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NON-CONFORMING DOMAIN DECOMPOSITION WITH HYBRID METHOD*[1)]

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Abstract

We present a non-conforming domain decomposition technique for solving elliptic problems with the finite element method. Functions in the finite element space associated with this method may be discontinuous on the boundary of subdomains. The sizes of the finite meshes, the kinds of elements and the kinds of interpolation functions may be different in different subdomains. So, this method is more convenient and more efficient than the conforming domain decomposition method. We prove that the solution obtained by this method has the same convergence rate as by the conforming method, and both the condition number and the order of the capacitance matrix are much lower than those in the conforming case.

§1. Introduction

Along with the development of the parallel computer in recent years, there has been a growing interest in methods based on domain decomposition for the numerical solution of elliptic partial differential equations. The key idea of this method is that the domain of the problem is decomposed into smaller subdomains, and then a computer is used to solve the problem on each subdomain. This is an efficient method for solving the big problem of elliptic partial differential equations on the parallel computer.

Up to now, there are only conforming finite elements with domain decomposition methods, with which the function of the finite element space must be compatible on the whole domain of the problem. However, it will be more convenient and more efficient to adopt different sizes of meshes and different kinds of shape functions in different subdomains when solving practical problems in science and engineering. But this is impossible for conforming finite elements.

The aim of this paper is to put forward a non-conforming domain decomposition for elliptic problems. This method needs no compatibility on the boundary of subdomains, that is to say, the function of the finite element space may be discontinuous on the boundary of subdomains. With this property we can use different sizes of meshes, different kinds of elements in different subdomains. We will prove that the convergence rate of the solution obtained by this method is the same as by the conforming method; moreover, the condition number and the order of the capacitance matrix are much lower than in the conforming case. In this paper, we only consider the method itself as well as the error and the condition number estimates. Solution by this method of the algebraic system of equations, which arises from the discretization of elliptic equations, will be discussed in another paper.

In Section2, we will introduce the decomposition of the domain and the construction of the finite element space. Section 3 contains the non-conforming method and the matrix

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representation. The error estimate of the energy norm will be obtained in Section 4. Finally, in Section 5 the condition number of the capacitance matrix will be given.

§2. The Decomposition of the Domain and Finite Element Space

For simplicity, we only consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We suppose that the domain Ω is a polygon.

We first decompose the domain Ω into subdomains Ω_i; then we subdivide the subdomain Ω_i and its boundary into finite elements.

More precisely, we shall begin with the following assumption with regard to Ω.

A1: Ω is a polygonal domain.

A2: For each $d, d > 0$, as a parameter, the domain Ω is decomposed into quasi-uniform subdomains $\Omega_i (i = 1, 2, \cdots, n)$ with size d. By this we mean that there exists a positive constant c independent of d such that each subdomain Ω_i contains a ball of diameter cd and is contained in ball of diameter d.

A3: For each parameter h, $0 < h < d$, the subdomain Ω_i is subdivided into quasi-uniform finite elements with size h. The meaning of this assumption is as above.

Let $\Omega_i^{h_i}$ be the union set of all elements in Ω_i, and $\Omega^h = \bigcup_i \Omega_i^{h_i}$.

A4: Let Γ be the union set of all boundaries of the subdomains, that is $\Gamma = \bigcup_i \partial\Omega_i$. For each $H, 0 < h < H \le d, \Gamma$ is subdivided into quasi-uniform line segments with size H. Its meaning is similar to A2. The vertices of Ω_i must be the vertices of elements. Let Γ^H be the union set of all line segments in Γ.

We always suppose that $0 < h < H \le d$ and assume the asymptotic behavior

$$\lim_{h \to 0} \frac{h}{H} = 0 \tag{2.2}$$

where $h = \max_i (h_i)$.

Completing the decomposition of the domain, we now construct the space of the finite elements. We make the following supposition:

Let $S_{h_i}(\Omega_i)$ be the space of piecewise m-th polynomial functions which are continuously defined in subdomain $\Omega_i^{h_i}$ and vanish on $\partial\Omega \cap \partial\Omega_i$.

