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FINITE ELEMENT EIGENVALUE COMPUTATION ON DOMAINS WITH REENTRANT CORNERS USING RICHARDSON EXTRAPOLATION*

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Abstract

In the presence of reentrant corners or changing boundary conditions, standard finite element schemes have only a reduced order of accuracy even at interior nodal points. This "pollution effect" can be completely described in terms of asymptotic expansions of the error with respect to certain fractional powers of the mesh size. Hence, eliminating the leading pollution terms by Richardson extrapolation may locally increase the accuracy of the scheme. It is shown here that this approach also gives improved approximations for eigenvalues and eigenfunctions which are globally defined quantities.

§1. Introduction

On domains with reentrant corners standard finite element schemes usually suffer from a global loss of accuracy caused by the presence of the local corner singularities. Various methods are devised in the literature for suppressing this "pollution effect", mainly by using systematic mesh refinement near the corner points, or by incorporating "singular" shape functions into the scheme. All these procedures require significant overheads and are not always easy to combine with existing standard routines; see [1] for a survey of these methods. As an alternative, it is proposed in [3] to use Richardson extrapolation for eliminating the leading pollution terms in the error. This approach is based on asymptotic error expansions of the form

$$(u-u_h)(x)=\sum_{n=1}^{N}A_n(x)h_n^{2\alpha_n}+R_h(x)h^2|\log(h)|, \tag{1.1}$$

where h_n are local mesh size parameters, and $\alpha_n < 1$ are the exponents in the leading "singularities" at the reentrant corners. Such an expansion is established in [3] for the approximation of the 2-dimensional Poisson equation by linear finite elements on certain locally uniform meshes. On the basis of (1.1), a simple extrapolation procedure yields improved approximations $\tilde{u}_h$ to u, satisfying

$$(u-\tilde{u}_h)(x)=R_h(x)h^2|\log(h)|, \quad x\in\Omega. \tag{1.2}$$

The implementation of this method is relatively easy, compared with the use of singular functions or systematic mesh refinements, particularly for three-dimensional problems. A further advantage is that one can work with (piecewise) uniform meshes which, at least in

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the case of constant coefficients, allows for very economical storage techniques. This makes it possible to solve two- and also certain three-dimensional "corner problems" with high accuracy on small computers.

The remainder term R_h in the expansion (1.1) is integrable on Ω, uniformly as $h \to 0$. Hence, the pointwise error estimate (1.2) implies similar ones also for certain globally defined quantities. For example, in approximating torsion moments one obtains

$$\int_\Omega u\,dx - \int_\Omega \tilde{u}_h dx = O(h^2|\log(h)|). \tag{1.3}$$

In the present paper, we shall extend this result to the approximation of eigenvalues and eigenfuncitons. In particular, an expansion of the form

$$\bar{\lambda}_h - \lambda = \sum_{n=1}^{N} C_n h_n^{2\alpha_n} + O(h^2) \tag{1.4}$$

will be derived for appropriate mean values $\bar{\lambda}_H$ of the discrete eigenvalues, which implies that the extrapolated values $\tilde{\lambda}_h$ satisfy

$$\tilde{\lambda}_h - \lambda = O(h^2). \tag{1.5}$$

This approach may largely improve the eigenvalue approximations already on relatively coarse meshes; some numerical results are given at the end of the paper. Consider, for example, the eigenvalue problem of the Laplacian, $-\Delta$, first on the unit square, and then on the unit square with two slits (see Fig. 2 below). Then, for linear finite elements an error in the smallest eigenvalue of less than 1% requires about 225 nodes for the regular domain, but more than 10,000 nodes for the slit domain. In the latter case, one step of h-extrapolation gives the same accuracy already on a mesh with 900 nodes.

In the following, $L_p(\Omega), 1 \le p \le \infty$, and $H^m(\Omega), H_0^m(\Omega), m \in I\!N$, are the usual Lebesgue and Sobolev spaces on Ω. The norm of $L_p(\Omega)$ is denoted by $\|\cdot\|_p$. The notation $\|\cdot\|_{p;\mathrm{loc}}$ refers to the L_p-norm on any fixed subdomain $\Omega' \subset \Omega$ having positive distance to all of the corner points of $\partial\Omega$. The L_2-inner product and norm on Ω are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Finally, the symbol c is used for a generic positive constant which may vary with the context, but is always independent of the mesh size parameter and of the particular functions involved.

§2. The Pollution Effect

Let $\Omega \subset I\!R^2$ be a polygonal domain with reentrant corners or slits. As model situations, we consider the first boundary value problem of the Laplacian operator

$$-\Delta u = f, \quad \text{in } \Omega, \quad u = b, \quad \text{on } \partial\Omega, \tag{2.1}$$

and the corresponding eigenvalue problem

$$-\Delta w = \lambda w, \quad \text{in } \Omega, \quad w = 0, \quad \text{on } \partial\Omega. \tag{2.2}$$

The data f and b are assumed to be smooth, say $f \in C^\alpha$ and $b \in C^{2+\alpha}$. Below, we shall largely use the notation of [3]. Let $\hat{Z} \equiv \{z_n, n = 1, \cdots, \hat{N}\}$ be the set of all corner points of

