Journal of Computational Mathematics, Vol.8, No.4, 1990, 333–341.

INCOMPLETE SEMIITERATIVE METHODS FOR SOLVING OPERATOR EQUATIONS IN BANACH SPACE*[1)]

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Abstract

There are several methods for solving operator equations in a Banach space. The successive approximation methods require the spectral radius of the iterative operator be less than 1 for convergence.

In this paper, we try to use the incomplete semiiterative methods to solve a linear operator equation in Banach space. Usually the special semiiterative methods are convergent even when the spectral radius of the iterative operator of an operator equation is greater than 1.

§1. Introduction

Let X be a complex Banach space. The set of all bounded linear operators from X into X is denoted by $B[X]$ which is also a Banach space. We consider the linear operator equation

$$Ax = b, \tag{1.1}$$

where $A \in B[X]$ and $b \in X$ is given. If $A^{-1} \in B[X]$ then the solution $\hat{x} = A^{-1}b$ of equation (1.1) exists uniquely. To study the successive approximation methods and semiitterative methods, we rewrite equation (1.1) in a fixed point form

$$x = Tx + f, \tag{1.2}$$

where $T \in B[X]$ and $f \in X$.

Let $\sigma(T)$ be the spectrum of T. Then for equation (1.2) and therefore equation (1.1) there exists an unique solution if and only if $1 \bar{\in} \sigma(T)$. We assume $1 \bar{\in} \sigma(T)$ and apply the successive approximation method to solve the following operator equation

$$x = Tx + f.$$

The iterative sequence

$$x_{m+1} = Tx_m + f = T^{m+1}x_0 + \Big(\sum_{i=0}^{m} T^i\Big)f, \quad m \geq 0, \quad x_0 \in X \tag{1.3}$$

converges for any $x_0 \in X$ if and only if the solution of equation (1.2) has a Neumann expansion

$$\hat{x} = \Big(\sum_{i=0}^{\infty} T^i\Big)f. \tag{1.4}$$

* Received January 22, 1988.
1) The Project Supported by Institute for Computational Mathematics of Kent State University.

But the Neumann series

$$\sum_{i=0}^{\infty} T^i$$

converges if and only if the spectral radius $r\sigma(T)$ of T satisfies

$$r\sigma(T) < 1. \tag{1.5}$$

This is a very strict condition for operator T.

We denote by $\mathcal{F}(T)$ the family of all functions which are analytic on some neighbourhood of $\sigma(T)$ (the neighbourhood need not be connected, and can depend on the particular function $f \in \mathcal{F}(T)$). Let $T \in B[X], f \in \mathcal{F}(T)$ and let V be an open subset of C whose boundary B consists of a finite number of rectifiable Jordan curvers. We assume that B is oriented. Suppose $V \supseteq \sigma(T)$ and $V \cup B$ is contained in the analytic domain of f. Then the operator $f(T)$ is defined by equation

$$f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda - T)^{-1} d\lambda. \tag{1.6}$$

Proposition 1[9]. Let $T \in B[X]$ and let $f \in \mathcal{F}(T)$. Then

$$f(\sigma(T)) = \sigma(f(T)), \tag{1.7}$$

and hence

$$r\sigma(T)^n = r\sigma(T^n), \quad n = 1, 2, \cdots. \tag{1.8}$$

§2. Incomplete Semiiterative Methods

Given a linear equation $Ax = b$, where $A \in B[X], b \in X$, we rewrite $Ax = b$ in a fixed point form

$$x = Tx + f, \tag{2.1}$$

where $T \in B[X], f \in X$ and $1\bar{\in}\sigma(T)$. Corresponding to the successive approximation method

$$x_m = Tx_{m-1} + f, \tag{2.2}$$

if we define the error vector and the residual vector as

$$e_m := \hat{x} - x_m, \quad r_m := f - (I - T)x_m, \tag{2.3}$$

then there holds

$$e_m := T^m e_0, \quad r_m := T^m r_0. \tag{2.3'}$$

Following Varge[4] we define a semiiterative method (SIM) with respect to iterative method (2.2) by

