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AN IMBEDDING METHOD FOR COMPUTING THE GENERALIZED INVERSES*

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Abstract

This paper deals with a system of ordinary differential equations with known conditions associated with a given matrix. By using analytical and computational methods, the generalized inverses of the given matrix can be determined. Among these are the weighted Moore-Penrose inverse, the Moore-Penrose inverse, the Drazin inverse and the group inverse. In particular, a new insight is provided into the finite algorithms for computing the generalized inverse and the inverse.

§1. Introduction

In [1, 2], the imbedding method for nonlinear matrix eigenvalue problems and for computational linear algebra are presented.

In many engineering problems we must find the generalized inverses of a given matrix.

Let $A \in C^{m\times n}$. Throughout this paper, let M and N be positive definite matrices of order m and n respectively. Then, there is a unique matrix $X \in C^{n\times m}$ satisfying

$$AXA = A, \quad XAX = X, \quad (MAX)^* = MAX, \quad (NXA)^* = NXA. \tag{1.1}$$

This X is called the weighted Moore-Penrose inverse of A, and is denoted by $X = A^+_{MN}$. In particular, when $M = I_m, N = I_n$, the matrix X that satisfies (1.1) is called the Moore-Penrose inverse of A, and is denoted by $X = A^+$, i.e., $A^+ = A^+_{I_mI_n}$.

Let $A \in C^{n\times n}$. The smallest nonnegative integer k such that

$$\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1}) \tag{1.2}$$

is called the index of A, and is denoted by $\mathrm{Ind}(A)$.

Let $A \in C^{n\times n}$. With $\mathrm{Ind}(A) = k$ and if $X \in C^{n\times n}$ is such that

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA \tag{1.3}$$

then X is called the Drazin inverse of A, and is denoted by $X = A_d$. In particular, when $\mathrm{Ind}(A) = 1$, the matrix X that satisfies (1.3) is called the group inverse of A, and is denoted by $X = A^\#$.

An imbedding method for the Moore-Penrose inverse is given in [3]. In this paper, the imbedding methods for the weighted Moore-Penrose inverse Moore-Penrose inverse, the Drazin inverse and the group inverse are presented, and these methods have a uniform formula.

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First, we show the generalized inverses can be characterized in terms of a limiting process. These expressions involve the inverse of the matrix $B_t(z)$, where $B_t(z)$ is a matrix of z. Secondly, we show how this problem may be reduced to integrating a system of ordinary differential equations subject to initial conditions. In particular, a new insight is provided into a series of finite algorithms for computing the generalized inverses and the inverse in [4–6, 9].

§2. Generalized Inverses as a Limit

In this section, we will show how the generalized inverses A^+, A^+_{MN}, A_d and $A^\#$ can be characterized in terms of a limiting process respectively.

Theorem 2.1. *Let $A \in C^{m\times n}$, $\text{rank}A = r$. Then*

$$A^+_{MN} = \lim_{z\to 0}(N^{-1}A^*MA - zI)^{-1}N^{-1}A^*M \tag{2.1}$$

where z tends to zero through negative values.

Proof. From the (M,N)-singular value decomposition theorem[7], there exists an M-unitary matrix $U \in C^{m\times m}$ and an N^{-1}-unitary matrix $V \in C^{n\times n}$ such that

$$A = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}V^* \tag{2.2}$$

where

$$U^*MU = I_m, \quad V^*N^{-1}V = I_n, \tag{2.3}$$

$$D = \text{diag}(d_1, d_2, \cdots, d_r), \quad d_i > 0, \quad i = 1, 2, \cdots, r \tag{2.4}$$

and

$$A^+_{MN} = N^{-1}V\begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*M. \tag{2.5}$$

Let

$$N^{-1/2}V = \tilde{V} = (v_1, v_2, \cdots, v_n), \tag{2.6}$$

$$M^{1/2}U = \tilde{U} = (u_1, u_2, \cdots, u_m), \tag{2.7}$$

then

$$\tilde{V}^* = \tilde{V}^{-1}, \quad \tilde{U}^* = \tilde{U}^{-1} \tag{2.8}$$

and

$$A^+_{MN} = N^{-1/2}\Big(\sum_{i=1}^{r} d_i^{-1}v_iu_i^*\Big)M^{1/2}. \tag{2.9}$$

