solution $u \in \mathcal{H}_0^1(\Omega)$ of (2.3) is also in*

$$\mathcal{V}_m^p = \left\{ v \in \mathcal{W}^{m+2,p}(\Omega) | v = 0, \quad on\ \partial\Omega \right\}. \tag{2.5}$$

Note. The case $m = 0$ was proved in [3]. The proof of the case $m > 0$ is based on the proof of the case $m = 0$, but the result of Agmon , Douglis and Nirenberg [4] should be taken into account.

Theorem 2.4. *Suppose $\partial\Omega$ is sufficiently smooth, $\lambda > 0, \mu > 0$, and $f \in \mathcal{W}^{m,p}(\Omega), p > 3$. Then there exist constants $\delta > 0$ and $M > 0$, such that there exists $u \in \mathcal{W}^{m+2,p}(\Omega)$, which satisfies $Au = f$, and*

$$\|u\|_{m+2,p} \le M \|f\|_{m,p} \tag{2.6}$$

provided that $\|f\|_{m,p} \le \delta$.

Note. The proof for the case $m = 0$ is given in [3]. The proof for the case $m > 0$ is similar if we make use of Theorem 2.3 in showing the regularity of the linearized operator.

Theorem 2.5 (Inverse function theorem [6]). *Let B_1, B_2 be Banach spaces. Let $U \subset B_1, V \subset B_2$ be open sets in B_1, B_2 respectively. Assume $u \in U, A : U \to V$ is a differentiable map, $Au = v \in V$.*

Suppose $A'(u)$ is invertible, and $\|A'(u)^{-1}\| \le C_1$. Suppose $A'(\tilde{u})$ is uniformly Lipschitz continuous with respect to $\tilde{u} \in U$,

$$\|(A'(\tilde{u}) - A'(\bar{u}))w\|_{B_2} \le C_2\|\tilde{u} - \bar{u}\|_{B_1}\|w\|_{B_1}.$$

Then for $\forall \Delta v \in B_1$ which satisfies $v + \Delta v \in V, \|\Delta v\|_{B_2} \le \frac{1}{4C_1^2C_2}$, there is a $\Delta u \in B_1$, such that $u + \Delta u \in U, A(u + \Delta u) = v + \Delta v$, and

$$\|\Delta u\|_{B_1} \le 2C_1\|\Delta v\|_{B_2}. \tag{2.7}$$

Here we assume $U \supset \left\{\tilde{u} \in B_1 \middle| \|\tilde{u} - u\|_{B_1} \le \frac{1}{2C_1C_2}\right\}$.

§3. Formulation of the finite element methods and the properties of solutions

Let Ω, Ω_h, J_h satisfy the conditions given in §1.

$$V_h = \left\{u \in C(\Omega) | u|_{\Omega\setminus\Omega_h} = 0, u|_K \in P_2(K), \forall K \in J_h\right\}.$$

In the following we will investigate the problem in these finite element spaces. First we define an operator ϕ_h on V_h as follows :

Definition 3.1. *Suppose $u \in V_h$. If $g_h \in V_h$ satisfies*

$$\int_\Omega (\sigma_{ij} + \sigma_{kj}\partial_k u_i)\partial_j v_i dx = \int_\Omega \nabla g_h \cdot \nabla v dx \equiv [g_h, v] \tag{3.1}$$

for $\forall v \in V_h$, then we define $\phi_h(u) = g_h$.

Note 3.1. It is obvious that the left-hand side of (3.1) defines on V_h a continuous linear functional, and V_h is a Hilbert space with $[\cdot,\cdot]$ as its inner product. Hence for any $u \in V_h$, there exists a unique $g_h \in V_h$ satisfying (3.1). So ϕ_h is well defined.

Note 3.2. Let $u \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$. Suppose $g \in H_0^1(\Omega)$ satisfies (3.1) for $\forall v \in H_0^1(\Omega)$. Then we define $\phi : W^{1,\infty}(\Omega) \cap H_0^1(\Omega) \to H_0^1(\Omega)$ as $\phi(u) = g$. With the same argument as above, it follows that ϕ is well defined.

It is easy to verify that the operator ϕ_h is continuously differentiable. We have

$$\left.\begin{aligned}[\phi_h'(u)v,w]=\int_\Omega\Big\{&\lambda\varepsilon_{kk}(v)\delta_{ij}+2\mu\varepsilon_{ij}(v))\partial_j w_i\\&+\Big[\tfrac{\lambda}{2}(\partial_k u_l\partial_k v_l+\partial_k v_l\partial_k u_l)\delta_{ij}+\mu(\partial_j u_k\partial_i v_k+\partial_i u_k\partial_j v_k)\\&+(\lambda\varepsilon_{kk}(v)\delta_{mj}+2\mu\varepsilon_{mj}(v))\partial_m u_i+\lambda\partial_k u_l\partial_k v_l\delta_{mj}\partial_m u_i\\&+\mu(\partial_m u_k\partial_j v_k+\partial_m v_k\partial_j u_k)\partial_m u_i+\sigma_{mj}(u)\partial_m v_i\Big]\partial_j w_i\Big\}dx\end{aligned}\right\}\qquad(3.2)$$

(3.2) can also be regarded as a definition of $\phi_h'(u)v$.