$\partial\Omega$, with the corresponding interior angles $\omega_n \in (0, 2\pi]$, and let $Z \equiv \{z_n, n = 1, \cdots, N\}$ be the subset of $\hat{Z}$, consisting of all reentrant corners. Corresponding to the points $z_n \in Z$, we introduce polar coordinates (r_n, θ_n), and define the singular functions

$$s_{n;i} \equiv (|i|\pi)^{-1/2} r_n^{\alpha_{n;i}} \sin(\alpha_{n;i}\theta_n), \quad i \in \mathbb{Z} \setminus \{0\},$$

where $\alpha_{n;i} \equiv i\pi \setminus \omega_n$. Further, we set $\alpha_* \equiv \min\{\alpha_{n;1}, n = 1, \cdots, N\}$. Then, the weak solution $u \in b + H_0^1(\Omega)$ of problem (2.1) admits an asymptotic expansion of the form (see Kondrat'ev [4])

$$u = \sum_{n=1}^{N} \sum_{i=1}^{I_n} K_{n;i}(u) s_{n;i} + U \tag{2.3}$$

with a remainder term $U \in C^{2+\alpha}(\Omega) \cap H^2(\Omega)$ and certain numbers $K_{n;i}(u)$, the so-called "stress intensity factors". Here, we take $I_n = 0$ if $\omega_n < \pi$, $I_n = 1$ if $\pi < \omega_n < \frac{3}{2}\pi$, $I_n = 2$ if $\frac{3}{2}\pi \le \omega_n < 2\pi$, and $I_n = 3$ if $\omega_n = 2\pi$. Then, the remainder U satisfies

$$\mu\nabla^2 U \in L^\infty(\Omega), \quad \mu = \prod_{\omega_n > \pi/2} r^{2-\alpha_{n;I_n+1}} \prod_{\omega_n = \pi/2} \log(r_n)^{-1}. \tag{2.4}$$

For $0 < \omega_n < \frac{\pi}{2}$, u has bounded derivatives at z_n. In the case of a right-angled corner, the logarithmic growth in (2.4) corresponds to the leading "singular" term $\frac{1}{4}\{f(z_n) + \Delta b(z_n)\} r_n^2 \log(r_n) \sin(2\theta_n)$ in u. For the purposes of the present paper it suffices to separate from u only those singular parts which correspond to the reentrant corners, $z_n \in Z$.

As in [3], to discretize (2.1) and (2.2), we consider a standard finite element scheme using piecewise linear elements. Let $\Omega_n \subset \bar{\Omega}$ be polygonal neighborhoods (relative to $\bar{\Omega}$) of the reentrant corners $z_n, n = 1, \cdots, N$, of $\partial\Omega$, such that $\Omega_n \cap \Omega_m = \emptyset$ for $n \ne m$. Further, let $\Omega_0 \equiv \Omega \setminus \cup\{\bar{\Omega}_n, n = 1, \cdots, N\}$. For a mesh size vector $H \equiv \{h_0, h_1, \cdots, h_N\}$, satisfying

$$0 < \gamma h \le h_n \le h \le \frac{1}{2}, \quad n = 0, 1, \cdots, N,$$

with some $\gamma > 0$, and $h \equiv \max\{h_n, n = 0, \cdots, N\}$, let $\{T_H\}$ be a uniformly regular family of triangulations of $\bar{\Omega}$, such that each T_H is of width h_n in Ω_n. Corresponding to T_H, we introduce the finite element space

$$S_H \equiv \{v \in H^1(\Omega), v \text{ piecewise linear with respect to } T_H\},$$

and set $S_{H,0} \equiv S_H \cap H_0^1(\Omega)$. Then, for functions $v \in H^1(\Omega)$ which are continuous up to the boundary, the Ritz projection $\mathcal{R}_H v$ into S_H is defined through the conditions $\mathcal{R}_H v - \varphi_H v \in S_{H,0}$, and

$$(\nabla \mathcal{R}_H v, \nabla \varphi_H) = (\nabla v, \nabla \varphi_H), \quad \forall \varphi_H \in S_{H,0}. \tag{2.5}$$

Here, φ_H denotes the usual pointwise interpolation operator into S_H. For the error $u - \mathcal{R}_H u$, there holds (see [3] and [1] for references)

$$\|u - \mathcal{R}_H u\| + \|u - \mathcal{R}_H u\|_{\infty;\text{loc}} = O(h^{2\alpha_*}). \tag{2.6}$$

In general, these estimates are optimal with respect to the power of h.

Corresponding to (2.2), we consider the discrete eigenvalue problems

$$(\nabla w_H \cdot \nabla \varphi_H) = \lambda_H (w_H, \varphi_H), \quad \forall \ \varphi_H \in S_{H,0}. \tag{2.7}$$

For the ordered eigenvalues $\lambda^1 \leq \lambda^2 \leq \cdots$ and $\lambda_H^1 \leq \lambda_H^2 \leq \cdots$ of the problems (2.2) and (2.7) (counted according to their multiplicity), there holds

$$\lambda_H^k - \lambda^k = O(h^{2\alpha_*}). \tag{2.8}$$

In order to describe the convergence of the eigenvectors, we fix some eigenvalue λ with multiplicity m. Let $\lambda_H^1, \cdots, \lambda_H^m$ denote the eigenvalues of (2.7) converging to λ, according to (2.8). Then, for any orthonormal sets $\{w_H^1, \cdots, w_H^m\}$ of corresponding eigenvectors there exists an orthonormal basis $\{w^{1,H}, \cdots, w^{m,H}\}$ of the eigenspace $E(\lambda)$ of λ, such that

$$\|w^{i,H} - w_H^i\| + \|w^{i,H} - w_H^i\|_{\infty;\text{loc}} = O(h^{2\alpha_*}), \quad i = 1, \cdots, m, \tag{2.9}$$

analogously to (2.6); see Strang-Fix [7] and the proof of Theorem 2 below.

To prove our error expansions, we need to impose some uniformity conditions on the triangulations, namely that they are "locally stretching invariant and symmetric" (see [3]):

(T) The family of triangulations $\{T_H\}$ is generated from some macro-triangulation T_* such that, for the reference sets $C_*^n \equiv \cup\{K \in T_*, z_n \in K\}$, $n = 1, \cdots, N$, there hold:

a) The vertices z_n are the only reentrant corners of $C_*^n, n = 1, \cdots, N$.

b) (Local stretching invariance) For any $H = (h_0, \cdots, h_N)$, $K = (k_0, \cdots, k_n)$, with $k_n \leq h_n$, the scaling $x \to k_n h_n^{-1}(x - z_n)$ maps the subtriangulations $T_H(C_*^n) \equiv \cup\{K \in T_H, K \subset C_*^n\}$, $n = 1, \cdots, N$, into the corresponding subtriangulations $T_K(C_*^N)$.

c) (Local symmetry) Each C_n^*, as well as the subtriangulation $T_H(C_*^n)$, is symmetric with respect to the bisector of the angle at z_n.