Let $S_h^0(\Omega)$ be the space of functions defined in $\Omega^h = \bigcup_i \Omega_i^{h_i}$, which are continuous and piecewise m-th polynomials in $\Omega_i^{h_i}$ and vanish on $\partial\Omega$. We emphasize that the functions in $S_h^0(\Omega)$ are only continuous in Ω_i but may be discontinuous on Ω.

Let $S_H(\Gamma)$ be the space of piecewise n-th polynomial functions continuously defined on Γ^H and vanishing on $\partial\Omega$. $S_H(\partial\Omega_i)$ is the space of piecewise n-th polynomial functions continuously defined on $\Gamma^H \cap \partial\Omega_i$ and vanishing on $\partial\Omega \cap \partial\Omega_i$.

We define the finite element space $S_{h\times H}$ as follows:

$S_{h\times H} \subset S_h^0(\Omega) \times S_H(\Gamma)$, $(u, \varphi) \in S_{h\times H}$ is and only if $(u, \varphi) \in S_h^0(\Omega) \times S_H(\Gamma)$ and $u = \varphi$ on the nodes emerging during subdividing the subdomain Ω_i into elements.

Space $S_{h\times H}$ is a subspace of $S_h^0(\Omega) \times S_h(\Gamma)$. It can be easily seen that $S_h(\Gamma) = (\varphi | (u, \varphi) \in S_{h\times H})$.

§3. The Method and Matrix Representation

We introduce the functional

$$J(u)=\sum_i\left\{\frac{1}{2}(\nabla u,\nabla u)_{\Omega_i}-(f,u)_{\Omega_i}\right\}, \tag{3.1}$$

which is defined on $H(\Omega)=H^1(\Omega_1)\oplus H^1(\Omega_2)\oplus\cdots\oplus H^1(\Omega_{n_d})$, where the inner product

$$(u,v)_{\Omega_i}=\int_{\Omega_i}uv\,dx.$$

The finite element problem associated to problem (2.1) is: Find $u^h\in S$ such that

$$J(u^h)=\min_{v^h\in S}J(v^h),$$

where $S=(u|\exists\varphi\in S_H(\Gamma),(u,\varphi)\in S_{h\times H})$. This problem is equivalent to: Find $(u^h,\varphi^H)\in S_{h\times H}$ such that

$$\sum_i((\nabla u^h,\nabla v)_{\Omega_i}-(f,v)_{\Omega_i}\}=0,\quad\forall(v,\psi)\in S_{h\times H}. \tag{3.2}$$

We have two ways to carry out the non-conforming domain decomposition method:

(1) Compute the stiffness matrix on each subdomain, get the total stiffness matrix by assembling, and solve the problem by using CGM on the whole domain Ω. It can be parallelly computed; see [1].

(2) Compute the capacitance matrix on subdomains, which is the boundary stiffness matrix of subdomains, by using the direct method, and minimize the functional (3.1) for inner variables of each subdomain. Use the iterative methods to seek the unknowns on the boundary of subdomain; then compute the inner unknowns of each subdomain by the boundary variables.

The second way of solving the problem is similar to the substructure method. The main difference lies in the capacitance metrix. Now, we discuss computation the stiffness matrix. We consider the basic functions of the following form: (u_{ij}) is the set of basic functions of space $S_{h_i}(\Omega_i)$, and (φ_{ij}) is the set of basic functions of space $S_{h_i}(\Omega_i)$. Using those basic functions, we can obtain the matrix representation of functional $J_i(u)$, where $J_i(u)=\frac{1}{2}(\nabla u,\nabla u)_{\Omega_i}-(f,u)_{\Omega_i}$, that is