Lemma 3.1. *There exist constants $\alpha_0>0$ and $\eta>0$ which only depend on λ,μ and Ω, such that*

$$[\phi_h'(u)v,v]\geq\frac{\alpha_0}{2}|v|_{1,2}^2,\quad\forall v\in\mathcal{H}_0^1(\Omega),$$

if $|u|_{1,\infty}\leq\eta$. The inequality holds especially for $\forall\ v\in V_h$.

Proof. By Theorem 2.1 , *i.e.* Korn's inequality, there is a constant $\alpha_0>0$, such that

$$\int_\Omega(\lambda\varepsilon_{kk}(v)\delta_{ij}+2\mu\varepsilon_{ij}(v))\partial_j v_i dx\geq\alpha_0|v|_{1,2}^2,\quad\forall\ \ v\in\mathcal{H}_0^1(\Omega).$$

From (3.2) it follows easily that there is a constant $\eta>0$, such that the lemma holds for $|u|_{1,\infty}\leq\eta$. QED.

Define

$$U_h=\Big\{u\in V_h\Big||u|_{1,\infty}\leq\eta\Big\}.$$

Then Corollary 3.1 implies that $[\phi_h'(u)v,v]$ is a continuous positive definite bilinear form defined on the space $V_h\times V_h$, whose norm is induced from $[\cdot,\cdot]$, provided that $u\in U_h$. In other words, $\phi_h'(u):V_h\to V_h$ is a uniform elliptic linear operator for $u\in U_h$.

Lemma 3.2. *Suppose $u\in U_h,g_h\in V_h$. Then there exists a unique $v\in V_h$, such that*

$$[\phi_h'(u)v,w]=[g_h,w],\quad\forall\ w\in V_h$$

and

$$|v|_{1,2}\leq\beta|g_h|_{1,2},\quad\beta=2/\alpha_o.$$

Proof. Note that V_h is a Hilbert space with $[\cdot,\cdot]$ as its inner product. The lemma follows from Lemma 3.1 and the Lax-Milgram theorem (cf. [8]). QED.

Remark 3.3. By means of the inverse inequalities (cf. [7]) on V_h, we also have the estimate

$$|v|_{1,\infty}\leq\beta_1 h^{-3/2}|g_h|_{1,2},$$

where β_1 is a constant dependent only on λ,μ, and Ω.

By Theorem 2.4, we know that for any $f\in\mathcal{W}^{1,p}(\Omega)$ with $\|f\|_{m,p}$ sufficiently small, there exists a unique $u\in\mathcal{W}^{m+2,p}(\Omega)\cap\mathcal{H}_0^1(\Omega)$, such that

$$Au=f\text{ and }\|u\|_{m+2,p}\leq M\|f\|_{m,p}.$$

The weak form of $Au = f$ is

$$\int_\Omega (\sigma_{ij} + \sigma_{kj}\partial_k u_i)\partial_j v_i dx = \int_\Omega f \cdot v dx, \quad \forall\ v \in \mathcal{H}_0^1(\Omega). \tag{3.3}$$

It would be inconvenient to discuss (3.3) on V_h, so we introduce the following definition :

Definition 3.2. *Suppose $f \in \mathcal{L}^2(\Omega)$. We define $\psi(f) = g$ if $g \in \mathcal{H}_0^1(\Omega)$ and*

$$[g, v] = \int_\Omega f \cdot v dx \quad , \quad \forall v \in \mathcal{H}_0^1(\Omega).$$

ψ is well defined because $[\cdot,\cdot]$ is an inner product of $\mathcal{H}_0^1(\Omega)$ and $\int_\Omega f \cdot v dx$ is a continuous linear functional defined on $\mathcal{H}_0^1(\Omega)$. We have in addition that $|\psi(f)|_{1,2} \le \gamma \|f\|_{L^2}$, there $\gamma > 0$ is a constant.

Now we can write the weak form of $Au = f$ as

$$\int_\Omega (\sigma_{ij} + \sigma_{kj}\partial_k u_i)\partial_j v_i dx = [\psi(f), v], \forall\ v \in \mathcal{H}_0^1(\Omega). \tag{3.3'}$$

By the definition of ϕ, we have $\phi(u) = \psi(f)$ (cf. Remark 3.2). Let π_h : $\mathcal{W}^{m+2,p}(\Omega) \to V_h$ be an interpolation operator. Denote $u_h = \pi_h u$.