$$y_m := \sum_{i=0}^{m} \pi_{m,i} x_i, \quad m \geq 0, \tag{2.4}$$

where the infinite lower triangular matrix

$$P := \begin{bmatrix} \pi_{00} & & & O \\ \pi_{10} & \pi_{11} & & \\ \pi_{20} & \pi_{21} & \pi_{22} & \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \tag{2.5}$$

satisfies

$$\sum_{i=0}^{m} \pi_{m,i} = 1. \tag{2.6}$$

The associated sequence of polynomial derived from the rows of triangular matrix P is

$$p_m(\lambda) := \sum_{i=0}^{m} \pi_{m,i} \lambda^i, \quad m \geq 0, \tag{2.7}$$

and satisfies the condition

$$p_m(1) = 1.$$

If we introduce another error vector and another residual vector

$$\tilde{e}_m := \hat{x} - y_m, \quad \tilde{r}_m := f - (I - T)y_m, \quad m \geq 0, \tag{2.8}$$

then there holds

$$\tilde{e}_m = p_m(T)e_0, \quad \tilde{r}_m = p_m(T)r_0, \quad m \geq 0. \tag{2.9}$$

For a SIM, we can rewrite (2.4) in matrix form:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = P \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}.$$

As in the finite-dimensional space, it can be shown that

$$x_m = Tx_{m-1} + f = x_{m-1} + r_{m-1}, \quad m \geq 1,$$

i.e.,

$$x_m = x_0 + \sum_{i=0}^{m-1} r_i, \quad m \geq 1, \tag{2.10}$$

or

$$x_m = x_0 + \left\{ \sum_{i=0}^{m-1} T^i \right\} r_0, \quad m \geq 1 \tag{2.11}$$

and

$$\hat{x} = x_0 + (I - T)^{-1} r_0. \tag{2.12}$$

The connection between $\{x_m\}, m \geq 0$, and the terms of series $x_0 + \sum_{i=0}^{m-1} r_i$ can be expressed in matrix notation simply as

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = S \begin{bmatrix} x_0 \\ r_0 \\ r_1 \\ \vdots \end{bmatrix}, \quad \text{with } S = \begin{bmatrix} 1 & & & O & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}. \tag{2.13}$$

The connection between sequence $\{x_0, r_0, r_1, \cdots\}$ and sequence $\{y_0, y_1, y_2, \cdots\}$ can be also expressed in matrix notation

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = P \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = PS \begin{bmatrix} x_0 \\ r_0 \\ r_1 \\ \vdots \end{bmatrix}. \tag{2.14}$$

Let

$$Q = PS = \begin{bmatrix} 1 & & & & \\ t_0 & r_{00} & & O & \\ t_1 & r_{10} & r_{11} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \tag{2.15}$$

be another infinite lower triangular matrix. From Q we can introduce a sequence of polynomials

$$q_{m-1}(\lambda) = \sum_{i=0}^{m-1} r_{m-1,i}\lambda^i. \tag{2.16}$$

From (2.3′) and (2.14) we have

$$y_m = x_0 + q_{m-1}(T)r_0. \tag{2.17}$$

Using (2.12) and (2.17) we can conclude that the y_m defined in (2.17) is a good approximation for solution $\hat{x}$ if $q_{m-1}(T)$ approximates well

$$g(T) \equiv (I - T)^{-1}.$$

How to choose these polynomials $q_{m-1}(\lambda)$ is a very interesting problem in this paper. Before answering this question, we give the following

Definition 1. *A semiitterative method (2.4) is called an incomplete semiiterative method (ISIM) with respect to the subsequence $< m_j >$ of $\{0, 1, 2, \cdots\}$, if $p_m(\lambda) \neq 0$, for all $m \in < m_j >$, i.e., if there exist nonzero elements in the mth row of matrix P, then $m \in < m_j >$.*