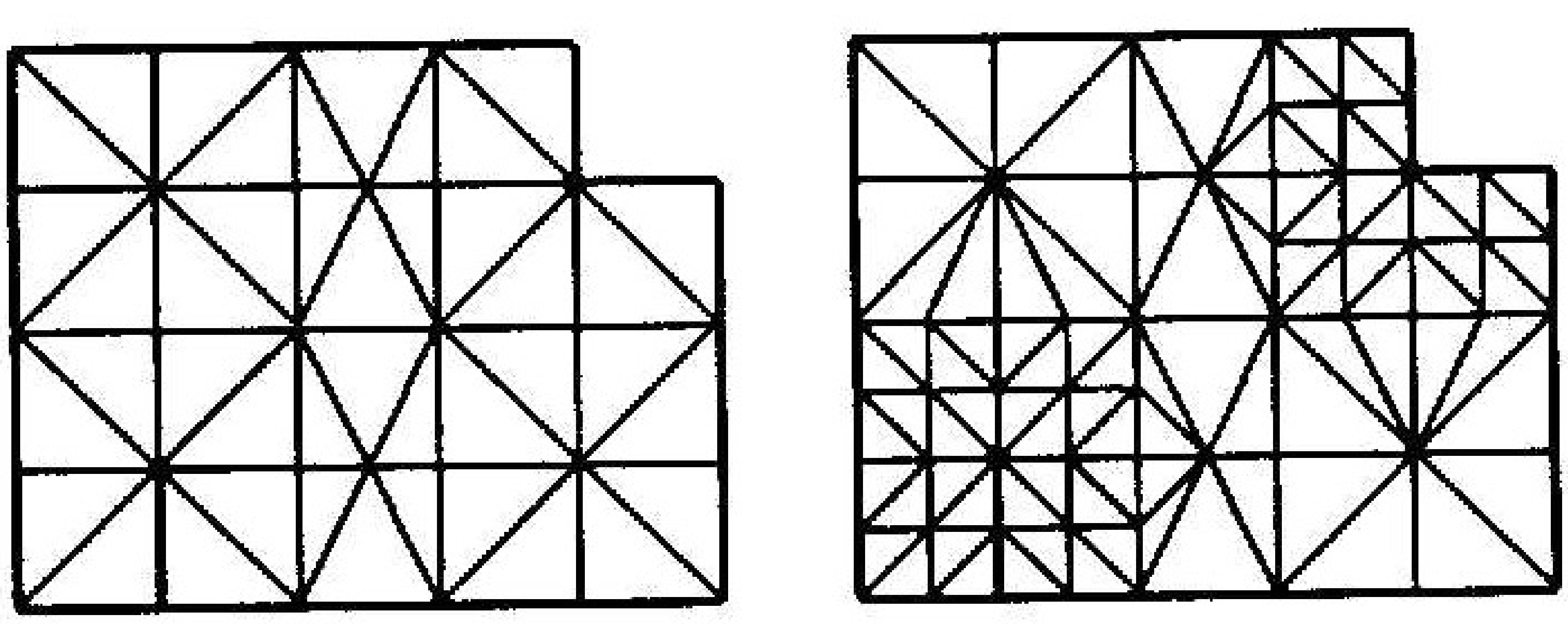

Fig. 1. Two members of a locally stretching invariant and symmetric family of triangulations

Without loss of generality, we set $\Omega_n \equiv C_*^n, n = 1, \cdots, N$. Below, the notation of "locally stretching invariance and symmetry" will synonymously be used also for a single triangulation. The central result of [3] is the following expansion theorem:

Theorem 1. *Let the family of triangulations $\{T_H\}$ satisfy condition* (T). *Then, there holds*

$$(u - R_{11}u)(x) = \sum_{n=1}^{N} A_n(x) h_n^{2\alpha_{n;1}} + R_H(x) h^2 |\log(h)|, \quad x \in \Omega, \tag{2.10}$$

with certain functions $A_n(\cdot) \in C(\bar\Omega \setminus \{z_n\})$ *and a remainder term satisfying* $R_H \in C(\bar\Omega \setminus \hat Z) \cap L_q(\Omega)$, *for some* $q > 1$, *and* $\|R_H\|_{\infty,\mathrm{loc}} + \|R_H\|_q \le c$.

We suspect that the local symmetry of the triangulations, and the technical condition (a) of (T), are not essential for the validity of Theorem 1. In order to describe the explicit form of the coefficients $A_n(x)$, some further notation is needed. With the solution $v_{n;-i}$ of problem (2.1), corresponding to the data $f \equiv -\Delta s_{n;-i} \equiv 0$ and $b \equiv s_{n;-i}$, we define the functions $s'_{n;-i} \equiv s_{n;-i} - v_{n;-i}, i \in I\!N$. Clearly, each of $s'_{n;-i}$ is harmonic in Ω, identically zero on $\partial\Omega$, and has the same singular behavior near z_n as $s_{n;-i}$. Using this notation, the coefficients in the expansion (2.10) have the form

$$A_n(x) = A_n K_{n;1}(u) s'_{n,-1}(x), \tag{2.11}$$

with certain numbers A_n which are determined by the local structure of the triangulations at the corners z_n. Further, for the stress intensity factors there holds the formula

$$K_{n,1}(u) = \int_\Omega -\Delta u s'_{n;-1} dx - \int_{\partial\Omega} u \frac{\partial}{\partial n} s'_{n;-1} ds. \tag{2.12}$$

In view of the expansion (2.10), the dominant error terms can be eliminated by Richardson extrapolation. Since the influence of the several corner points z_n is localized by the appearence of the "local" mesh sizes h_n, one may work with meshes which are refined only in the neighborhoods Ω_n of the corners z_n. From [3], we recall the following algorithm:

Step 1. Choose an appropriate locally stretching invariant and symmetric triangulation T_H with mesh size vector $H \equiv \{h_0, h_1, \cdots, h_N\}$, and compute the Ritz projection $\mathcal{R}_H u$.