Lemma 3.3. *Let $u \in \mathcal{W}^{m+2,p}(\Omega), u_h = \pi_h u$. Then*

$$|\phi(u) - \phi(u_h)|_{1,2} \le C|u - u_h|_{1,2} \tag{3.4}$$

where C is a constant dependent on $|u|_{1,\infty}, |u - u_h|_{1,\infty}, \lambda$ and μ.

Proof. By noting that operator ϕ is continuously differentiable and (3.2) defines its derivative operator, we have a $0 < \theta < 1$, such that

$$\phi(u) - \phi(u_h) = \phi'(u_h + \theta(u - u_h))(u - u_h).$$

Now, (3.4) follows from this equality and (3.2). QED.

Definition 3.3 *$(V_h, [\cdot,\cdot])$. is a subspace of $\mathcal{H}_0^1(\Omega)$. For any $g \in \mathcal{H}_0^1(\Omega)$, denote by $(g)_h$ the orthogonal projection of g on $(V_h, [\cdot,\cdot])$, i.e. $(g)_h \in V_h$, and*

$$\int_\Omega \nabla(g)_h \cdot \nabla v dx = \int_\Omega \nabla g \cdot \nabla v dx \quad , \quad \forall\ v \in V_h. \tag{3.5}$$

By Definition 3.3 , it is obvious that

$$(\phi(u))_h - (\phi(u_h))_h = (\phi(u) - \phi(u_h))_h.$$

Hence

$$|(\phi(u))_h - (\phi(u_h))_h|_{1,2} \le |\phi(u) - \phi(u_h)|_{1,2}. \tag{3.6}$$

From Definition 3.1, (3.3)' and (3.5), it follows that

$$\phi_h(u_h) = (\phi(u_h))_h. \tag{3.7}$$

Taking $f \in \mathcal{W}^{1,p}(\Omega)$, from Theorem 2.4 it follows that the solution of $Au = f$ is in $\mathcal{W}^{3,p}(\Omega) \cap \mathcal{H}_0^1(\Omega)$, and $\|u\|_{3,p} \le \frac{\eta}{2\tilde{\gamma}}$ if $\|f\|_{1,p} \le \frac{\eta}{2M\tilde{\gamma}}$. By the Sobolev imbedding theorem (cf. [9]), we have in addition $|u|_{1,\infty} \le \frac{\eta}{2}$, where $\tilde{\gamma}$ is the Sobolev imbedding norm . From the interpolation theory of the finite element spaces (cf. [7] , [10]), we have

$$|u - \pi_h u|_{1,\infty} \le Ch^{2-1/p}|u|_{3,p} \quad , \quad p > 3,$$

$$|u - \pi_h u|_{1,2} \le Ch^2|u|_{3,2}.$$

Combining with Theorem 2.4, we get

$$|u - \pi_h u|_{1,\infty} \le Ch^{2-1/p}\|f\|_{1,p}, \tag{3.8}$$

$$|u - \pi_h u|_{1,2} \le Ch^2\|f\|_{1,p}. \tag{3.9}$$

Hence $u_h \equiv \pi_h u \in U_h$ if h is sufficiently small and $\|f\|_{1,p} \le \frac{\eta}{2M\tilde{\gamma}}$.

Lemma 3.4. *$\phi_h'(u) \ : \ V_h \to V_h$ is Lipschitz continuous on U_h, and*

$$|[(\phi_h'(u_1) - \phi_h'(u_2))v, w]| \le \beta_2|u_1 - u_2|_{1,\infty}|v|_{1,2}|w|_{1,2} \tag{3.10}$$

for $\forall u_1, u_2 \in U_h$ and $\forall v, w \in V_h$.

Proof. We only need to verify (3.10). But this can be directly derived from (3.2). QED.

Remark 3.4. Let $w = (\phi_h'(u_1) - \phi_h'(u_2))v$ in (3.10). Then we have

$$|(\phi_h'(u_1) - \phi_h'(u_2))v|_{1,2} \le \beta_2|u_1 - u_2|_{1,\infty}|v|_{1,2} \le \beta_3|u_1 - u_2|_{1,\infty}|v|_{1,\infty}. \tag{3.10'}$$

Theorem 3.1. *Let $f \in \mathcal{W}^{1,p}(\Omega), \|f\|_{1,p} \le \frac{\eta}{2M\tilde{\gamma}}$. Suppose $Au = f, u_h = \pi_h u$. Then there exists a constant $\delta > 0$, which may depend on h, such that for $\forall g \in V_h$, there is a $\bar{u}_h \in U_h$ which satisfies $\phi_h(\bar{u}_h) = g$ and*

$$|\bar{u}_h - u_h|_{1,\infty} \le 2\beta_1 h^{-3/2}|g - (\phi(u_h))_h|_{1,2}$$

provided that $|g - (\phi(u_h))_h|_{1,2} \le \delta$. Here δ can be taken as $\frac{\eta}{16\beta_1}h^{3/2}$.