Proposition 2. Let $P := (\pi_{mj})_{m \geq 0}, 0 \leq j \leq m$, be an infinite lower triangular matrix satisfying $\sum_{i=1}^{m} \pi_{mi} = 1$ if $m \in < m_j >$ and $\pi_{mi} = 0 (i = 0, 1, \cdots, m)$ if $m \bar{\in} < m_j >$. Let $Q = PS$ be defined by (2.15). Then corresponding to the sequence of polynomials $\{p_m(\lambda)\}_{m \geq 0}, m \in < m_j >$, of matrix P and $\{q_{m-1}(\lambda)\}_{m \geq 1}, m \in < m_j >$, of matrix Q there hold

$$q_{m-1}(\lambda) = [1 - p_m(\lambda)]/(1 - \lambda), \quad m \geq 0, m \in < m_j >, \quad q_{-1}(\lambda) := 0,$$

and

$$p_m(\lambda) = 1 - (1 - \lambda)q_{m-1}(\lambda) \quad m \geq 0, \quad m \in < m_j >, \quad q_{-1}(\lambda) := 0. \tag{2.18}$$

The proof of Proposition 2 is similar to the proof in paper [1]. Let $g(\lambda) = 1/(1 - \lambda)$. Suppose $q_{m-1}(\lambda)$ $(m \in < m_j >)$ is an approximation polynomial of function $g(\lambda)$ in an open set $\Omega \subseteq C$ in some norm sense, then we denote $q_{m-1}(\lambda) \sim g(\lambda)$ in Ω.

Theorem 1. *Let $T \in B[X], 1\bar{\in}\sigma(T)$ and $q_{m-1}(\lambda) \sim g(\lambda)$ in a bounded region $\Omega \supseteq \sigma(T)$. Let $p_m(\lambda) = 1-(1-\lambda)q_{m-1}(\lambda) = \sum_{i=0}^{m} \pi_{mi}\lambda^i$ and $y_m = \sum_{i=0}^{m} \pi_{mi}x_i$ for any $m \in < m_j >$. Then there exists a positive number $C(\Omega)$ such that*

$$\|y_m - \hat{x}\| \le C(\Omega) \max_{\lambda\in\bar{\Omega}} |q_{m-1}(\lambda) - g(\lambda)| \, \|r_0\|. \tag{2.19}$$

Furthermore

$$\|y_m - \hat{x}\| \le \inf_{\Omega\supseteq\sigma(T)} \{C(\Omega) \max_{\lambda\in\bar{\Omega}} |q_{m-1}(\lambda) - g(\lambda)|\} \, \|r_0\|. \tag{2.20}$$

Proof. From (2.12) and (2.17), we have

$$\|y_m - \hat{x}\| \le \|q_{m-1}(T) - g(T)\| \, \|r_0\|.$$

Let $f(\lambda) = g(\lambda) - q_{m-1}(\lambda)$ and B be the boundary of Ω. Then, by (1.6) we have (assume that $1\bar{\in}\bar{\Omega}$)

$$f(T) = \frac{1}{2\pi i}\int_B f(\lambda)(\lambda - T)^{-1} d\lambda.$$

The resolvent $R_\lambda \equiv (\lambda - T)^{-1}$ is analytic on B which is a compact set of C. Hence $\|R_\lambda\| \le c_1$ for any B, where C_1 is a positive constant. Let Γ be the measure of B, then we have

$$\|f(T)\| \le \frac{1}{2\pi} C_1 \Gamma \max |q_{m-1}(\lambda) - g(\lambda)| \equiv C(\Omega) \max_{\lambda\in B} |q_{m-1}(\lambda) - g(\lambda)|.$$

Using the maximum modulus principle, we obtain inequality (2.19). Since the left-hand side of (2.19) is independent of $\sigma(T)$, inequality (2.19) implies inequality (2.20) and thus completes the proof of Theorem 1.

Now we will construct a sequence of polynomials which is a good approximation of $g(\lambda)$ for a suitable parameter α.

Let

$$\begin{aligned} \varphi_0 &= \alpha, \\ \varphi_1 &= \varphi_0[2 - (1-\lambda)\varphi_0], \\ &\vdots \\ \varphi_{m+1} &= \varphi_m[2 - (1-\lambda)\varphi_m], \quad m \ge 0. \end{aligned} \tag{2.21}$$

For the difference between $\varphi_m(\lambda)$ and $g(\lambda)$ we have

Theorem 2. *Let Ω be a compact set in C and $1\bar{\in}\Omega$. Suppose* $\sup\{|\alpha(1-\lambda) - 1| : \lambda \in \Omega\} = q$ *for fixed $\alpha \in C$. Then*

$$\|\varphi_m - g\|_{\Omega,\infty} \le \max_{\lambda\in\Omega}\{1/|1-\lambda|\} q^{2^m}, \tag{2.22}$$

where $g(\lambda) = 1/(1-\lambda)$.