Since

$$N^{-1}A^*MA = N^{-1/2}\Big(\sum_{i=1}^{r} d_i^2 v_iv_i^*\Big)N^{1/2} \tag{2.10}$$

and the vectors $v_1, v_2, \cdots, v_n$ form an orthogonal system in C^n,

$$I = \sum_{i=1}^{n} v_i v_i^* = N^{-1/2}\Big(\sum_{i=1}^{n} v_i v_i^*\Big)N^{1/2}, \tag{2.11}$$

therefore

$$N^{-1}A^*MA - zI = N^{-1/2}\Big(\sum_{i=1}^{r}(d_i^2 - z)v_i v_i^* - z\sum_{i=r+1}^{n} v_i v_i^*\Big)N^{1/2}.$$

Let $\tilde{A} = M^{1/2}AN^{-1/2}$; then

$$\tilde{A}^*\tilde{A} = N^{1/2}(N^{-1}A^*MA)N^{-1/2}. \tag{2.12}$$

Since $\tilde{A}^*\tilde{A}$ is symmetric and has nonnegative eigenvalues, $N^{-1}A^*MA$ has nonnegative eigenvalues too. The matrix $N^{-1}A^*MA - zI$ with $z < 0$ is therefore nonsingular. Its inverse is

$$(N^{-1}A^*MA - zI)^{-1} = N^{1/2}\Big(\sum_{i=1}^{r}(d_i^2 - z)^{-1}v_i v_i^* - \sum_{i=r+1}^{n} z^{-1}v_i v_i^*\Big)N^{1/2},$$

as is easily verified. Next, form

$$(N^{-1}A^*MA - zI)^{-1}N^{-1}A^*M = N^{-1/2}\Big(\sum_{i=1}^{r}(d_i/(d_i^2 - z))v_i v_i^*\Big)M^{1/2}.$$

Now, we take the limit and see that

$$\lim_{z\to 0}(N^{-1}A^*MA - zI)^{-1}N^{-1}A^*M = N^{-1/2}\Big(\sum_{i=1}^{r} d_i^{-1}v_i v_i^*\Big)M^{1/2} = A^+_{MN}.$$

Corollary 2.1. Let $A \in C^{m\times n}$, then

$$A^+ = \lim_{z\to 0}(A^*A - zI)^{-1}A^* \tag{2.13}$$

where z tends to zero through negative values.

Theorem 2.2. *Let $A \in C^{n\times n}$ with* Ind$(A) = k$. *Then*

$$A_d = \lim_{z\to 0}(A^{k+1} - zI)^{-1}A^k \tag{2.14}$$

where z tends to zero through negative values.

Proof. From the theorem of the canonical form representation for A and A_d [8], there exists a nonsingular matrix P such that

$$A = P\begin{pmatrix} C & O \\ O & N \end{pmatrix}P^{-1} \tag{2.15}$$

where C is nonsingular and N is nilpotent of index k, i.e.,

$$N^k = 0 \tag{2.16}$$

and

$$A_d = P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix} P^{-1}. \tag{2.17}$$

From (2.15), (2.16),

$$A^{k+1} - zI = P\begin{pmatrix} C^{k+1} - zI & O \\ O & -zI \end{pmatrix} P^{-1}.$$

Since C is nonsingular, C^{k+1} is also nonsingular, and z tends to zero through negative values, $C^{k+1} - zI$ is also nonsingular. Then

$$(A^{k+1} - zI)^{-1}A^k = P\begin{pmatrix} (C^{k+1} - zI)^{-1}C^k & O \\ O & O \end{pmatrix} P^{-1}.$$

Now, we take the limit and see that

$$\lim_{z\to 0}(A^{k+1} - zI)^{-1}A^k = P\begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix} P^{-1} = A_d.$$

Corollary 2.2. Let $A \in C^{n\times n}$ with $\mathrm{Ind}(A) = 1$. Then

$$A^{\#} = \lim_{z\to 0}(A^2 - zI)^{-1}A. \tag{2.18}$$

Let $A \in C^{n\times n}$ be nonsingular, then $\mathrm{Ind}(A) = 0$,

$$A^{-1} = \lim_{z\to 0}(A - zI)^{-1}. \tag{2.19}$$

§3. Imbedding Methods for Computing the Generalized Inverses

In order to find the generalized inverses, from (2.1), (2.13), (2.14) and (2.18), we must find the inverse of the matrix $B_t(z)$, where $B_t(z)$ is an $n \times n$ matrix of z.