Step 2. Choose some $\kappa > 0$ and refine T_H locally in $\Omega_n, n = 1, \cdots, N$, in such a way that the resulting triangulation, T'_H, corresponds to the mesh size vector

$$H' \equiv \{h_0, \gamma_1 h_1, \cdots, \gamma_N h_N\}, \quad \gamma_n \equiv 2^{-\kappa/(2\alpha_{n;1})}.$$

The connection between the refined subtriangulations in Ω_n and the fixed triangulation in Ω_0 may be realized in the usual way, either by properly adjusting the triangles in the contact zone or by introducing blind nodes.

Setp 3. Corresponding to T'_H compute the Ritz projection $\mathcal{R}'_H u$ and form the linear combination

$$\tilde{\mathcal{R}}_H u \equiv (2^\kappa - 1)^{-1}(2^\kappa \mathcal{R}'_H u - \mathcal{R}_H u).$$

In the presence of only one reentrant corner (or if all corners have the same angle) one usually takes $\kappa = 2\alpha_*$. As an immediate consequence of the expansion (2.10), there holds

$$\|u - \tilde{\mathcal{R}}_H u\|_{\infty;\mathrm{loc}} + \|u - \tilde{\mathcal{R}}_H u\|_q = O(h^2|\log(h)|). \tag{2.13}$$

Next, we consider the eigenvalue problems (2.2) and (2.7). Again, let λ be an eigenvalue of (2.2) with multiplicity m, and let $\lambda_H^1, \cdots, \lambda_H^m$ and $w_H^1, \cdots, w_H^m$ be the approximating discrete eigenvalues and corresponding (ortho-normalized) eigenfunctions. We denote by $\pi, \pi^\perp$, and π_H the orthogonal projections onto the eigenspace $E(\lambda)$ of λ, onto its orthogonal complement $E(\lambda)^\perp$, and onto $\mathrm{span}[w_H^1, \cdots, w_H^m]$, respectively. Finally, we introduce the notation $K \equiv (-\Delta)^{-1}$ for the solution operator of (2.2). Notice that the operator $(I - \lambda K) : E(\lambda)^\perp \to E(\lambda)^\perp$ is an isomorphism. Employing this notation, we can state the following results.

Theorem 2. *Let the family of triangulations $\{T_H\}$ satisfy condition* (T). *Then, for the mean values $\bar{\lambda}_H \equiv \frac{1}{m}\sum_{i=1}^{m}\lambda_H^i$, there holds*

$$\bar{\lambda}_H - \lambda = \frac{1}{m}\lambda^2 \sum_{n=1}^{N} A_n h_n^{2\alpha_{n;1}} \|\pi s_{n;-1}\|^2 + O(h^2). \tag{2.14}$$

Further, for any function $v \in L^\infty(\Omega)$, there holds

$$(\pi v - \pi_H v)(x) = \lambda \sum_{n=1}^{N} A_n h_n^{2\alpha_{n;1}} \Big\{ (v, \pi s'_{n;-1}) e_n(x) + (v, e_n)\pi s'_{n;-1}(x) \Big\}$$
$$+ R_H(v;x) h^2 |\log(h)|, \tag{2.15}$$

for $x \in \Omega$, where $e_n \equiv (I - \lambda K)^{-1}\pi^\perp s'_{n;1} \in E(\lambda)^\perp$. The remainder term satisfies $R_H(v;\cdot) \in C(\bar{\Omega}/\hat{Z}) \cap L_q(\Omega)$, for some $q > 1$, and

$$\|R_H(v;\cdot)\|_{\infty;\mathrm{loc}} + \|R_H(v;\cdot)\|_q \le c\|v\|_\infty.$$

We note that, as a particular consequence of (2.15), the error for any continuous eigenfunction w can be expanded in the form

$$(w - \pi_H w)(x) = \lambda \sum_{n=1}^{N} A_n h_n^{2\alpha_{n;1}} (w, s'_{n;-1}) e_n(x) + R(w;x) h^2 |\log|(h)||. \tag{2.16}$$

The proof of Theorem 2 will be given in the next section. The constants A_n are the same as in (2.11).

Now let T'_H be a refined mesh as described in the extrapolation procedure above. Then, using the mean value $\bar{\lambda}'_H$ of the corresponding discrete eigenvalues, we obtain from (2.14) that

$$\tilde{\lambda}_H \equiv (2^\kappa - 1)^{-1}(2^\kappa \bar{\lambda}'_H - \bar{\lambda}_H) = \lambda + O(h^2). \tag{2.17}$$

Analogously, by (2.15), for any $v \in L^\infty(\Omega)$, the linear combination

$$\tilde{v} \equiv (2^\kappa - 1)^{-1}(2^\kappa \pi'_H v - \pi_H v) = \pi v + O(h^2 |\log(h)|) \tag{2.18}$$

provides an improved approximation of the continuous eigenfunction $\pi v \in E(\lambda)$, on interior subdomains Ω' bounded away from all corner points. In practice, for v in (2.18) one may take an orthonormal system of discrete eigenfunctions on the triangulation T'_H, yielding a basis of an m-dimensional subspace which approximates $E(\lambda)$ with order $O(h^2|\log(h)|)$.