Proof. By Lemma 3.2 and Remark 3.3, we know that $\phi_h'(u) \ : \ (V_h, \mathcal{W}^{1,\infty}) \to (V_h, \mathcal{H}_0^1)$ is invertible provided that $u \in U_h$, and that $\|\phi_h'(u)^{-1}\| \le \beta_1 h^{-3/2}$. By Lemma 3.4 and Theorem 2.5, it follows that for $\forall g \in V_h$, if $|g - (\phi(u_h))_h|_{1,2} \le \frac{h^3}{4\beta_1^2\beta_3}$, then there exists a $\bar{u}_h \in V_h$, such that $\phi_h(\bar{u}_h) = g$, and

$$|\bar{u}_h - u_h|_{1,\infty} \le 2\beta_1 h^{-3/2}|g - (\phi(u_h))_h|_{1,2} \le \frac{h^{3/2}}{2\beta_1\beta_3}.$$

Hence for sufficiently small h, one has

$$\begin{aligned} |\bar{u}_h|_{1,\infty} &\le |u|_{1,\infty} + |u - u_h|_{1,\infty} + |u_h - \bar{u}_h|_{1,\infty} \\ &\le \frac{\eta}{2} + C\eta h^{2-\frac{1}{p}} + \frac{h^{3/2}}{2\beta_1\beta_3} < \eta \end{aligned}$$

i.e. $\bar{u} \in U_h$.

The above procedure can be continued if u_h and $(\phi(u_h))_h$ are substituted by $\bar{u}_h$ and $(\phi(\bar{u}_h))_h = g$ respectively.

Let k be the maximum integer satisfying

$$k\frac{h^{3/2}}{2\beta_1\beta_3} \le \frac{\eta}{2} - C\eta h^{2-\frac{1}{p}}.$$

Assume h_0 satisfies $C\eta h_0^{2-\frac{1}{p}} < \frac{\eta}{4}, \frac{\beta_1\beta_3\eta}{4}h_0^{-3/2} > 1$. Then for all $h \le h_0$ we have

$$k+1 > \frac{2\beta_1\beta_3}{h^{3/2}}(\frac{\eta}{2} - C\eta h^{2-\frac{1}{p}}) \ge \frac{\beta_1\beta_3\eta}{2}h^{-3/2}$$

or

$$k > \frac{\beta_1\beta_3\eta}{4}h^{-3/2}.$$

Hence the above procedure can iterate at least k times if $h \le h_0$, *i.e*, as soon as $g \in V_h$ and

$$|g - (\phi(u_h))_h|_{1,2} \le \frac{\eta}{16\beta_1}h^{3/2},$$

there will exist a $\bar{u}_h \in U_h$, such that $\phi_h(\bar{u}_h) = g$, and

$$|\bar{u}_h - u_h|_{1,\infty} \le 2\beta_1 h^{-3/2}|g - (\phi(u_h))_h|_{1,2}. \tag{3.11}$$

Let $\delta = \frac{\eta h^{3/2}}{16\beta_1}$; the theorem follows . QED.

Remark 3.5. In Theorem 3.1, η can be substituted by $CM\|f\|_{1,p}$.

Theorem 3.2. *Let $u_1, u_2 \in U_h$. Suppose $\phi_h(u_1) = g_1$, and $\phi_h(u_2) = g_2$. Then*

$$|u_1 - u_2|_{1,2} \le \beta|g_1 - g_2|_{1,2}, \tag{3.12}$$

$$|u_1 - u_2|_{1,\infty} \le \beta_1 h^{-3/2}|g_1 - g_2|_{1,2}. \tag{3.12'}$$

Proof. We have for $\forall v \in V_h$

$$[\phi_h(u_1) - \phi_h(u_2), v] = \int_0^1 [\phi_h'(u_2 + t(u_1 - u_2))(u_1 - u_2), v]dt.$$

Let $V = u_1 - u_2$. By Lemma 3.1, we get the following inequality and hence (3.12)

$$[g_1 - g_2, u_1 - u_2] = [\phi_h(u_1) - \phi_h(u_2), u_1 - u_2] \ge \frac{\alpha_0}{2}|u_1 - u_2|_{1,2}^2.$$

(3.12)' comes from (3.12) and the inverse inequalities on V_h (cf. [7]). QED.

Theorem 3.3. *There is a constant $\bar{h} > 0$, such that if $h \le \bar{h}$, then for $\forall f \in W^{1,p}(\Omega)$ satisfying $\|f\|_{1,p} \le \frac{\eta}{2M\gamma}$, there exists a unique $\bar{u}_h \in U_h$ which solves*

$$\phi_h(\bar{u}_h) = (\psi(f))_h$$

where $(\psi(f))_h$ is defined by Definition 3.2 and Definition 3.3.