$$B_t(z) = (b_{ij}^{(t)}) = \begin{cases} N^{-1}A^*MA - zI, & t = 1, \\ A^*A - zI, & t = 2, \\ A^{k+1} - zI, & t = 3, \\ A^2 - zI, & t = 4, \\ A - zI, & t = 5. \end{cases} \tag{3.1}$$

Let

$$F_t(z) = \mathrm{adj}\ B_t(z) = (B_{ij}^{(t)}), \quad g_t(z) = \det\ B_t(z). \tag{3.2}$$

where adj $B_t(z)$ is the adjoint of the matrix $B_t(z)$ whose elements $B_{ij}^{(t)}$ are the cofactors of the j-th row and i-th column element of $B_t(z)$. Then

$$(B_t(z))^{-1} = F_t(z)/g_t(z). \tag{3.3}$$

Theorem 3.1. *Let $F_t(z)$ and $g_t(z)$ satisfy (3.2). Then $F_t(z)$ and $g_t(z)$ satisfy the following ordinary differential equations:*

$$\begin{cases} \dfrac{dF_t}{dz} = \dfrac{-F_t \operatorname{tr}(F_t) + F_t^2}{g_t}, & (3.4) \\ \dfrac{dg_t}{dz} = -\operatorname{tr}(F_t). & (3.5) \end{cases}$$

Proof. Premultiplying both sides of (3.3) by the matrix B_t and then postmultiplying both sides by det B_t, we get

$$I\det B_t = B_t \operatorname{adj} B_t \tag{3.6}$$

where I is a unit matrix. By postmultiplying both sides of (3.3) by B_t det B_t, we have

$$I\det B_t = (\operatorname{adj} B_t)B_t. \tag{3.7}$$

Differentiate both sides of (3.6) with respect to the parameter z:

$$(B_t)_z \operatorname{adj} B_t + B_t(\operatorname{adj} B_t)_z = I(\det B_t)_z. \tag{3.8}$$

Premultiply both sides of (3.8) by adj B_t:

$$(\operatorname{adj} B_t)(B_t)_z \operatorname{adj} B_t + (\operatorname{adj} B_t)B_t(\operatorname{adj} B_t)_z = (\operatorname{adj} B_t)I(\det B_t)_z. \tag{3.9}$$

By making use of (3.7) in the second term of (3.9), we obtain

$$(\operatorname{adj} B_t)(B_t)_z \operatorname{adj} B_t + \det B_t(\operatorname{adj} B_t)_z = (\operatorname{adj} B_t)(\det B_t)_z. \tag{3.10}$$

Since det B_t is a scalar, form (3.10) we find

$$(\operatorname{adj} B_t)_z = ((\operatorname{adj} B_t)(\det B_t)_z - (\operatorname{adj} B_t)(B_t)_z(\operatorname{adj} B_t))/\det B_t. \tag{3.11}$$

Then differentiating det B_t with respect to z, we obtain

$$(\det B_t)_z = \sum_{i,j=1}^{n} \frac{\partial(\det B_t)}{\partial b_{ij}^{(t)}} \frac{db_{ij}^{(t)}}{dz}. \tag{3.12}$$

However,

$$\frac{\partial(\det B_t)}{\partial b_{ij}^{(t)}} = B_{ij}^{(t)} \tag{3.13}$$

and

$$\frac{dB_t}{dz} = -I. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) gives

$$(\det B_t)_z = \sum_{i=1}^{n} B_{ii}^{(t)} \frac{db_{ii}^{(t)}}{dz} = -\sum_{i=1}^{n} B_{ii}^{(t)} = -\operatorname{tr}(F_t). \tag{3.15}$$

By substituting (3.14) and (3.15) into the right hand side of (3.11), we have

$$(\operatorname{adj} B_t)_z = ((\operatorname{adj} B_t)(-\operatorname{tr}(F_t)) + (\operatorname{adj} B_t)^2)/\det B_t. \tag{3.16}$$

By substituting (3.2) into (3.16) and (3.15), we obtain (3.4) and (3.5) immediately.