The results of this paper are not restricted to linear finite elements. They easily carry over to more general triangular or quadrilateral elements, provided that the results from the local pointwise error analysis used in the proofs in [3] can be made available for them. For elements of higher order, or, for low order elements on certain uniform meshes (see Lin and Xie [5], and also [2] and [6]), one might seek to carry the expansions (2.14) and (2.5) further to higher order , $r > 2$, of the remainder terms. However, in the case of several reentrant corners, the order $r = 4\alpha_*$ seems to be the upper limit for the localization of the pollution effect to the neighborhoods of the corners.

§3. Proof of Theorem 2

First, we shall express the eigenvalue error $\lambda_H^i - \lambda$ in terms of the projection errors $w^{i,H} - \mathcal{R}_H w^{i,H}, i = 1, \cdots, m$. For abbreviation, we drop the superscripts i and H, setting $\lambda_H \equiv \lambda_H^i, w_H \equiv w_H^i$, and $w \equiv w^{i,H}$. Notice that $\|w\| = \|w_H\| = 1$. Then using equations (2.2) and (2.7), we see that

$$\begin{aligned}\|\nabla[w - w_H]\|^2 &= \|\nabla w\|^2 - 2(\nabla w, \nabla w_H) + \|\nabla w_H\|^2 \\ &= \lambda_H - \lambda + \lambda\{2 - 2(w, w_H)\} = \lambda_H - \lambda + \lambda\|w - w_H\|^2.\end{aligned}$$

Further, by the orthogonality properties of $\mathcal{R}_H$, there holds

$$\|\nabla[w - w_H]\|^2 = \|\nabla[w - \mathcal{R}_H w]\|^2 + \|\nabla[\mathcal{R}_H w - w_H]\|^2.$$

and, by analogous arguments,

$$\begin{aligned}\|\nabla[\mathcal{R}_H w - w_H]\|^2 &= (\nabla w, \nabla[\mathcal{R}_H w - w_H]) - (\nabla w_H, \nabla[\mathcal{R}_H w - w_H]) \\ &= \lambda(w, \mathcal{R}_H w - w_H) - \lambda_H(w_H, \mathcal{R}_H w - w_H) \\ &= \lambda(w - w_H, \mathcal{R}_H w - w_H) + (\lambda - \lambda_H)(w_H, \mathcal{R}_H w - w_H).\end{aligned}$$

Combining the foregoing identities, we obtain

$$\lambda_H - \lambda = \|\nabla[w - \mathcal{R}_H w]\|^2 + \lambda(w - w_H, \mathcal{R}_H w - w) + \{\lambda - \lambda_H\}(w_H, \mathcal{R}_H w - w_H). \quad (3.1)$$

Then, the low order error estimates (2.6), (2.8) and (2.9) imply that

$$\lambda_H - \lambda = \|\nabla[w - \mathcal{R}_H w]\|^2 + O(h^2). \quad (3.2)$$

Now, observing that $\|\nabla[w - \mathcal{R}_H w]\|^2 = \lambda(w, w - \mathcal{R}_H w)$, we might directly apply the pointwise expansion for $w - \mathcal{R}_H w$, given by Theorem 1. This, however, would result in an expansion with a remainder term of order $O(h^2|\log(h)|)$. In order to avoid the extra logarithm we proceed as follows.

Inserting the "singular" expansion (2.3), for w, into the identity (3.2), we obtain

$$\begin{aligned}\lambda_H - \lambda = &\sum_{n,m=1}^{N} \sum_{i=1}^{I_n} \sum_{j=1}^{I_m} K_{n;i}(w) K_{m;j}(w) (\nabla[s_{n;i} - \mathcal{R}_H s_{n;i}], \nabla[s_{m;j} - \mathcal{R}_H s_{m;j}]) \\ &+ 2 \sum_{n=1}^{N} \sum_{i=1}^{I_n} (\nabla[s_{n;i} - \mathcal{R}_H s_{n;i}], \nabla[w - \mathcal{R}_H w]) + \|\nabla[w - \mathcal{R}_H w]\|^2. \end{aligned} \quad (3.3)$$

The several terms on the right of (3.3) will be handled separately. First, since $w \in H^2(\Omega)$, there holds

$$\|\nabla[w - \mathcal{R}_H w]\|^2 = O(h^2). \quad (3.4)$$

In order to estimate the several inner products in (3.3), we shall use some of the technical notation and results from [3].

Let $R \equiv \frac{1}{4} \min\{\text{dist}(z_n, z_m), z_n, z_m \in \hat{Z}, z_n \neq z_m\}$, and, for $0 < a \leq R$,

$$\Omega_n^a \equiv \{x \in \Omega, \text{dist}(x, z_n) < a\}, \quad n = 1, \cdots, \hat{N}, \quad \Omega_0^a \equiv \Omega \setminus \bigcup_{n=1}^{N} \Omega_n^a.$$

Further, let $\{\psi_n, n = 0, \cdots, N\} \subset C_0^\infty(\mathbb{R}^2)$ be a partition of unity on Ω, such that

$$\sum_{m=0}^{N} \psi_m \Big|_{\Omega} \equiv 1, \quad \psi_n \Big|_{\Omega_n^R} \equiv 1, \quad \psi_n \Big|_{\Omega_0^{3R/2}} \equiv 0, \quad n = 1, \cdots, N.$$

Correspondingly, we introduce the weight functions

$$\rho_n \equiv \psi_0 + \psi_n (r_n^2 + \kappa^2 h^2)^{1/2}, \quad n = 1, \cdots, N,$$

and, for multi-indices $\gamma = (\gamma_1, \cdots, \gamma_N)$,

$$\rho_\pi^\gamma \equiv \prod_{m=1}^{N} \rho_m^{\gamma_m}.$$

The constant κ is taken sufficiently large, $\kappa \geq \kappa_0 \geq 1$, such that

$$\max_{K \in T_H} \left\{ \max_{y \in K} \rho_\pi^\gamma(y) / \min_{y \in K} \rho_\pi^\gamma(y) \right\} \leq c(\gamma). \tag{3.5}$$