$$J_i(u)=\frac{1}{2}(U_i^T,X_i^T)\begin{pmatrix}A_i & B_i\\ B_i^T & C_i\end{pmatrix}\begin{pmatrix}U_i\\ X_i\end{pmatrix}-(U_i^T,X_i^T)\begin{pmatrix}f_i\\ g_i\end{pmatrix} \tag{3.3}$$

where vector variable U_i (the values of u at the nodes of Ω_i) and X_i (the values of φ at the nodes of $\partial\Omega_i$) correspond to inner variables of Ω_i and boundary variables respectively. As on the nodes of $\partial\Omega_I$, we have $u=\varphi$, where $\varphi\in S_H(\Gamma)$ with $(u,\varphi)\in S_{h\times H},\varphi|_{\partial\Omega_i}$ must be the linear combination of basis $(\varphi_{ij}),\varphi=(\varphi_{i1},\varphi_{i2}\cdots\varphi_{ii_i})Y_i$, where $Y_i=(Y_{i1},y_{i2}\cdots y_{ii_i})^T$, so we have $X_i=L_iY_i$, where L_i is a rectangular matrix. Representing X_i in (3.3) by $X_i=L_iY_i$, we have

$$J_i(u)=\frac{1}{2}(U_i^T,Y_i^T)\begin{pmatrix}A_i & B_iL_i\\ (B_iL_i)^T & L_i^TC_iL_i\end{pmatrix}\begin{pmatrix}U_i\\ Y_i\end{pmatrix}-(U_i^T,Y_i^T)\begin{pmatrix}f_i\\ L_i^Tg_i\end{pmatrix}, \tag{3.4}$$

where the matrix

$$\begin{pmatrix} A_i & B_i L_i \\ (B_i L_i)_i^T & L_i^T C_i L_i \end{pmatrix}$$

is the subdomain stiffness matrix of algebraic system (3.2); the total stiffness matrix is obtained by assembling.

Minimizing (3.4) about variables U_i, we have

$$A_i U_i + B_i L_i Y_i - f_i = 0, \quad U_i = A_i^{-1}(-B_i L_i Y_i + f_i), \tag{3.5}$$

Substituting (3.5) into (3.4), we get the energy representation on the boundary as

$$\frac{1}{2} Y_i^T L_i^T (C_i - B_i^T A_i B_i) L_i Y_i - Y_i^T L_i^T (g_i - B_i^T A_i^T f_i). \tag{3.6}$$

$L_i^T(C_i - B_i^T A_i B_i) L_i$ is the boundary stiffness matrix. We obtain the capacitance matrix by assembling.

Algorithm.

(1) According to L_i, A_i, B_i, C_i, and vectors g_i, f_i, compute matrix $L_i^T(C_i - B_i^T A_i B_i)L_i$ and vector $L_i^T(g_i - B_i^T A_i^T f_i)$.

(2) Using the iterative method, solve the equation $CY = g$, where Y obtained by assembling Y_i is the unknown on Γ, and C is the capacitance matrix.

(3) Compute the inner unknown U_i of Ω_i by equations (3.5).

The algorithm can be done in parallel.

§4 The Convergence Theorem

The aim of this section is to give the error estimate of the energy norm.

Let $\hat{\pi}$ be the projection operator of $H_0^1(\Omega)$ onto $S_h = (u|(u,\varphi) \in S_{h\times H})$, π be the projection operator of $H_0^1(\Omega)$ onto $S_h^0(\Omega)$ and $\bar{\pi}$ be the projection operator of $\bigcup_i H^{1/2}(\partial\Omega_i)$ onto $S_H(\Gamma)$. With d as in A. 2 (roughly the diameter of Ω_i), we define the weighted norm on $H^{1/2}(\partial\Omega_i)$ by

$$|w|_{1/2,\partial\Omega_i} = \left(\int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{(w(x)-w(y))^2}{|x-y|^2} ds(x)ds(y) + d^{-1}|w|_{0,\partial\Omega_i}^2\right)^{1/2}, \tag{4.1}$$

where s is the curvilinear length along $\partial\Omega_i$.