For a value of z suitably less than zero, $z = z_0$, we can determine the determinant and the adjoint of the matrix $B_t(z_0)$ accurately by , e.g., Gaussian elimination. This provides initial conditions at $z = z_0$ for the differential equations in (3.4)–(3.5) which can now be integrated numerically with z going from z_0 toward zero.

For convenience, denote

$$A^+_{MN} = A_{(1)}, \quad A^+ = A_{(2)}, \quad A_d = A_{(3)}, \quad A^\# = A_{(4)}, \quad A^{-1} = A_{(5)}, \tag{3.17}$$

and if $A \in C^{m\times n}$, let

$$D_1 = N^{-1}A^*M, \quad D_2 = A^*, \tag{3.18}$$

and if $A \in C^{n\times n}$, $\mathrm{Ind}(A) = k$, let

$$D_3 = A^k, \quad D_4 = A, \quad D_5 = I. \tag{3.19}$$

Then, with z close to zero, $(F_t(z)/g_t(z))D_t$ yields an approximation to $A_{(t)}$.

Let us summarize this in the form of a theorem.

Theorem 3.2. *Let the matrix F_t and the scalar g_t be determined by the differential equations*

$$\begin{cases} \dfrac{dF_t}{dz} = \dfrac{F_t^2 - F_t \operatorname{tr}(F_t)}{g_t}, & (3.4)\\[2mm] \dfrac{dg_t}{dz} = -\operatorname{tr}(F_t) & (3.5) \end{cases}$$

and the initial conditions

$$\begin{cases} F_t(z_0) = \operatorname{adj}(D_tA - z_0I), & (3.20)\\ g_t(z_0) = \det(D_tA - z_0I) & (3.21) \end{cases}$$

where $z_0 < 0, |z_0| < \min\limits_{i\in S}|z_i|; S = \{i/z_i \neq 0$ is the eigenvalue of $D_tA\}$. By integrating this system from z_0 to $z = 0$ and forming

$$(F_t(z)/g_t(z))D_t, \quad t = 1,2,3,4,5 \tag{3.22}$$

we obtain, in the limit, $A_{(t)}$.

§4. New Insight for the Finite Algorithms

A series of the finite algorithms for computing the generalized inverses and the inverses are given in [4–6, 9]. In this section, a new insight for the finite algorithms is presented.

Theorem 4.1. *If $A \in C^{m\times n}$, let $A_{(1)} = A^+_{MN}$, $A_{(2)} = A^+$ and*

$$D_1 = N^{-1}A^*M, \quad D_2 = A^*; \tag{3.18}$$

if $A \in C^{n\times n}$ with $\mathrm{Ind}(A) = k$, let $A_{(3)} = A_d, A_{(4)} = A^\#, A_{(5)} = A^{-1}$ and

$$D_3 = A^k, \quad D_4 = A, \quad D_5 = I \tag{3.19}$$

and let

$$\mathrm{rank} D_t = r \le n, \quad t = 1,2,3,4,; \quad \mathrm{rank} D_5 = n \tag{4.1}$$

and

$$F_t(z) = \mathrm{adj}\ (D_t A - zI) = (-1)^{n-1}(F_1^{(t)} z^{n-1} + \cdots + F_{n-1}^{(t)} z + F_n^{(t)}), \tag{4.2}$$

$$g_t(z) = \det\ (D_t A - zI) = (-1)^n (g_0^{(t)} z^n + g_1^{(t)} z^{n-1} + \cdots + g_n^{(t)}) \tag{4.3}$$

where $F_1^{(t)}, F_2^{(t)}, \cdots, F_n^{(t)}$ *are constant* $n \times n$ *matrices and* $g_0^{(t)} = 1, g_1^{(t)}, \cdots, g_n^{(t)}$ *are scalars. Then*

$$A_{(t)} = (-F_r^{(t)}/g_r^{(t)}) D_t, \quad t = 1, 2, \cdots, 5. \tag{4.4}$$

Proof. From (2.1), (2.13), (2.14), (2.18), (4.2) and (4.3), we have

$$(D_t A - zI)^{-1} = F_t(z)/g_t(z). \tag{4.5}$$

Hence

$$A_{(t)} = \lim_{z \to 0} (D_t A - zI)^{-1} D_t = \lim_{z \to 0} \left(- \left(\frac{F_1^{(t)} z^{n-1} + \cdots + F_{n-1}^{(t)} z + F_n^{(t)}}{g_0^{(t)} z^n + g_1^{(t)} z^{n-1} + \cdots + g_n^{(t)}} \right) \right) D_t$$

