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APPROXIMATE SEVERAL ZEROES OF A CLASS OF PERIODICAL COMPLEX FUNCTIONS*

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Abstract

This paper discussed the number of zeroes of the complex function $F : C \to C$ defined by

$$F(Z) = \sum_{k=1}^{n} (a_k \cos(kZ) + b_k \sin(kZ)) + \alpha_0 + \alpha_1 \mathrm{Im}(Z) + \cdots + \alpha_m (\mathrm{Im}(Z))^m,$$

where Im(Z) is the imaginary part of $Z, |a_n| + |b_n| \neq 0$. Let $n_1 = \max_{1 \le k \le n} \{0, k | b_k \neq -ia_k\}$ and $n_2 = \max_{1 \le k \le n} \{0, k | b_k \neq ia_k\}$. We prove that if 0 is a regular value of F and $n_1 n_2 \neq 0$, then F has at least $n_1 + n_2$ zeroes in domain $(0, 2\pi] \times R$ and $n_1 + n_2$ of them can be located with the homotopy method simultaneously. Furtheromore, if $\alpha_1 = \cdots = \alpha_m = 0$ and $n_1 n_2 \neq 0$, then F has exactly $n_1 + n_2$ zeroes in domain $(0, 2\pi] \times R$.

§1. Introduction

Let C be the complex plane. We regard C as R^2 by identifying $Z = x + iy \in C, x, y \in R$ with $(x, y) \in R^2$. Define a complex function $F : C \to C$ by

$$F(Z) = T(Z) + f(Z), \tag{1.1}$$

where T is a triangular polynomial with degree n and f is a polynomial of Im(Z) with degree m. That is

$$T(Z) = \sum_{k=1}^{n} (a_k \cos(kZ) + b_k \sin(kZ)),$$

$$f(Z) = \alpha_0 + \alpha_1 \mathrm{Im}(Z) + \cdots + \alpha_m (\mathrm{Im}(Z))^m,$$

where a_k, b_k, α_j are all complex numbers and $\alpha_m \neq 0, |a_n| + |b_n| \neq 0$.

By the definition of F, F is a periodical function of Z with period 2π. So we need only to discuss the zero distribution of F in domain $(0, 2\pi] \times R$. Section 2 studies the number of zeroes of F and develops a method to calculate several zeroes of F. Section 3 gives some numerical examples.

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§2. Approximate the Zeroes

Let $\phi : R^p \to Q^q$ be a smooth mapping. Let $x \in R^p$ be a regular point if the Jacobian matrix of ϕ at x is of full rank. We call $y \in R^q$ a regular value of ϕ if $\phi^{-1}(y) + \{x \in R^p | \phi(x) = y\}$ contains only regular points of ϕ.

Lemma 1[1]. *Let $\phi : R^p \times R^q \to R^r$ be a smooth mapping. If 0 is a regular value of ϕ, then for almost all $d \in R^q$, 0 is a regular value of the mapping $\phi(\cdot, d) : R^p \to R^r$.*

Consider the function F of form (1.1). Since $\frac{\partial F}{\partial \alpha_0} = 1$, by Lemma 1, for almost all $\alpha_0 \in C$, 0 is a regular value of F. In this section, we always assume that 0 is a regular value of F.

Lemma 2[2]. *Let $H : R^n \times [0,1] \to R^n$ be a smooth mapping. Suppose 0 is a regular value of $H, H(\cdot, 0) : R^n \to R^n$ and $H(\cdot, 1) : R^n \to R^n$. Let (x^1, t^1) and (x^2, t^2) be two boundary points of a component of $H^{-1}(0)$.*

(a) If $t^1 = t^2$, then

$$\text{sgn det } \frac{\partial H}{\partial x}(x^1, t^1) = -\text{sgn det } \frac{\partial H}{\partial x}(x^2, t^2).$$

(b) If $t^1 \neq t^2$, then

$$\text{sgn det } \frac{\partial H}{\partial x}(x^1, t^1) = \text{sgn det } \frac{\partial H}{\partial x}(x^2, t^2),$$

where sgn is the sign funciton.

Let $F = T + f : C \to C$ be as in (1.1). T is a triangular polynomial with degree n. Define the auxiliary function $G : C \to C$ by

$$G(Z) = c(e^{in_1 Z} + e^{-in_2 Z}), \tag{2.1}$$

where c is a nonzero complex number. It is easy to know that G has exactly $n_1 + n_2$ zeroes in domain $(0, 2\pi] \times R$; they are

$$Z = \frac{2k+1}{n_1 + n_2}\pi, \quad k = 0, 1, \cdots, n_1 + n_2 - 1,$$

and 0 is a regular value of G.