For simplicity, we make the following assumption in this section and Section 5:

$$m_i = m < n, \quad h_i = h, \quad i = 1, 2, \cdots, n_d.$$

Lemma 1. *There are two positive constants c_1 and c_2, independent of h, H, and d, such that for any $(v,\psi) \in S_{h\times H}$,*

$$c_1|\psi|_{0,\partial\Omega_i} \le |v|_{0,\partial\Omega_i} \le c_2|\psi|_{0,\partial\Omega_i} \tag{4.2}$$

and

$$c_1|\psi|_{1/2,\partial\Omega_i} \le |v|_{1/2,\partial\Omega_i} \le C_2|\psi|_{1/2,\partial\Omega_i}. \tag{4.3}$$

Proof. Noting that $v|_{\partial\Omega_i}$ is the m-th interpolating polynomial function and using the inverse inequality, we have

$$|\psi|_{1/2,\partial\Omega_i} \le |v|_{1/2,\partial\Omega_i} + |\psi - v|_{1/2,\partial\Omega_i} \le |v|_{1/2,\partial\Omega_i} + ch^{m+1/2}|\psi|_{m+1,\partial\Omega_i}$$
$$\le |v|_{1/2,\partial\Omega_i} + c(h/H)^{m+1/2}|\psi|_{1/2,\partial\Omega_i}.$$

By assumption (2.2) we can let $c(h/H)^{m+1/2} < \frac{1}{2}$, so

$$|\psi|_{1/2,\partial\Omega_i} < 2|v|_{1/2,\partial\Omega_i};$$

on the other hand,

$$|v|_{1/2,\partial\Omega_i} \le |\psi|_{1/2,\partial\Omega_i} + |v - \psi|_{1/2,\partial\Omega_i} < (1 + c(h/H)^{m+1/2})|\psi|_{1/2,\partial\Omega_i} < c|\psi|_{1/2,\partial\Omega_i}.$$

This completes the proof of (4.3). (4.2) can be proved in the same way.

Lemma 2. *If $v \in S_h^0(\Omega)$ and vanishes at all interior nodes of Ω_k, then*

$$c_1 h^{-1}|v|_{0,\partial\Omega_k}^2 \le |v|_{1,\partial\Omega_k}^2 \le c_2 h^{-1}|v|_{0,\partial\Omega_k}, \tag{4.4}$$

where c_1 and c_2 are constants independent of h, H and d, and

$$|v|_{1,\partial\Omega_k}^2 = (\nabla v, \nabla v)_{\partial\Omega_k} = \int_{\partial\Omega_k} \nabla v \cdot \nabla v ds.$$

As for Lemma 2, see [2].

Theorem 1. *There is a constant c independent of h, H and d, such that*

$$\|u - u^h\|_h \le c\Big\{h^m\sqrt{\sum_i |u|_{m+1,\partial\Omega_i}^2} + h^{1/2}H^{n+1}\sqrt{\sum_i |u|_{n+1,\partial\Omega_i}^2}$$
$$+ch^m\sqrt{h\sum_i |u|_{m+1,\partial\Omega_i}^2} + (h/H)^{m+1}(H/D)^{1/2}\|u\|_{2,\Omega}\Big\},$$

where u, u^h are respectively the solutions of problems (2.1) and (3.2); energy norm $\|v\|_h$ is defined by

$$\|v\|_h^2 = \sum_i (\nabla v, \nabla v)_{\Omega_i}.$$

Proof. For each $(v^h, \psi^H) \in S_{h\times H}$, we have

$$\sum_i \{(\nabla u, \nabla v^h)_{\Omega_i} - (f, v^h)_{\Omega_i}\} = \sum_i \int_{\partial\Omega_i} (\partial u/\partial n) v^h ds$$
$$= \sum_i \int_{\partial\Omega_i} (\partial u/\partial n)(v^h - \psi^H) ds \le \sum_i |\partial u/\partial n|_{0,\partial\Omega_i} \cdot |v^h - \psi^H|_{0,\partial\Omega_i}. \tag{4.5}$$