Let u be the solution of $Au = f$. We have estimates

$$|\bar{u}_h - u|_{1,2} \leq Ch^2\|f\|_{1,p}, \tag{3.13}$$

$$|\bar{u}_h - u|_{1,\infty} \leq Ch^{1/2}\|f\|_{1,p}, \tag{3.14}$$

$$|\bar{u}_h|_{1,\infty} \leq C\|f\|_{1,p}. \tag{3.15}$$

Proof. From (3.4) and (3.6), it follows that

$$|(\phi(u))_h - (\phi(u_h))_h|_{1,2} \leq Ch^2\|f\|_{1,p}.$$

Hence

$$|(\phi(u))_h - (\phi(u_h))_h|_{1,2} < \frac{\eta}{16\beta_1}h^{3/2}, \quad \text{for } h \leq \left(\frac{M\tilde{\gamma}}{8\beta_1 C}\right)^2.$$

By Theorem 3.1, there is a $\bar{u}_h \in U_h$, such that $\phi_h(\bar{u}_h) = (\phi(u))_h$, and

$$|\bar{u}_h - u_h|_{1,\infty} \leq 2\beta_1 h^{-3/2}|(\phi(u))_h - (\phi(u_h))_h|_{1,2} \leq Ch^{1/2}\|f\|_{1,p}. \tag{3.16}$$

By Theorem 3.2 and (3.12), we also have

$$|\bar{u}_h - u_h|_{1,2} \leq C\|f\|_{1,p}h^2. \tag{3.17}$$

On the other hand, it follows from (3.3)' that $\phi(u) = \psi(f)$. Hence $\phi_h(\bar{u}_h) = (\psi(f))_h$.

The uniqueness is a simple corollary of Theorem 3.2.

By the definition of $(\psi(f))_h$ and the definition of $\phi_h(\bar{u}_h) = (\psi(f))_h$, we have

$$\int_\Omega (\sigma_{ij} + \sigma_{kj}\partial_k \bar{u}_{hi})\partial_j v_i dx = [(\psi(f))_h, v] = [\psi(f), v], \quad \forall v \in V_h.$$

By the definition of $\psi(f)$, the equality can be further expressed as

$$\int_\Omega (\sigma_{ij}(\bar{u}_h) + \sigma_{kj}(\bar{u}_h)\partial_k \bar{u}_{hi})\partial_j v_i dx = \int_\Omega f \cdot v dx, \quad \forall v \in V_h. \tag{3.18}$$

By comparing (3.18) with (3.3) , we conclude that $\bar{u}_h \in V_h$ is the finite element solution of $Au = f$ on V_h. Combining (3.8),(3.9) with (3.16), (3.17), we get (3.13) and (3.14). (3.15) now follows from the fact $|u|_{1,\infty} < C\|f\|_{1,p}$. QED.

§4. Incremental methods, their convergence and error estimates

Assume $f \in \mathcal{W}^{1,p}(\Omega), \|f\|_{1,p} \leq \frac{\eta}{2M\tilde{\gamma}}$. For the sake of convenience, we denote $(\psi(f))_h = \bar{f}$ (cf. Definitions 3.2 and 3.3), and denote $u(t\bar{f})$ as the solution of

$Au = tf$, and $u_h(t\bar{f})$ as the solution of $\phi_h(u) = t\bar{f}$. From §3 we have for $0 \le t \le 1$

$$|u_h(t\bar{f}) - u(t\bar{f})|_{1,\infty} \le Ch^{1/2}t\|f\|_{1,p}, \tag{4.1}$$

$$|u_h(t\bar{f}) - u(t\bar{f})|_{1,2} \le Ch^2 t\|f\|_{1,p}. \tag{4.2}$$

Now we are going to expound the incremental method :

Fix V_h. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of [0,1] satisfying $\dfrac{\max_k(t_{k+1} - t_k)}{\min_k(t_{k+1} - t_k)} \le \rho$ with $\rho \ge 1$ a given constant. Denote $t = \max_k(t_{k+1} - t_k)$.

Define $u_h^i, i = 0, 1, \cdots, n$, in the following way :

$$\begin{cases} u_h^0 = 0, \\ u_h^{i+1} = u_h^i + v_h^i, \quad i = 0, 1, \cdots, n-1 \end{cases} \tag{4.3}$$

where $v_h^i \in V_h$ satisfy the equations

$$\phi_h'(u_h^i)v_h^i = (t_{i+1} - t_i)\bar{f}, \quad 0 \le i \le n-1. \tag{4.4}$$

We take u_h^n as an approximate finite element solution of $u(\bar{f})$.

Theorem 4.1. *Take* $t \le \frac{\eta}{2C}h^{3/2}e^{-c\rho h^{-3/2}}$, *whrer* $C = 2\beta\beta_1\beta_3\bar{\bar{\gamma}}\|f\|_{1,p}$ *is a constant (* $\bar{\bar{\gamma}}$ *is a constant,* $|\bar{f}|_{1,2} \le \bar{\bar{\gamma}}\|f\|_{1,p}$*). Then* $u_h^i, 0 \le i \le n$, *defined by (4.3) and (4.4) are in* U_h, *and*

$$|u_h^i - u_h(t_i\bar{f})|_{1,\infty} \le C(1 + Ch^{-3/2}t)^i h^{-3/2}t, \tag{4.5}$$

$$|u_h^i - u_h(t_i\bar{f})|_{1,2} \le C(1 + Ch^{-3/2}t)^i t. \tag{4.6}$$

Proof. We will use induction to verify (4.5) and (4.6). If (4.5), (4.6) are verified , then $u_h^i \in U_h$ follows easily from the limitation of t.