Define homotopy $E : C \times [0,1] \times C \to C$ by

$$E(Z, t, \alpha) = tF(Z) + (1-t)G(Z) + t(1-t)\alpha. \tag{2.2}$$

Then, $E(\cdot, 0, \cdot) = G(\cdot)$ and $E(\cdot, 1, \cdot) = F(\cdot)$. Since 0 is a regular value of F and G, and

$$\frac{\partial E}{\partial \alpha} = t(1-t),$$

by Lemma 1, for almost all $\alpha \in C$, 0 is a regular value of $H(\cdot, \cdot) = E(\cdot, \cdot, \alpha) : C \times [0,1] \to C$. Fix $\alpha \in C$ such that 0 is a regular value of H. $H^{-1}(0) = \{(Z, t) \in C \times [0,1] | H(Z, t) = 0\}$ is a one-dimensional manifold. That is, $H^{-1}(0)$ consists only of simple smooth curves.

Let G_1, G_2 be respectively the real part and the imaginary part of G. Since G is an analytic function, G satisfies the Gauchy-Riemann equations

$$\frac{\partial G_1}{\partial x} = \frac{\partial G_2}{\partial y}, \quad \frac{\partial G_1}{\partial y} = -\frac{\partial G_2}{\partial x}. \tag{2.3}$$

Hence, the Jacobian determinant of the real mapping $(G_1, G_2) : R^2 \to R^2$ is positive at its zeroes. By Lemma 2, $H^{-1}(0)$ contains no curves with both boundary points in $C \times \{0\}$. We have

Lemma 3. *Let F, H be as above. Then for almost all $\alpha \in C, H^{-1}(0)$ is a one-dimensional manifold and any curve in $H^{-1}(0)$ starting at $C \times \{0\}$ must either intersect $C \times \{1\}$ at a zero of F or go to infinity.*

Now we prove the boundedness of the curves of $H^{-1}(0)$.

Lemma 4. *Let F be as in (1.1) satisfying $|a_n| + |b_n| \neq 0$, $n_1 = \max_{1\le k\le n}\{0, k|b_k \neq -ia_k\}$, $n_2 = \max_{1\le k\le n}\{0, k|b_k \neq ia_k\}$. Let G be as in (2.1) and H be as above. If $n_1 n_2 \neq 0$, then for almost all $c, \alpha \in C$, any curve in $H^{-1}(0)$ is bounded.*

Proof. First, we prove that $H^{-1}(0)$ is bounded in direction y. Notice that

$$\cos(kZ) = \frac{1}{2}(e^{ikZ} + e^{-ikZ}), \quad \sin(kZ) = \frac{1}{2i}(e^{ikZ} - e^{-ikZ}).$$

We have

$$\begin{aligned} H(Z,t) &= (1-t)c(e^{in_1 Z} + e^{-in_2 Z}) + t(a_n \cos(nZ) + b_n \sin(nZ)) + \cdots \\ &= ((1-t)c + \frac{1}{2}t(a_{n_1} - ib_{n_1}))e^{in_1 Z} + ((1-t)c + \frac{1}{2}t(a_{n_2} + ib_{n_2}))e^{-in_2 Z} + \cdots. \end{aligned}$$

Since for all $t \in [0,1]$ and for almost all $c \in C$,

$$(1-t)c + \frac{1}{2}t(a_{n_1} - ib_{n_1}) \neq 0, \quad (1-t)c + \frac{1}{2}t(a_{n_2} + ib_{n_2}) \neq 0,$$

that is, for almost all $c \in C$, the coefficients of the terms $e^{in_1 Z}$ and $e^{-in_2 Z}$ in H are nonzero, if $\{(Z(k), t(k))\}_{k=1}^{\infty} \subset H^{-1}(0)$ and $y(k) \to \infty$, $t(k) \to t_0 \in [0,1]$ as $k \to \infty$, without loss of generality, we assume $y(k) \to +\infty$ as $k \to \infty$, then

$$\lim_{k\to\infty} \frac{H(Z(k), t(k))}{e^{-in_2 Z(k)}} = (1-t_0)c + \frac{1}{2}t_0(a_{n_2} + ib_{n_2}) = 0.$$

This is a contradiction. Hence, for almost all $c \in C, H^{-1}(0)$ is bounded in direction y.

Now, we prove that every curve in $H^{-1}(0)$ is bounded in direction x. Suppose in contrary that some component of $H^{-1}(0)$ is not bounded in direction x. By the periodicity of H and the boundedness of $H^{-1}(0)$ in direction y, there exists a positive number M such that $[0, 2\pi] \times [-M, M] \times [0,1]$ contains an infinite number of curves of $H^{-1}(0)$. This contradicts that 0 is a regular value of H.

Now, we are ready to prove our main result.