Let $\bar{v}_i$ denote the average value of v^h on Ω_i. Using the Poincare inequality and the inverse inequality, we have

$$|v^h - \psi^H|_{0,\partial\Omega_i} = |(v^h - \bar{v}_i) - (\psi^H - \bar{v}_i)|_{0,\partial\Omega_i} < ch^{m+1}|\psi^H - \bar{v}_i|_{m+1\partial\Omega_i}$$
$$< c(h/H)^{m+1}H^{1/2}|\psi^H - \bar{v}_i|_{1/2,\partial\Omega_i} < c(h/H)^{m+1}H^{1/2}|v^h - \bar{v}_i|_{1/2,\partial\Omega_i}$$
$$< c(h/H)^{m+1}H^{1/2}|v^h|_{1\Omega_i}. \tag{4.6}$$

Combining (4.5) and using the Schwarz inequality, we have

$$\sum_i (\nabla(u^h - u), \nabla v^h)_{\Omega_i} \leq c(h/H)^{m+1} H^{1/2} \sqrt{\sum_i |\partial u/\partial n|^2_{0,\partial\Omega_i}} \|v\|_h.$$

Noting that $u^h - \hat{\pi} u = u - \hat{\pi} u + u^h - u$, we get

$$\sum_i (\nabla(u^h - \hat{\pi} u), \nabla v^h)_{\Omega_i} \leq \sum_i (\nabla(u - \hat{\pi} u), \nabla v^h)_{\Omega_i}$$
$$+ c(h/H)^{m+1} H^{1/2} \|v^h\|_h \sqrt{\textstyle\sum_i |\partial u/\partial n|^2_{0,\partial\Omega_i}}.$$

Letting $v^h = u^h - \hat{\pi} u$ gives

$$\|u^h - \hat{\pi} u\|_h \leq \|u - \hat{\pi} u\|_h + c(h/H)^{m+1} H^{1/2} \sqrt{\sum_i |\partial u/\partial n|^2_{0,\partial\Omega_i}}, \tag{4.7}$$

$$\|u - \hat{\pi} u\|_h \leq \|u - \pi u\|_h + \|\pi u - \hat{\pi} u\|_h < ch^m \sqrt{\sum_i |u|^2_{m+1,\Omega_i}} + \|\pi u - \hat{\pi} u\|_h. \tag{4.8}$$

Applying Lemma 2 to function $\pi u - \hat{\pi} u$, we get

$$\begin{aligned}
\|\pi u - \hat{\pi} u\|_h^2 &\leq ch^{-1} \sum_i \int_{\partial\Omega_i} |\pi u - \hat{\pi} u|^2 ds \\
&< ch^{-1} \sum_i \int_{\partial\Omega_i} (|\hat{\pi} u - \bar{\pi} u|^2 + |\bar{\pi} u - u|^2 + |u - \pi u|^2) ds \\
&< ch^{-1} \sum_i \int_{\partial\Omega_i} |u - \bar{\pi} u|^2 ds + ch^{2m+1} \sum_i |u|^2_{m+1,\partial\Omega_i} + ch^{-1} \sum_i \int_{\partial\Omega_i} |\hat{\pi} u - \bar{\pi} u|^2 ds \\
&< ch^{-1} H^{2(n+1)} \sum_i |u|^2_{n+1,\partial\Omega_i} + ch^{2m+1} \sum_i |u|^2_{m+1,\partial\Omega_i} + ch^{-1} \sum_i \int_{\partial\Omega_i} |\hat{\pi} u - \bar{\pi} u|^2 ds.
\end{aligned} \tag{4.9}$$