(4.5), (4.6) are obviously true for $i = 0$. Assume (4.5), (4.6) are true for i. By the inverse inequality , to complete the induction, it is enough to prove that (4.6) holds for $i + 1$.

Let $\bar{v}_h^i$ be the solution of

$$\phi_h'(u_h(t_i\bar{f}))\bar{v}_h^i = (t_{i+1} - t_i)\bar{f}. \tag{4.7}$$

By definition , we have

$$\phi_h'(u_h^i)v_h^i - \phi_h'(u_h(t_i\bar{f}))\bar{v}_h^i = 0$$

or

$$\phi_h'(u_h(t_i\bar{f}))(v_h^i - \bar{v}_h^i) = (\phi_h'(u_h(t_i\bar{f})) - \phi_h'(u_h^i))\bar{v}_h^i.$$

Hence

$$v_h^i - \bar{v}_h^i = \phi_h'(u_h(t_i\bar{f}))^{-1}(\phi_h'(u_h(t_i\bar{f})) - \phi_h'(u_h^i))\bar{v}_h^i.$$

By the assumption of the induction , we have $u_h^i \in U_h$. Then, it follows from Lemma 3.2 and (3.10) that

$$|v_h^i - \bar{v}_h^i|_{1,2} \le \beta\beta_2 |u_h(t_i\bar{f}) - u_h^i|_{1,\infty} |\bar{v}_h^i|_{1,2}. \tag{4.8}$$

Again by the assumption of the induction, Lemma 3.2 and (4.7), we get

$$|v_h^i - \bar{v}_h^i|_{1,2} \le \frac{1}{2} C^2 h^{-3/2} (1 + Ch^{-3/2}t)^i t^2. \tag{4.9}$$

On the other hand, we have

$$\phi_h(u_h(t_{i+1}\bar{f})) - \phi_h(u_h(t_i\bar{f})) - \phi_h'(u_h(t_i\bar{f}))\bar{v}_h^i = 0.$$

Hence there is a $\theta = \tau(u_h(t_{i+1}\bar{f}) - u_h(t_i\bar{f}))$ such that

$$\phi_h'(u_h(t_i\bar{f}) + \theta)(u_h(t_{i+1}\bar{f}) - u_h(t_i\bar{f})) - \phi_h'(u_h(t_i\bar{f}))\bar{v}_h^i = 0$$

or

$$\begin{aligned} u_h(t_{i+1}\bar{f}) - u_h(t_i\bar{f}) - \bar{v}_h^i &= \phi_h'(u_h(t_i\bar{f}) + \theta)^{-1}(\phi_h'(u_h(t_i\bar{f})) \\ &\quad - \phi_h'(u_h(t_i\bar{f}) + \theta))\bar{v}_h^i. \end{aligned}$$

Then, it follows from Lemma 3.2 and (3.10) that

$$|u_h(t_{i+1}\bar{f}) - u_h(t_i\bar{f}) - \bar{v}_h^i|_{1,2} \le \beta\beta_2 |u_h(t_{i+1}\bar{f}) - u_h(t_i\bar{f})|_{1,\infty} |\bar{v}_h^i|_{1,2}. \tag{4.10}$$

By (3.12)' and Lemma 3.2, we get

$$|u_h(t_{i+1}\bar{f}) - u_h(t_i\bar{f}) - \bar{v}_h^i|_{1,2} \le \frac{1}{2} C^2 h^{-3/2} t^2. \tag{4.11}$$

Because $|u_h^i + v_h^i - u_h(t_{i+1}\bar{f})| \le |u_h^i - u_h(t_i\bar{f})| + |v_h^i - \bar{v}_h^i| + |\bar{v}_h^i - u_h(t_{i+1}\bar{f}) + u_h(t_i\bar{f})|$, it follows from (4.9), (4.11) and the assumption of the induction that

$$|u_h^{i+1} - u_h(t_{i+1}\bar{f})|_{1,2} \le C(1 + Ch^{-3/2}t)^{i+1} t.$$

These complete the induction. QED.

Remark 4.1. $(1+Ch^{-3/2}t)^n = [(1+Ch^{-3/2}t)^{\frac{h^{3/2}}{ct}}]^{ch^{-3/2}tn}$, but $nt \le \frac{\max_k(t_{k+1} - t_k)}{\min_k(t_{k+1} - t_k)}$.