Theorem 5. *Let $F = T + f : C \to C$ be as in (1.1), T be a triangular polynomial satisfying $|a_n| + |b_n| \neq 0$ with degree n, $n_1 = \max_{1\le k\le n}\{0, k|b_k \neq -ia_k\}$, $n_2 = \max_{1\le k\le n}\{0, k|b_k \neq$*

$ia_k\}$. If 0 is a regular value of F and $n_1 n_2 \neq 0$, then F has at least n_1+n_2 zeroes in domain $(0,2\pi] \times R$ and n_1+n_2 of them can be located with the homotopy method.

Proof. Let G be as in (2.1) and H be as above. By Lemma 3, for almost all $\alpha \in C$, $H^{-1}(0)$ is a one-dimensional manifold, and $H^{-1}(0)$ contains no curves with both boundary points in $C\times\{0\}$. By Lemma 4, for almost all $c \in C$, any curve in $H^{-1}(0)$ is bounded. Hence, we need only to show that any two curves of $H^{-1}(0)$ starting respectively at $(\eta^1,0),(\eta^2,0) \in (0,2\pi] \times R \times \{0\}$ must intersect $C \times \{1\}$ at different zeroes $(\varsigma^1,1),(\varsigma^2,1)$ of F. That is, $\varsigma^2-\varsigma^1 \neq 2k\pi$ for all integers k. Otherwise, suppose for some integer $k_0, \varsigma^2-\varsigma^1 = 2k_0\pi$; then by the periodicity of H, the curve in $H^{-1}(0)$ starting at $(\varsigma^1+2k_0\pi,0)$ must intersect $C\times\{1\}$ at $(\varsigma^2,1)$ too. So there are two curves in $H^{-1}(0)$ with $(\varsigma^2,1)$ as an end point. This is a contradiction.

Corollary 6. Let $F: C \to C$ be defined by

$$F(Z) = a_0 + \sum_{k=1}^{n}(a_k \cos(kZ) + b_k \sin(kZ))$$

with $|a_n|+|b_n| \neq 0, n_1 = \max_{1\le k\le n}\{0,k|b_k \neq -ia_k\}$, $n_2 = \max_{1\le k\le n}\{0,k|b_k \neq ia_n\}$. If 0 is a regular value of F and $n_1n_2 \neq 0$, then F has exactly n_1+n_2 zeroes in $(0,2\pi]\times R$.

Proof. Since F is analytic and 0 is a regular value of F, F satisfies the Cauchy-Riemann equations, and the real Jacobian determinant of F is positive at its zeroes. Since $n_1n_2 \neq 0$, by Lemma 2 and the proof of Theorem 5, the corollary is obvious.

§3. Numerical Experiments

A program was written for zeroes of the class of periodical complex functions based on the algorithm of [3]. The following are some examples calculated with homotopy (2.2).

Example 1. $F: C \to C$ is defined by

$$F(Z) = 2\sin(4Z) + \cos(3Z) + 2(\mathrm{Im}(Z))^{100} + i(\mathrm{Im}(Z))^2 + 8i.$$

The eight resulting zeroes of F are

$$\begin{array}{ll}(0.737398267, 0.482591212), & (1.57079601. -0.466710865),\\ (2.40419388,\ 0.482591212), & (3.21657562, -0.527321279),\\ (3.98593998,\ 0.583417416), & (4.71239090, -0.610647082),\\ (5.43883705,\ 0.583416760), & (6.20820236, -0.527321696).\end{array}$$

Example 2. Let $F: C \to C$ be

$$F(Z) = \sin(6Z) + \cos(4Z) + \cos(2Z) + \sin(Z) + (\mathrm{Im}(Z))^8 \\ +(\mathrm{Im}(Z))^4 + i((\mathrm{Im}(Z))^5 + (\mathrm{Im}(Z))^2) + 20.1 + 9i$$

The twelve resulting zeroes of F are

$$(0.716865185, 0.592638135),\quad (0.868974626, -0.596008837),$$
$$(1.79889965, 0.647369623),\quad (1.93854427), -0.656512976),$$
$$(2.77105713, 0.676286399),\quad (2.90113831, -0.662934899),$$
$$(3.86054516, 0.577746332),\quad (4.02312183, -0.585952878),$$
$$(4.93177986, 0.631447732),\quad (5.08513451), -0.639616251),$$
$$(5.90183067, 0.675030220),\quad (6.03476429, -0.656130612).$$

Example 3. $F: C \to C$ is defined by

$$F(Z) = \sin(3Z) + i\cos(3Z) + \cos(2Z) + \sin(Z) + (\mathrm{Im(Z)})^8 + (\mathrm{Im(Z)})^4 + \mathrm{i}((\mathrm{Im(Z)})^5 + (\mathrm{Im(Z)})^2) + 0.1 + 0.3\mathrm{i}.$$

The five resulting zeroes of F are

$$(1.20212746, -0.182225823),\quad (1.95877171, -0.556550562),$$
$$(3.44832802, -0.803195238E-01),\quad (4.78542042, 0.244741678),$$
$$(5.88517857, -0.223386347).$$

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