Since

$$\begin{aligned}
h^{-1} |\hat{\pi} u - \pi u|^2_{0,\partial\Omega_i} &\leq ch^{2m+1} |\bar{\pi} u|^2_{m+1,\partial\Omega_i} \\
&< ch^{2m+1} (|\bar{\pi} u - u|^2_{m+1,\partial\Omega_i} + |u|^2_{m+1,\partial\Omega_i}) < ch^{2m+1} |u|^2_{m+1,\partial\Omega_i},
\end{aligned} \tag{4.10}$$

combining (4.7), (4.8), (4.9) and (4.10), we get

$$\|u^h - u\|_h \leq ch^m \sqrt{\sum_i |u|^2_{m+1,\Omega_i}} + c(h/H)^{m+1} H^{1/2} \sqrt{\sum_i |\partial u/\partial n|^2_{0,\partial\Omega_i}}$$
$$+ ch^{-1/2} H^{n+1} \sqrt{\sum_i |u|^2_{n+1,\Omega_i}} + ch^m \sqrt{h \sum_i |u|^2_{m+1,\partial\Omega_i}}.$$

Since

$$|\partial u/\partial n|^2_{0,\partial\Omega_i} \leq c\{d^{-1} |u|^2_{1,\Omega_i} + d|u|^2_{2,\Omega_i}\} < cd^{-1} \|u\|^2_{2,\Omega_i},$$

the proof of Theorem 1 is completed.

It is evident that the convergence rate of the non-conforming method is the same as that of the conforming method under the condition of Theorem 2, by the following Theorem 1.

Theorem 2. *If* $(m+1)(m+1/2) \le n+1/2$, *and* $c_1 h^{\frac{1}{m+1}} \le H \le c_2 h^{(2m+1)/(2n+1)}$, *where* c_1 *and* c_2 *are positive constants, then we have*

$$\|u-u^h\| \le ch^m\{\sqrt{\sum_i |u|^2_{m+1,\Omega_i}} + \sqrt{\sum_i |u|^2_{n+1,\partial\Omega_i}}\sqrt{h\sum_i |u|^2_{m+1,\partial\Omega_i}}\}$$

with constant c *independent of* h, H, d *and the solution* u.

§5. The Condition Number of the Capacitance Matrix

When we carry out the algorithm in Section 3, we solve the equation $CY = g$ with the iterative method. The conjugate gradient iteration is usually used. The convergence rate is dependent on the condition number of matrix C. We estimate the condition number of the capacitance system in this section.

Lemma 3. *There is a constant* c *independent of* h, H *and* d, *Such that*

$$|v^h|_{0,\Omega} \le c\|v^h\|_h \quad \text{for all } (v^h,\psi^H) \in S_{h\times H}. \tag{5.1}$$

Proof. Considering the equation

$$\begin{cases} -\Delta w = v^h & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

we have

$$(v^h,v^h)_\Omega = (-\Delta w, v^h)_\Omega = \sum_i (\nabla w, \nabla v^h)_{\Omega_i} + \sum_i \int_{\partial\Omega_i} (\partial w/\partial n)(v^h - \psi^H)ds. \tag{5.2}$$

Applying (4.6), we get

$$\begin{aligned} \Big|\sum_i \int_{\partial\Omega_i} (\partial w/\partial n)(v^h-\psi^H)ds\Big| &\le \sum_i |(\partial w/\partial n)|_{0,\partial\Omega_i}\cdot|v^h-\psi^H|_{0,\partial\Omega_i} \\ &< c(h/H)^{m+1}H^{1/2}\|v^h\|_h\sqrt{\textstyle\sum_i |(\partial w/\partial n)|^2_{0,\partial\Omega_i}}. \end{aligned} \tag{5.3}$$

By applying the imbedding theorem of Sobolev space and the theory of elliptic equations[3], we get

$$\begin{aligned} \Big|\sum_i\Big| (\partial w/\partial n)|^2_{0,\partial\Omega_i} &\le c\sum\{d^{-1}\|\nabla w\|^2_{0,\Omega_i} + d|\nabla w|^2_{1,\Omega_i}\} \\ &< c\sum_i (d^{-1}\|\nabla w\|^2_{0,\Omega_i} + d|w|^2_{2,\Omega_i}) < c(d+d^{-1})|v^h|_{0,\Omega}. \end{aligned} \tag{5.4}$$