In virtue of (3.5), the usual L_2-error estimates for the interpolation operator S_H carry over to weighted norms,

$$\|\rho_\pi^\gamma \nabla^r (\varphi - S_H \varphi)\| \leq c(\gamma) h^{2-r} \|\rho_\pi^\gamma \nabla^2 \varphi\|, \quad r \in \{0, 1\}. \tag{3.6}$$

Using this notation, we recall the following weighted norm error estimate from [3], Lemma 2.1,

$$\|\rho_\pi^\gamma \nabla [v - \mathcal{R}_H v]\| \leq c(\gamma) \{ \|\rho_\pi^\gamma \nabla [v - S_H v]\| + h^{-1} \|\rho_\pi^\gamma [v - S_H v]\| \}, \tag{3.7}$$

where $\gamma \equiv (\alpha_{l;1} - 1 - \varepsilon, \cdots, \alpha_{N;1} - 1 - \varepsilon)$ and $0 < \varepsilon << \alpha_*$. This holds for an arbitrary function $v \in H^1(\Omega) \cap C(\bar{\Omega})$, provided that the constant κ is chosen sufficiently large (independent of h).

Now, we are prepared to analyze the inner products in (3.3). Using the local approximation properties of S_H, we conclude that

$$\begin{aligned} |(\nabla [s_{n;i} - \mathcal{R}_H s_{n;i}], \nabla (w - \mathcal{R}_H w])| &= |(\nabla [s_{n;i} - S_H s_{n;i}], \nabla [w - \mathcal{R}_H w])| \\ &\leq \|\rho_\pi^{-\gamma} \nabla [s_{n;i} - S_H s_{s;i}]\| \|\rho_\pi^\gamma \nabla [w - \mathcal{R}_H w]\| \leq c(\gamma) h \|\rho_\pi^\gamma \nabla [w - \mathcal{R}_H w]\|, \end{aligned}$$

where γ is as defined in (3.7). Consequently, by (3.7) and by (3.6), we obtain in view of (2.4) that

$$\begin{aligned} |(\nabla [s_{n;i} - \mathcal{R}_H s_{n;i}], \nabla [w - \mathcal{R}_H w])| &\leq c(\gamma) h \{ \|\rho_\pi^\gamma \nabla [w - S_H w]\| \\ &+ h^{-1} \|\rho_\pi^\gamma [w - S_H w]\| \} \leq c h^2 w \|\rho_\pi^\gamma \nabla^2 w\| = O(h^2). \end{aligned} \tag{3.8}$$

From Lemma 3.1 of [3], we recall the estimate

$$(\nabla [s_{n;i} - \mathcal{R}_H s_{n;i}], \nabla [s_{m;j} - \mathcal{R}_H s_{m;j}]) = O(h^2), \tag{3.9}$$

for $m \neq n$, and

$$(\nabla [s_{n;i} - \mathcal{R}_H s_{n;i}], \nabla [s_{n;j} - \mathcal{R}_H s_{n;j}]) = O(h^{\min\{\alpha_{n;i} + \alpha_{n;j}, 2\}} |\log(h)|^r); \tag{3.10}$$

where $r = 1$, for $\alpha_{n;i} + \alpha_{n;j} = 2$, and $r = 0$, else. Finally, Theorem 3 of [3] states that, for $\alpha_{n;i} + \alpha_{n;j} < 2$,

$$A_n^{(i,j)}(H) \equiv h_n^{-\alpha_{n;i}-\alpha_{n;j}}(\nabla[s_{n;i} - \mathcal{R}_H s_{n;i}], \nabla[s_{n;j} - \mathcal{R}_H s_{n;j}]) = A_n^{(i,j)} + O(h^{2-\alpha_{n;i}-\alpha_{n;j}}), \tag{3.11}$$

with certain numbers $A_n^{(i,j)}$. If the triangulations are locally symmetric, then $A_n^{(i,j)} = 0$, for $i \neq j$. Collecting the foregoing results and setting $A_n \equiv A_n^{(1,1)}$, we obtain the following expansion for a single eigenvalue

$$\lambda_H - \lambda = \sum_{n=1}^{N} A_n K_{n;1}(w)^2 h_n^{2\alpha_{n;1}} + O(h^2). \tag{3.12}$$

From the representation formula (2.12), we see that

$$K_{n;1}(w) = \lambda(w, s'_{n;-1}). \tag{3.13}$$

Therefore, reintroducing the superscripts i, H, and summing up the identity (3.12), we arrive at

$$\sum_{i=1}^{m} \lambda_H^i - m\lambda = \lambda^2 \sum_{n=1}^{N} A_n \sum_{i=1}^{m} (w^{i,H}, s'_{n;-1})^2 h_n^{2\alpha_{n;1}} + O(h^2) = \lambda^2 \sum_{n=1}^{N} A_n \|\pi s'_{n;-1}\|^2 h_n^{2\alpha_{n;1}} + O(h^2).$$

This proves the eigenvalue expansion (2.14).

Next, we prove the expansion (2.15) for the error between the projections π and π_H. First, for any function $v \in L^\infty(\Omega)$, there holds

$$\begin{aligned} \pi v - \pi_H v &= \sum_{i=1}^{m} \left\{ (v, w^{i,H}) w^{i,H} - (v, w_H^i) w_H^i \right\} \\ &= \sum_{i=1}^{m} \left\{ (v, \pi^\perp[w^{i,H} - w_H^i]) w^{i,H} + (v, w^{i,H}) \pi^\perp [w^{i,H} - w_H^i] \right\} \\ &+ \sum_{i=1}^{m} \left\{ (v, \pi[w^{i,H} - w_H^i]) w^{i,H} + (v, w^{i,H}) \pi[w^{i,H} - w_H^i] \right\} \\ &+ \sum_{i=1}^{m} (v, w_H^i - w^{i,H}) \{ w^{i,H} - w_H^i \}. \end{aligned} \tag{3.14}$$