Proof. Since

$$\begin{aligned} (1-\lambda)\varphi_{m+1}(\lambda) - 1 &= (1-\lambda)\varphi_m(\lambda)[2 - (1-\lambda)\varphi_m(\lambda)] - 1 \\ &= -[(1-\lambda)\varphi_m(\lambda) - 1]^2 \\ &= -[(1-\lambda)\varphi_{m-1}(\lambda) - 1]^{2^2} \\ &\vdots \\ &= -[(1-\lambda)\varphi_0(\lambda) - 1]^{2^{m+1}}, \end{aligned} \tag{2.22}$$

we have

$$(1-\lambda)[\varphi_{m+1}(\lambda)-g(\lambda)] = -[(1-\lambda)\varphi_0 - 1]^{2^{m+1}},$$

$$|\varphi_{m+1}(\lambda)-g(\lambda)| \le \max_{\lambda\in\Omega}\{1/|1-\lambda|\}q^{2^{m+1}},$$

and

$$\|\varphi_{m+1}(\lambda)-g\|_{\Omega,\infty} \le \max_{\lambda\in\Omega}\{1/|1-\lambda|\}q^{2^{m+1}}.$$

This is the required inequality.

From (2.21) we can define an incomplete semiiterative method:

Let $q_{m-1}(\lambda)=\varphi_{m-1}(\lambda)(q_{-1}:=0)$ be a sequence of polynomials. The $p_m(\lambda)$ is defined as

$$p_m(\lambda) = 1-(1-\lambda)q_{m-1}(\lambda) = \pi_{\tilde{m}0}+\pi_{\tilde{m}1}\lambda+\cdots+\pi_{\tilde{m}2^{m-1}}\lambda^{2^{m-1}}, m\ge 1, \tilde{m}=2^{m-1}.$$
$$p_0(\lambda):=1.$$

Hence the subset $<m_j>=\{0,1,2,4,8,\cdots\}$ and the associated infinite lower triangular matrix is

$$P=\begin{bmatrix} \pi_{00} & & & & & \\ \pi_{10} & \pi_{11} & & & O & \\ \pi_{20} & \pi_{21} & \pi_{22} & & & \\ 0 & 0 & 0 & 0 & & \\ \pi_{40} & \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.23}$$

Theorem 3. *Let the* ISIM *be defined by* (2.4) *for* $m\in<m_j>$ *and the relative infinite lower triangular matrix* P *be given as in* (2.23). *Suppose* $T\in B[H], 1\bar{\in}\sigma(T)$ *and* $T^*=T$. *Then*

$$\|y_m-\hat{x}\| \le \max_{\lambda\in\sigma(T)}\{1/|1-\lambda|\}q^{2^{m-1}}\|r_0\|, \tag{2.24}$$

where H *is a Hilbert space.*

Proof. From (2.12) and (2.17) we have

$$\|y_m-\hat{x}\| \le \|q_{m-1}(T)-g(T)\|\ \|r_0\|.$$

Since $f(\lambda)=g(\lambda)-q_{m-1}(\lambda)$ is a continuous function on $\sigma(T)$, so

$$\|g(T)-q_{m-1}(T)\| = \max_{\lambda\in\sigma(T)}|g(\lambda)-q(\lambda)|.$$

Then, using Theorem 2 we obtain the required inequality (2.24).

Theorem 4. *Let the* ISIM *be defined as in* (2.4) *and the relative infinite lower triangular matrix* P *given as in* (2.23). *Suppose* $T\in B[X];\Omega$ *is a bounded region in* C *and* $1\bar{\in}\bar{\Omega}\supseteq\sigma(T)$. *Then there exists a positive constant* C*, such that*

$$\|y_m-\hat{x}\| \le C\max_{\lambda\in\bar{\Omega}}\{1/|1-\lambda|\}q^{2^{m-1}}\|r_0\|.$$

Proof. By Theorem 1 and 2.