By (5.2), (5.3) and (5.4), we get

$$\begin{aligned} |v^h|^2_{0,\Omega} &\le \sum_i |w|_{1,\Omega_i}\cdot|v^h|_{1,\Omega_i} + c(h/H)^{m+1}|v^h|_{0,\Omega}\|v^h\|_h \\ &\le \sqrt{\sum_i |w|^2_{1,\Omega}}\|v^h\|_h + c(h/H)^{m+1}|v^h|_{0,\Omega}\|v^h\|_h < c\|v^h\|_h\cdot|v^h|_{0,\Omega}. \end{aligned}$$

This completes the proof of Lemma 3.

Theorem 3. *For the algebraic system* (3.2), *the condition number of the capacitance matrix is* $O(d^{-1}H^{-1})$.

Proof. For each $(v, \psi) \in S_{h\times H}$, let $V \in S_h$ be the discrete harmonic function defined by

$$(\nabla V, \nabla \phi)_{\Omega_i} = 0, \quad \text{for all } \phi \in S_h, \quad V = v \text{ on } \partial\Omega_i.$$

It is easily seen that

$$\min_{\substack{w\in S_h \\ w|_{\partial\Omega_i}=v}} (\nabla w, \nabla w)_{\Omega_i} = (\nabla V, \nabla V)_{\Omega_i}. \tag{5.5}$$

Let $U \in H^1(\Omega_1) \oplus H^1(\Omega_2) \oplus \cdots \oplus H^1(\Omega_{nd})$ be the harmonic function defined by

$$\begin{cases} (\nabla U, \nabla \Phi)_{\Omega_i} = 0, & \text{for all } \phi \in H^1(\Omega_i), \\ U = v & \text{on } \partial\Omega_i. \end{cases}$$

By applying the triangle inequality and the theory of elliptic equations [3], we get

$$(\nabla V, \nabla V)_{\Omega_i} \le (\nabla U, \nabla U)_{\Omega_i} + (\nabla(V-U), \nabla(V-U))_{\Omega_i} < c|v|^2_{1/2,\partial\Omega_i} + ch^{2\epsilon}\|U\|_{1+\epsilon,\Omega_i},$$

where $0 < \epsilon < 1/2$. Using Lemma 1 and the inverse inequality, we have

$$(\nabla V, \nabla V)_{\Omega_i} \le c|v|^2_{1/2,\partial\Omega_i} + ch^{2\epsilon}|v|^2_{1/2+\epsilon,\partial\Omega_i} < c|v|^2_{1/2,\partial\Omega_i} < c|\psi|^2_{1/2,\partial\Omega_i} < cH^{-1}|\psi|^2_{0,\partial\Omega_i}. \tag{5.6}$$

On the other hand,

$$\sum_i (\nabla V, \nabla V)_{\Omega} \ge c\sum_i \|V\|^2_{1,\Omega_i} \tag{5.7}$$

By Lemma 3. Using Lemma 1, we have

$$|\psi|^2_{0,\partial\Omega_i} \le c|v|^2_{0,\partial\Omega_i} < \{d|V|^2_{1,\Omega_i} + d^{-1}\|V\|^2_{0,\Omega_i}\} < cd^{-1}\|V\|^2_{1,\Omega_i}. \tag{5.8}$$

By (5.7) and (5.8), we get

$$cd\sum_i |\psi|^2_{0,\partial\Omega_i} \le \sum_i (\nabla V, \nabla V)_{\Omega_i}. \tag{5.9}$$

Then by (5.9) and (5.6), we have

$$cd\sum_i |\psi|^2_{0,\partial\Omega_i} \le \sum_i (\nabla V, \nabla V)_{\Omega_i} \le cH^{-1}\sum_i |\psi|^2_{0,\partial\Omega_i}.$$

This completes the proof of the theorem by noting (5.5).

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