In view of the basic estimates (2.9), the last sum on the right is of the order $O(h^2)$ in $L^2(\Omega) \cap L^\infty_{\text{loc}}$. Further, using the identity

$$(w^{i,H} - w_H^i, w^{j,H}) + (w^{j,H} - w_H^j, w^{i,H}) = (w^{i,H} - w_H^i, w^{j,H} - w_H^j), \quad j = 1, \cdots, m,$$

we obtain for the second sum on the right of (3.14) that

$$\sum_{i=1}^{m} \left\{ (v, \pi(w^{i,H}, w_H^i)) w^{i,H} + (v, w^{i,H}) \pi(w^{i,H} - w_H^i) \right\}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m}\left\{(v, w^{j,H})(w^{i,H} - w_H^i, w^{j,H})w^{i,H} + (v, w^{i,H})(w^{i,H} - w_H^i, w^{j,H})w^{j,H}\right\}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m}(v, w^{j,H})(w^{i,H} - w_H^i, w^{j,H} - w_H^j)w^{i,H} = O(h^2), \tag{3.15}$$

in $L^2(\Omega) \cap L^\infty_{\text{loc}}$.

Therefore, it remains to expand the error $\pi^\perp(w^{i,H} - w_H^i)$ for each single eigenfunction. As above, we drop the superscripts i and H, and simply set $w = w^{i,H}$. Following an idea of Q. Lin, we use the operators $K \equiv (-\Delta)^{-1}$ and $K_H \equiv \mathcal{R}_H K$, to write

$$w - w_H = \lambda K w - \lambda_H K_H w_H = \lambda K(w - w_H) + B_H + C_H, \tag{3.16}$$

where

$$B_H \equiv (\lambda - \lambda_H)Kw + \lambda(K - K_H)w = \frac{1}{\lambda}(\lambda - \lambda_H)w + w - \mathcal{R}_H w,$$

$$C_H \equiv (\lambda - \lambda_H)(K_H - K)w + (\lambda - \lambda_H)K(w_H - w) + \lambda_H(K - K_H)(w_H - w).$$

Consequently, there holds

$$(I - \lambda K)(w - w_H) = B_H + C_H. \tag{3.17}$$

Analogously to the basic estimate (2.6) we have that

$$\|(K - K_H)f\|_{\infty;\text{loc}} + \|(K - K_H)f\| \le Ch\|f\|, \quad f \in L^2(\Omega).$$

Using this together with (2.8) and (2.9), we see that C_H satisfies

$$\|C_H\|_{\infty;\text{loc}} + \|C_H\|_q = O(h^2|\log(h)|). \tag{3.18}$$

This is the standard estimate for all of the remainder terms in the subsequent identities. For B_H, we apply Theorem 1, (2.11), and (3.12), to obtain

$$B_H = \lambda \sum_{n=1}^{N} A_n(w, s'_{n;-1})h_n^{2\alpha_{n;1}}\{s'_{n;-1} - (w, s'_{n;-1})w\} + D_H, \tag{3.19}$$

where D_H also satisfies (3.18). Next, using (3.17), (3.18), and (3.19), we see that, for any of the basis functions $w_H^k, k = 1, \cdots, m$, there holds

$$0 = (w^{k,H}, (I - \lambda K)(w - w_H))w^{k,H}$$

$$= \lambda \sum_{n=1}^{N} A_n(w, s'_{n;-1})h_n^{2\alpha_{n;1}}\{(w^{k,H}, s'_{n;-1}) - (w, s'_{n;-1})(w^{k,H}, w)\}w^{k,H} + E_H^k,$$

where E_H^k satisfies (3.18). Summing this over k and observing that $\|w\|^2 = 1$ and $(w^{k,H}, w) = 0$, for $w \ne w^{k,H}$, we obtain that

$$\lambda \sum_{n=1}^{N} A_n(w, s'_{n;-1})h_n^{2\alpha_{n;1}}\pi s'_{n;-1} = \lambda \sum_{n=1}^{N} A_n(w, s'_{n;-1})h_n^{2\alpha_{n;1}}(w, s'_{n;-1})w + E_H, \tag{3.20}$$

where E_H also satisfies (3.18). We note that the identity (3.20) implicitly contains a condition to be satisfied by the stress intensity factors of the eigenfunctions of multiple eigenvalues. Using (3.20) together with (3.19) we arrive at the expansion

$$(I - \lambda K)(w - w_H) = \lambda \sum_{n=1}^{N} A_n(w, s'_{n;-1}) h_n^{2\alpha_{n;1}} \pi^{\perp} s'_{n;-1} + F_H, \tag{3.21}$$

where F_H satisfies (3.18). Notice that $F_H \in E(\lambda)^{\perp}$. This implies that

$$\pi^{\perp}(w - w_H) = \lambda \sum_{n=1}^{N} A_n(w, s'_{n;-1}) h_n^{2\alpha_{n;1}} e_n + G_H. \tag{3.22}$$

where $e_n = (I - \lambda K)^{-1}\pi^{\perp} s'_{n;-1}$, and the remainder $G_H \equiv (I - \lambda K)^{-1} F_H$ satisfies (3.18), as is guaranteed by the open mapping theorem.

Now, we reintroduce the superscripts i and H, obtaining

$$\pi^{\perp}(w^{i,H} - w_H^i) = \lambda \sum_{n=1}^{N} A_n(w^{i,H}, s'_{n;-1}) h_n^{2\alpha_{n;1}} e_n + G_H^i, \tag{3.23}$$

where the remainder G_H^i satisfies (3.18). Inserting this into (3.14) we arrive at the expansion

$$\pi v - \pi_H v = \lambda \sum_{i=1}^{m} A_n h_n^{2\alpha_{n;1}} \{(v, e_n)\pi s'_{n;-1} + (v, \pi s'_{n;-1}) e_n\} + O(h^2|\log(h)|), \tag{3.24}$$

in $L^q(\Omega) \cap L^\infty_{\text{loc}}$. This completes the proof of the theorem.