§3. The Determination of Best Parameter $\alpha_{\rm opt}$

From Theorem 3 and the Theorem 4 we have seen that the rate of convergence of $\{y_m\}_{m\geq 0}$ depends on the size of q. If $y(\lambda) = \alpha(\lambda - 1) + 1$, then our problem is to solve the minimizing problem:

$$\min_{\alpha} \max_{\lambda\in\Omega} |y(\lambda)|, \tag{3.1}$$

where $y(1) = 1$ and Ω is a bounded open subset of C.

Case 1. Suppose λ is a real variable and α is a real parameter. Then the above problem can be simplified as

$$\min_{\alpha\in I\!R} \max_{m\leq\lambda\leq M} |\alpha\lambda + 1 - \alpha|, \quad 1\bar{\in}[m, M].$$

Using the condition $y(1) = 1$, the above problem can be solved by Chebyshev polynomial. Let

$$\tilde{T}_n(\lambda) = \frac{T_n\left(\frac{2\lambda-(m+M)}{M-m}\right)}{T_n\left(\frac{2-(m+M)}{M-m}\right)}$$

where $T_n(\lambda)$ is a first kind Chebyshev polynomial of degree n. For $n = 1$,

$$\tilde{T}_1(\lambda) = \{[2\lambda - (M+m)]/[2-(M+m)]\}.$$

Then we obtain the best parameter

$$\alpha_{\rm opt} = 2/[2-(M+m)].$$

Theorem 5. *Suppose $T \in B[H], T^* = T$. Let $m = \inf_{\|x\|=1} \langle Tx, x\rangle, M = \sup_{\|x\|=1} \langle Tx, x\rangle$ and $1\bar{\in}[m, M]$. Then*

$$\min_{\alpha\in I\!R} \max_{m\leq\lambda\leq M} |\alpha\lambda + 1 - \alpha| = \max_{m\leq\lambda\leq M} |\alpha_{\rm opt}\lambda + 1 - \alpha_{\rm opt}| < 1,$$

where H is a Hilbert space.

Proof. We can show $|\alpha_{\rm opt}M + 1 - \alpha_{\rm opt}| = \max_{m\leq\lambda\leq M} |\alpha_{\rm opt}\lambda + 1 - \alpha_{\rm opt}|$ if $m < M < 1$. Then

$$|\alpha_{\rm opt}M + 1 - \alpha_{\rm opt}| = |M-m|/|2-(M+m)| = (M-m)/[2-(m+M)] < 1.$$

Similarly to the case of $m < M < 1$, we have

$$\max|\alpha_{\rm opt}\lambda + 1 - \alpha_{\rm opt}| = (M-m)/(M+m-2) < 1$$

if $1 < m < M$. This completes the proof of Theorem 5.

Case 2. Let $T \in B[X]$, X be a complex Benach space, and Ω be a bounded open subset of C. Suppose λ is a complex variable and α be a complex parameter. Then, the minimizing problem becomes

$$\min_{\alpha\in C} \max_{\lambda\in\Omega} |y(\lambda)| = \min\{\max_{\lambda\in\Omega} |p_1(\lambda) : p_1 \in P_1, \quad p_1(1) = 1\}, \tag{3.2}$$

where P_1 is the set of all polynomials of degree not exceeding 1. A classical result[5] shows that the problem (3.2) has a solution $y^*(\lambda)$.

Suppose $\bar{\Omega}$ is a closed bounded set in the z-plane. In the class $\mathcal{P}_n[z_0]$ of polynomials $p_n(z)$ of degree not exceeding $n \geq 0$ with $p_n(z_0) = 1$, where z_0 is a fixed point in the z-plane, we find a polynomial such that it is closest to zero on $\bar{\Omega}$. Let

$$\mu_n(\bar{\Omega}, z_0) := \inf_{p_n \in \mathcal{P}[z_0]} \sup_{z \in \bar{\Omega}} |p_n(z)|. \tag{3.3}$$