where $z < 0$. If $g_n^{(t)} \ne 0$, then

$$A_{(t)} = (-F_n^{(t)}/g_n^{(t)}) D_t.$$

Next, consider that $g_n^{(t)} = 0$ but $g_{n-1}^{(t)} \ne 0$. Since the above limit exists, according to Theorems 2.1, 2.2 and Corollaries 2.1, 2.2, we must have

$$F_n^{(t)} D_t = 0$$

and then

$$A_{(t)} = -(F_{n-1}^{(t)}/g_{n-1}^{(t)}) D_t.$$

We know

$$\mathrm{rank}(D_2 A) = \mathrm{rank}(A^* A) = \mathrm{rank} A^* = \mathrm{rank} D_2 = r.$$

Similarly,

$$\begin{aligned} \mathrm{rank}(D_1 A) &= \mathrm{rank}(N^{-1} A^* M A) = \mathrm{rank}(N^{1/2}(N^{-1} A^* M A) N^{-1/2}) \\ &= \mathrm{rank}((M^{1/2} A N^{-1/2})^* (M^{1/2} A N^{-1/2})) = \mathrm{rank}(M^{1/2} A N^{-1/2}) \\ &= \mathrm{rank} A = \mathrm{rank} D_1 = r. \end{aligned}$$

Since $\mathrm{Ind}(A) = k$,

$$\begin{aligned} \mathrm{rank}(D_3 A) &= \mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k) = \mathrm{rank} D_3 = r, \\ \mathrm{rank}(D_4 A) &= \mathrm{rank}(A^2) = \mathrm{rank}(A) = \mathrm{rank} D_4 = r, \\ \mathrm{rank}(D_5 A) &= \mathrm{rank}(A) = \mathrm{rank} D_5 = n, \end{aligned}$$

the number of the nonzero eigenvalues of $D_t A$ should be r; we assume $z_1, z_2, \cdots, z_r$ are different from zero and $z_{r+1} = z_{r+2} = \cdots = z_n = 0$. Since $g_t(z)$ is the characteristic polynomial of D_tA, according to Vieta's relations between the roots and coefficients of a polynomial, we have

$$g_r^{(t)} \neq 0, \quad g_{r+1}^{(t)} = \cdots = g_n^{(t)} = 0. \tag{4.6}$$

Therefore

$$A_{(t)} = (-F_r^{(t)}/g_r^{(t)})D_t.$$

Theorem 4.2. *The quantities $F_1^{(t)}, g_1^{(t)}, F_2^{(t)}, g_2^{(t)}, \cdots, F_r^{(t)}, g_r^{(t)}$ are determined by the recurrence relations*

$$\begin{cases} F_{i+1}^{(t)} = D_tAF_i^{(t)} + g_i^{(t)}I, & (4.7) \\ & i = 1, 2, \cdots, r-1. \\ g_{i+1}^{(t)} = -(i+1)^{-1} \text{ tr } (D_tAF_{i+1}^{(t)}), & (4.8) \end{cases}$$

The initial conditions are

$$\begin{cases} F_1^{(t)} = I, & (4.9) \\ g_1^{(t)} = -\text{ tr } (D_tA). & (4.10) \end{cases}$$

Proof. From (4.5) we have

$$(F_1^{(t)}z^{n-1} + \cdots + F_{n-1}^{(t)}z + F_n^{(t)})(D_tA - zI) = -(z^n + g_1^{(t)}z^{n-1} + \cdots + g_{n-1}^{(t)}z + g_n^{(t)})I. \tag{4.11}$$