§4. Numerical Examples

In order to illustrate the results of the paper, we report some numerical test calculations for the model eigenvalue problem (2.2). Figure 2, below, shows the meshes corresponding to the mesh size $h = 1/4$. The successively refined meshes are obtained, for simplicity, by globally subdividing each triangle into four congruent subtriangles. The tables, below, show the errors in the approximation of the smallest eigenvalue.

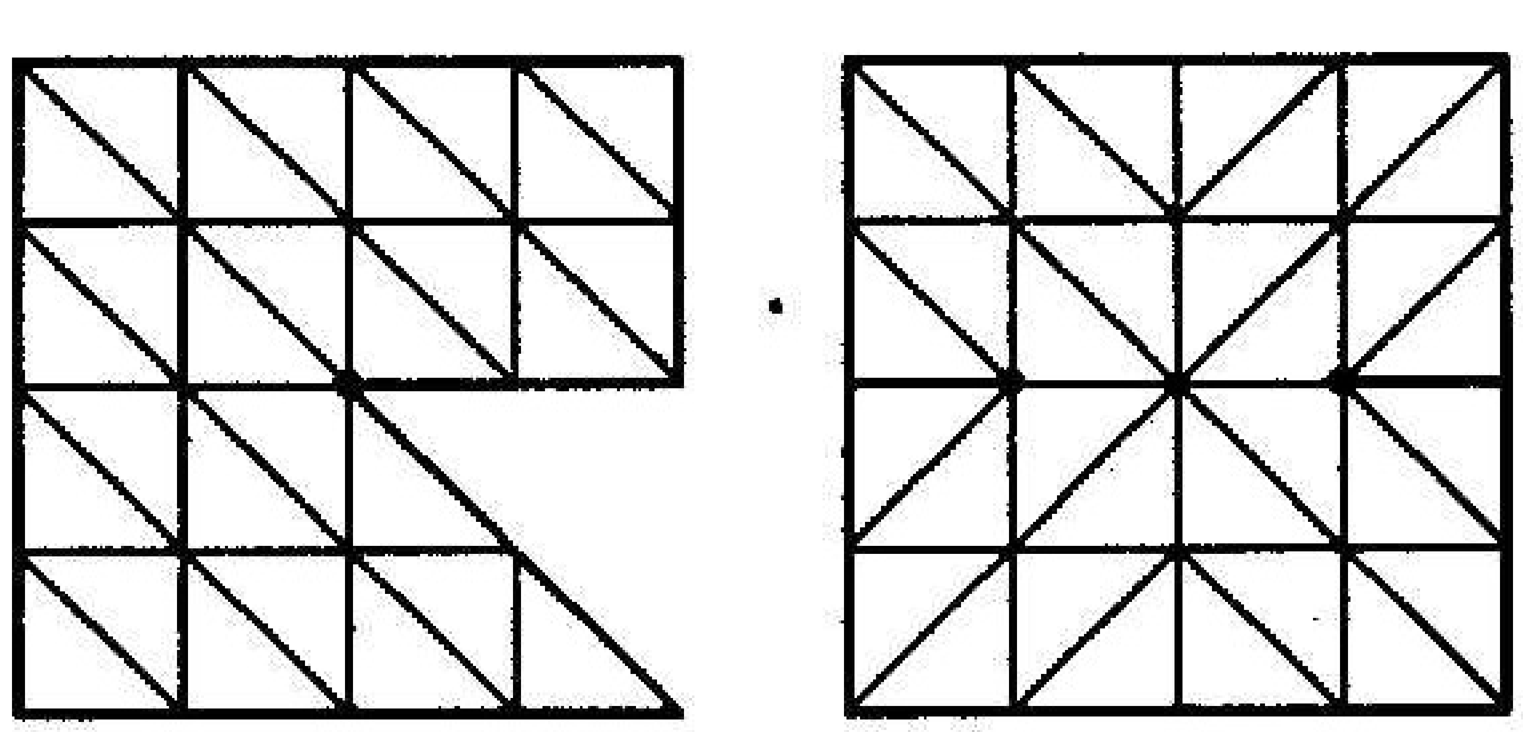

Fig 2. Model problem $-\Delta u = f$

Example 1. For a domain with a 45°-crack, $\omega = \frac{7}{4}\pi$, the relevant singular exponent is $\alpha_1 = \frac{4}{7}$. The expected order of convergence for the eigenvalues is $O(h^{8/7})$. One step of $h^{8/7}$-extrapolation raises the order to $O(h^2)$.

h^{-1}	λ_H	$(\lambda_H - \lambda)/\lambda$	ratio	$\tilde{\lambda}_H$	$(\tilde{\lambda}_H - \lambda)/\lambda$	ratio
8	40.318	.1315				
16	37.189	.04372	.332	34.599	−.02898	
32	36.208	.01620	.370	35.403	−.00641	.221
64	35.862	.00647	.399	35.575	−.00157	.245
128	35.728	.00272	.420	35.617	−.00038	.245
∞	35.631		.453	35.631		.250

Example 2. For a domain with two slits, $\omega = 2\pi$, the relevant singular exponent is $\alpha_1 = \frac{1}{2}$. The expected order of eigenvalue convergence is $O(h)$. One step of h-extrapolation eliminates the pollution effect of both slits.

h^{-1}	λ_H	$(\lambda_H - \lambda)/\lambda$	ratio	$\tilde{\lambda}_H$	$(\tilde{\lambda}_H - \lambda)/\lambda$	ratio
8	33.627	.1954				
16	30.318	.07773	.397	27.008	−.03991	
32	29.100	.03445	.443	27.883	−.00882	.221
64	28.586	.01619	.470	28.072	−.00207	.235
128	28.351	.00784	.484	28.117	−.00050	.242
∞	28.131		.500	28.131		.250

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