Hence $(1 + Ch^{-3/2}t)^n \le e^{c\rho h^{-3/2}}$.

Define $u_h^t = u_h^n$. We have inequalities

$$|u_h^t - u_h(\bar{f})|_{1,\infty} \le Ch^{-3/2} e^{c\rho h^{-3/2}} t \|f\|_{1,p}, \tag{4.12}$$

$$|u_h^t - u_h(\bar{f})|_{1,2} \le Ce^{c\rho h^{-3/2}} t \|f\|_{1,p}. \tag{4.13}$$

(4.12) and (4.13) imply that $u_h^t \to u_h(\bar{f})$ as $t \to 0$.

Theorem 4.2. *Suppose $f \in \mathcal{W}^{1,p}(\Omega)$ and $\|f\|_{1,p} \le \frac{\eta}{2M\bar{\gamma}}$. Then, if $h \le \bar{h}$ and $t < \frac{\eta}{2C} h^{3/2} e^{-c\rho h^{-3/2}}$, we can get an approximate finite element solution u_h^t by* *means of the incremental method defined by (4.3) and (4.4), with error estimates*

$$|u_h^t - u(\bar{f})|_{1,\infty} \le C(th^{-3/2}e^{c\rho h^{-3/2}} + h^{1/2})\|f\|_{1,p}, \tag{4.14}$$

$$|u_h^t - u(\bar{f})|_{1,2} \le C(te^{c\rho h^{-3/2}} + h^2)\|f\|_{1,p}. \tag{4.15}$$

by now, we have proved the convergence of the discrete incremental method, and also offered error estimates. But in the error estimates, the coefficient of t is an exponential function of h since there exists $h^{-3/2}$ in the estimate of $\|\phi_h'(u)^{-1}\|$. This is more or less unpleasant.

Now, we are going to give an improved incremental method, the estimates of which will have an h-independent coefficient of t.

Fix V_h. Let $0 = t_0 < t_1 \cdots < t_n = 1$ be a partition of [0,1]. Denote $t = \max_k(t_{k+1} - t_k)$. Define $\{u_h^i\}_{i=0}^n$ in the following way :

$$\begin{cases} u_h^0 = 0, \\ \phi_h'(u_h^i)v_h^i = t_{i+1}\bar{f} - \phi_h(u_h^i) \quad , \quad 0 \le i \le n-1, \\ u_h^{i+1} = u_h^i + v_h^i \quad , \quad 0 \le i \le n-1. \end{cases} \tag{4.16}$$

Taking u_h^n as an approximate solution of $u_h(\bar{f})$, we have

Theorem 4.3. *Suppose $f \in W^{1,p}(\Omega), \|f\|_{1,p} \le \frac{\eta}{2M\bar{\bar{\gamma}}}, h < \bar{h}$ (cf. Theorem 3.3). Let $C = 2\beta\beta_1\beta_2\bar{\bar{\gamma}}\|f\|_{1,p}$ (where $\bar{\bar{\gamma}}$ is a constant, such that $|\bar{f}|_{1,2} \le \bar{\bar{\gamma}}\|f\|_{1,p}$). Then $u_h^i, 0 \le i \le n$, defined by (4.16) are in U_h, and have estimates*

$$|u_h^i - u_h(t_i\bar{f})|_{1,\infty} \le 2C^2t^2h^{-3}, \quad 0 \le i \le n, \tag{4.17}$$

$$|u_h^i - u_h(t_i\bar{f})|_{1,2} \le 2C^2t^2h^{-3/2}, \quad 0 \le i \le n. \tag{4.18}$$

We have in particular

$$|u_h^i - u_h(t_i\bar{f})|_{1,\infty} \le Ct \quad , \quad 0 \le i \le n, \tag{4.19}$$

$$|u_h^i - u_h(t_i\bar{f})|_{1,2} \le Ct \quad , \quad 0 \le i \le n \tag{4.20}$$

if $t \le \frac{h^3}{2C}$ with $h < C\eta^{-1}$.

Proof. Only (4.17), (4.18) are to be verified. We will complete the proof by induction.

If (4.17), (4.18) hold for i, then it is easy to show that $u_h^i \in U_h$. It is also clear that (4.17) is a direct conclusion of the inverse inequality [7] and (4.18). Therefore, we only need to prove (4.18). It is obvious that (4.18) holds for $i = 0$.