From Theorem 4.1, we have

$$g_{r+1}^{(t)} = \cdots = g_n^{(t)} = 0 \quad \text{and} \quad F_j^{(t)}D_t = 0, \quad j = r+1, \cdots, n$$

so that

$$D_tAF_j^{(t)} = F_j^{(t)}D_tA = 0, \quad j = r+1, \cdots, n.$$

By comparing the identical power of z on both sides of (4.11), we see that (4.7) holds. It is also true that

$$F_{r+1}^{(t)} = D_tAF_r^{(t)} + g_r^{(t)}I \tag{4.12}$$

and

$$F_{r+2}^{(t)} = F_{r+3}^{(t)} = \cdots = F_n^{(t)} = 0. \tag{4.13}$$

To obtain (4.8), from (3.5) we have

$$\begin{aligned} &(-1)^n \ (nz^{n-1} + (n-1)g_1^{(t)}z^{n-2} + \cdots + (n-r)g_r^{(t)}z^{n-r-1}) \\ &\quad = -(-1)^{n-1}(z^{n-1} \text{ tr } (F_1^{(t)}) + \cdots + z^{n-r} \text{ tr } (F_r^{(t)}) + z^{n-r-1} \text{ tr } (F_{r+1}^{(t)})). \end{aligned}$$

Equating coefficients of the like power of z, we see that

$$(n-i)g_i^{(t)} = \text{ tr } (F_{i+1}^{(t)}).$$

Now take the trace of both sides of (4.7) to obtain

$$\text{tr } (F_{i+1}^{(t)}) = \text{ tr } (D_tAF_i^{(t)}) + ng_i^{(t)}.$$

It follows that

$$g_{i+1}^{(t)} = -(i+1)^{-1} \text{ tr } (D_tAF_{i+1}^{(t)}),$$

which completes the proof.

§5. Examples

Example 1. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$N^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \ N^{-1}A^*M = \begin{pmatrix} 4 & -2 & 8 \\ -2 & 2 & -4 \end{pmatrix}, \ N^{-1}A^*MA = \begin{pmatrix} 12 & -2 \\ -6 & 2 \end{pmatrix},$$

$$F_1^{(1)} = I, \ g_1^{(1)} = -14, \quad F_2^{(1)} = \begin{pmatrix} -2 & -2 \\ -6 & -12 \end{pmatrix}, \ g_2^{(1)} = -12, \quad F_3^{(1)} = 0, \ g_3^{(1)} = 0,$$

$$A_{MN}^{+} = (-F_2^{(1)}/g_2^{(1)})N^{-1}A^*M = 1/3 \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}.$$

Example 2. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$F_1^{(2)} = I, \ g_1^{(2)} = -2, \quad F_2^{(2)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ g_2^{(2)} = 1,$$

$$F_3^{(2)} = 0, \ g_3^{(2)} = 0, \quad A^{+} = (-F_2^{(2)}/g_2^{(2)})A^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 3. Let

$$A = \begin{pmatrix} 1 & 0.1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{Ind}(A) = 2.$$

Then

$$A^2 = \begin{pmatrix} 1 & 0.2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0.3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F_1^{(3)} = I, \ g_1^{(3)} = -2, \quad F_2^{(3)} = \begin{pmatrix} -1 & 0.3 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \ g_2^{(3)} = 1,$$

$$F_3^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ g_3^{(3)} = 0, \quad A_d = (-F_2^{(3)}/g_2^{(3)})A^2 = \begin{pmatrix} 1 & -0.1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 4. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix}, \quad \mathrm{Ind}(A) = 1.$$

Then

$$A^2 = \begin{pmatrix} 3 & 8 & 3 \\ 0 & 1 & 0 \\ 6 & 16 & 6 \end{pmatrix},$$

$$F_1^{(4)} = I,\ g_1^{(4)} = -10, \quad F_2^{(4)} = \begin{pmatrix} -7 & 8 & 3 \\ 0 & -9 & 0 \\ 6 & 16 & -4 \end{pmatrix},\ g_2^{(4)} = 9,$$

$$F_3^{(4)} = \begin{pmatrix} 6 & 0 & -3 \\ 0 & 0 & 0 \\ -6 & 0 & 3 \end{pmatrix},\ g_3^{(4)} = 0, \quad A^{\#} = (-F_2^{(4)}/g_2^{(4)})A = (1/9)\begin{pmatrix} 1 & -6 & 1 \\ 0 & 9 & 0 \\ 2 & -12 & 2 \end{pmatrix}.$$

Example 5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

be nonsingular. Then

$$F_1^{(5)} = I,\ g_1^{(5)} = -1, \quad F_2^{(5)} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, \quad g_2^{(5)} = -2,$$

$$F_3^{(5)} = 0,\ g_3^{(5)} = 0, \quad A^{-1} = (-F_2^{(5)}/g_2^{(5)}) = (1/2)\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}.$$

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