We need to find a polynomial $\pi_n(z) \in \mathcal{P}_n[z_0]$ such that

$$\sup_{z \in \bar{\Omega}} |\pi_n(z)| = \mu_n(\bar{\Omega}, z_0). \tag{3.4}$$

Proposition 3[5]. Suppose that $Q_m(z)$ is a given polynomial of degree $m \geq 1$, that $\bar{\Omega}$ is the bounded set of points z satisfying the inequality $|Q_m(z)| \leq r$, where $r > 0, z_0 \bar{\in} \bar{\Omega}$ and that n is an arbitrary natural number. Then

$$\mu_{m,n}(\bar{\Omega}, z_0) = r^n / |Q_m(z_0)|, \tag{3.5}$$

and $\mu_{mn}(\bar{\Omega}, z_0)$ is attained only for the polynomial

$$\pi_{mn}(z) = \{Q_m(z)/Q_m(z_0)\}^n. \tag{3.6}$$

Using Proposition 3, the above problem (3.2) becomes calculating

$$\mu_1(\bar{\Omega}, 1) = \inf_{p_1 \in \mathcal{P}[1]} \sup_{z \in \bar{\Omega}} |p_1(z)|, \tag{3.7}$$

or constructing a polynomial $Q_1(z)$ such that

$$|Q_1(z)| \leq r \quad \text{for all } z \in \bar{\Omega}.$$

Then

$$\pi_1(z) = Q_1(z)/Q_1(1), \tag{3.8}$$

and

$$\min\{\max_{z \in \bar{\Omega}} |p_1(z)| : p_1 \in \mathcal{P}_1[1]\} = r/|Q_1(1)|. \tag{3.9}$$

Example. Let $\bar{\Omega} = \{z : |1+z| \leq 1\}$. Then $Q_1(z) = 1 + z$ and $r = 1$. Thus $\mu_1(\bar{\Omega}, 1) = 1/|Q_1(1)| = 1/2$ and $\pi_1(z) = (1+z)/2$ and $\alpha_{\text{opt}} = 1/2$.

From the above example, we can see a obvious but useful result:

Theorem 6. *Let $T \in B[X]$, where X is a complex Banach space. Suppose $\Omega \supseteq \sigma(T)$ is an open set which can be contained by a disk $C_R = \{z : |z - z_1| \leq R\}$ and $1 \bar{\in} C_R$. Then*

$$\min_{\alpha \in C} \max_{\lambda \in \bar{\Omega}} |\alpha\lambda + 1 - \alpha| \leq \mu_1(C_R, 1) = R/|Q_1(1)| < 1, \tag{3.10}$$

where $Q_1(z) = z - z_1$.

Theorem 6 shows that, if the ISIM introduced by infinite lower triangular matrix (2.23) and for some open set $\Omega \supseteq \sigma(T)$ which can be contained by a disk $C_R (1 \bar{\in} C_R)$ then the ISIM is always convergent for $\alpha = \alpha_{\text{opt}}$.

§4. Numerical Example

To illustrate the effectiveness of the ISIM, we simply consider the following example:

Example. To solve the linear equation

$$Ax = b, \tag{4.1}$$

we let $A = I - T$ and rewrite equation (5.1) in a fixed point form

$$x = Tx + b, \tag{4.2}$$

where

$$T = \begin{bmatrix} 10 & 1 & 0 \\ 1 & 11 & 1 \\ 0 & 1 & 12 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -10 \\ -12 \\ -12 \end{bmatrix}.$$

Since the spectrum $\sigma(T) = \{11 - \sqrt{3}, 11, 11 + \sqrt{3}\}$, so $m = 11 - \sqrt{3}$, $M = 11 + \sqrt{3}$ and $\alpha_{\rm opt} = 2/[2 - (m + M)] = -1/10$. Using (2.21) we obtain recursively $\varphi_3(\lambda) = q_3(\lambda)$ and $p_4(\lambda) = 1 - (1 - \lambda)q_3(\lambda) = \sum_{i=0}^{8} \pi_{8,i}\lambda^i$. Thus from (2.17) or (2.4) with $x_0 = (1, 0, 0)$, $r_0 = b - Ax_0$ and $x_{i+1} = Tx_i + b$, we have

$$y_4 = x_0 + q_3(T)r_0 = \sum_{i=0}^{8} \pi_{8,i}x_i = (1.00000000, 0.99999919, 0.99999919).$$

The error vector is $\hat{x} - y_4 = (0, 8.1 \times 10^{-7}, 8.1 \times 10^{-7})$, and it shows that the rate of convergence of the ISIM is very high.

Of course, the above conclusion is true for an operator equation in an infinite Hilbert space or a Banach space.

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