Assume (4.18) is true for i. Then it follows from (4.16) and $t_{i+1}\bar{f} = \phi_h(u_h(t_{i+1}\bar{f}))$ that

$$\phi_h'(u_h^i)v_h^i = \phi_h(u_h(t_{i+1}\bar{f})) - \phi_h(u_h^i).$$

And there is a $0 < \tau < 1$, such that

$$\phi_h(u_h(t_{i+1}\bar{f})) - \phi_h(u_h^i) = \phi_h'(u_h^i + \tau(u_h(t_{i+1}\bar{f}) - u_h^i))(u_h(t_{i+1}\bar{f}) - u_h^i).$$

Denoting $u_h^i + \tau(u_h(t_{i+1}\bar{f}) - u_h^i) = u^\tau$, we have

$$v_h^i - (u_h(t_{i+1}\bar{f}) - u_h^i) = \phi_h'(u_h^i)^{-1}(\phi_h'(u^\tau) - \phi_h'(u_h^i))(u_h(t_{i+1}\bar{f}) - u_h^i).$$

By Lemmas 3.2 and 3.4, we have

$$|u_h^{i+1} - u_h(t_{i+1}\bar{f})|_{1,2} \leq \beta\beta_2|u_h(t_{i+1}\bar{f}) - u_h^i|_{1,\infty}|u_h(t_{i+1}\bar{f}) - u_h^i|_{1,2}.$$

But $|u_h(t_{i+1}\bar{f}) - u_h^i| \leq |u_h(t_{i+1}\bar{f}) - u_h(t_i\bar{f})| + |u_h(t_i\bar{f}) - u_h^i|$, so it follows from Theorem 3.2 and the assumption of the induction that

$$|u_h(t_{i+1}\bar{f}) - u_h^i|_{1,2} \leq \beta t|\bar{f}|_{1,2} + 2C^2t^2h^{-3/2}.$$

Because t is taken to be not greater than $\frac{h^{3/2}}{2C^2}\min\{\eta, 1\}$, we get

$$|u_h^{i+1} - u_h(t_{i+1}\bar{f})|_{1,2} \leq 2C^2h^{-3/2}t^2.$$

This completes the induction. QED.

Corollary. Define $u_h^t = u_h^n$. Then , if $t \leq \frac{h^3}{2C}$, we have $u_h^t \to u_h(\bar{f})$ as $t \to 0$, and

$$|u_h^t - u_h(\bar{f})|_{1,\infty} \leq 2C^2t^2h^{-3} \leq Ct, \tag{4.21}$$

$$|u_h^t - u_h(\bar{f})|_{1,2} \leq 2C^2h^{-3/2}t^2 \leq Ct. \tag{4.22}$$

Theorem 4.4. *Suppose $f \in W^{1,p}(\Omega)$, $\|f\|_{1,p} \leq \frac{\eta}{2M\gamma}$, $h \leq \bar{h}$, $t \leq \frac{h^3}{2C}$. Then u_h^t is an approximate finite element solution of $Au = f$. And we have error estimates*

$$|u_h^t - u(\bar{f})|_{1,\infty} \leq C\|f\|_{1,p}(t + h^{2/1}), \tag{4.23}$$

$$|u_h^t - u(\bar{f})|_{1,2} \leq C\|f\|_{1,p}(t + h^2). \tag{4.24}$$

Proof. The conclusion comes from Theorem 4.3 and its corollary, and (3.13), (3.14). QED.

Remark 4.2. From (4.21) and (4.22) we may find that the algorithm defined by (4.16) is virtually a second order method. We would be able to show that u_h^t converges to $u_h(\bar{f})$ with rate Ct^2, where C is independent of h, if we could prove that the norm $\|\phi_h'(u)^{-1}\|$ is independent of h.

§5. A More general form of σ

All results in this paper are proved for the case where

$$\sigma_{ij} = \lambda E_{kk}(u)\delta_{ij} + 2\mu E_{ij}(u), \quad \lambda > 0, \mu > 0.$$

But in fact, they also hold for the case when $\sigma^* \in C^{m+2}(\mathcal{M}^3, \mathcal{M}^3)$, $\mathcal{M}^3$ is the space of third-order matrices, and

$$\sigma^*(E) = \sigma(E) + O(E).$$

We have $E_{ij}(u) \in \mathcal{W}^{m+1,p}(\Omega)$ provided that $u \in \mathcal{W}^{m+2,p}(\Omega)$. Hence it can be showed that $\sigma^*_{ij}(E), D\sigma^*_{ij}(E) \in W^{m+1,p}(\Omega)$ (cf. Theorem 2.3–2 of [3]). Thus $A_1 = \text{div}\,((I + \nabla u)\sigma^*(E)) \in C^1(\mathcal{W}^{m+2,p}(\Omega), \mathcal{W}^{m,p}(\Omega))$. It is easy to verify that $A_1'(0) = A'(0)$. These ensure that $A_1'(u)$ and A_1 are invertible when $|u|_{1,\infty}$ is sufficiently small (cf. Theorem 2.3–3 of [3]).

We can show that $E(u)$ is small provided that u is small, and thus that $O(E)$ and its first derivatives will also be small. Hence there exists a constant η^*, such that $\phi, \phi_h, \phi_h'(u)$ will keep their properties when σ is substituted by σ^* , provided that $|u|_{1,\infty} < \eta^*$. Thus all proofs in Sections 3 and 4 still work for the σ^* case, *i.e.* all results in this paper hold if A is substituted by A_1.

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