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AN ITERATIVE ALGORITHM FOR THE COEFFICIENT INVERSE PROBLEM OF DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, an iterative algorithm for solving a coefficient inverse problem is submitted. The key of the method is to project an unknown coefficient function on a finite dimensional function space. Thus, the inverse problem can be changed into a nonlinear algebraic system of equations.

§1. Introduction

Now, the coefficient inverse problem of differential equations is becoming more and more noticeable in the fields of economics, science and technology, national defence and so on. It can be applied to many aspects such as resources prospecting, system identification, telemetering and remote sensing, etc. Many authors studied the problem and presented a lot of valuable results in theoretical analysis and numerical computation[1,2,3]. However, since the inverse problem is nonlinear and ill-posed, much difficulty remains in theoretical analysis and numerical solution. So, there are still a lot of problems to be solved in both respects.

We will take the coefficient inverse problem of the one-dimensional convection-diffusion equation for example and derive an algorithm to solve it.

Consider the initial-boundary problem

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\Big(k(x)\frac{\partial u}{\partial x}\Big)+v(x,t)\frac{\partial u}{\partial x}=f(x,t),\quad 0<x<1,\quad t>0,\tag{1.1}$$

$$u(x,0)=u_0(x),\quad 0<x<1,\tag{1.2}$$

$$u(0,t)=g_0(t),\quad u(1,t)=g_1(t),\quad t>0,\tag{1.3}$$

with additional measured condition

$$Pu(\cdot,t)=h(t),\quad t>0,\tag{1.4}$$

where $k\in æ=\{k\in H^1[0,1];k(x)\ge k_*>0,x\in(0,1)\}, v,f,g_0,g_1$ and h are given functions, and P is an operator, for example

$$Pu(\cdot,t)=u(x_0,t),\quad x_0\in(0,1)\tag{1.5}$$

or

$$Pu(\cdot,t)=\frac{\partial u}{\partial x}(0,t),\tag{1.6}$$

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and so on.

By the general theory of differential equations, the (classical or generalized) solution of problem (1.1)–(1.3) exists and is unique under certain conditions. But now, we aim at finding $k \in æ$, such that the solution $u = u(k; x, t)$ of problem (1.1)–(1.3) satisfies the additional measured condition (1.4). This is the "inverse problem".

This problem has an important background in physics. If $v = 0$, (1.1) is a heat conduction equation, the problem is to find the thermal conductivity of an inhomogeneous material[5]. If $v \neq 0$, it is a problem of identification of the diffusion coefficient in a water quality model for a river[4]. The existence and uniqueness of the solution of the inverse problem have been discussed in [6]; we only put forward an algorithm to solve it.

§2. An Iterative Algorithm

Through a simple map, (1.1)–(1.4) can be changed into the following form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}(k(x)\dfrac{\partial u}{\partial x}) + v(x,t)\dfrac{\partial u}{\partial x} = f(x,t) - g(t)k'(x), \quad 0 < x < 1, \quad t > 0, & (2.1) \\ u(x,0) = u_0(x), \quad 0 < x < 1, & (2.2) \\ u(0,t) = u(1,t) = 0, \quad t > 0 & (2.3) \end{cases}$$

and

$$Pu(\cdot, t) = h(t), \tag{2.4}$$

where v, f, g, u_0, and h are given functions.

2.1. Approximation of the inverse problem

Suppose $e_1, \cdots, e_n$ are linearly independent functions in $H_0^1[0,1]$, and $E_n =$ span $\{e_1, \cdots, e_n\}$. For any $k \in æ$, let

$$u^n = \sum_{j=1}^{n} c_j(t)e_j(x)$$

be an approximate solution of problem (2.1)–(2.3).

Taking the inner product of $e_j (j = 1, \cdots, n)$ with equation (2.1) in $L^2[0,1]$, we have

$$(\frac{\partial u}{\partial t}, e_j) + (k\frac{\partial u}{\partial x}, e_j') + (v\frac{\partial u}{\partial x}, e_j) = (f, e_j) + g(t)(k, e_j'), \quad j = 1, \cdots, n.$$

Substituting u by $u^n = \sum_{j=1}^{n} e_j(x)c_j(t)$ in the above formula and making a simple arrangement, we obtain

$$A_n\frac{dC}{dt} + [B_n(k) + V_n(t)]C(t) = F_n(t) + g(t)R_n(k),$$

where

$$\begin{cases} A_n = [(e_i, e_j)]_{n\times n}, \\ B_n(k) = [(ke_i', e_j')]_{n\times n}, \\ V_n(t) = [(v(t)e_i', e_j)]_{n\times n}, \\ F_n(t) = ((f, e_1), \cdots, (f, e_n))^T, \\ R_n(k) = ((k, e_1), \cdots, (k, e_n))^T, \\ C(t) = (c_1(t), \cdots, c_n(t))^T; \end{cases} \tag{2.5}$$

$(\cdot, \cdot)$ is the inner product in $L^2(0,1)$. Let $C^0 = (c_1^0, \cdots, c_n^0)$ such that $u_0^n = \sum_{j=1}^{n} c_j^0 e_j(x)$ is an orthogonal projection of $u_0(x)$ onto E_n. Then

$$\begin{cases} A_n \dfrac{dC}{dt} + (B_n(k) + V_n(t))C(t) = F_n(t) + g(t)R_n(k), & (2.6) \\ C(0) = C^0 & (2.7) \end{cases}$$

can be take as an approximation of problem (2.1)–(2.3). Obviously, the solution C of problem (2.6)–(2.7) depends not only on time t but also on $k \in æ$, i.e. $C = C(k;t)$.

Define the nonlinear operator Q^0 as follows: $\forall k \in æ$,

$$Q^0(k;t) = \sum_{j=1}^{n} c_j(k;t) Pe_j(\cdot),$$

where $C(k;t) = (c_1(k;t), \cdots, c_n(k,t))^T$ is the solution of problem (2.6)–(2.7) for a given $k \in æ$.

Thus, the coefficient inverse problem presented in §1 can be replaced approximately by

Problem (I). Find $k \in æ$, such that

$$Q^0(k;t) = h(t). \tag{2.8}$$

Because of the nonlinearity of the operator equation (2.8), it is required to use an iterative algorithm to find its solution. Up to now, the Fréchet derivative of the nonlinear operator $Q^0(k;t)$ with respect to k, $Q_k^{0\prime}(k;t)$ is needed in most methods. So, how to find this kind of Fréchet derivative is the key of many algorithms, such as PST method[5], the perturbation method[2], and TCC method[7]. In this paper, we will give another one.

2.2. Solving problem (I) approximately

Suppose $\psi_0, \cdots, \psi_m$ are linearly independent functions, and $\sum_{j=0}^{m} \psi_j = 1, \psi_j \geq 0, j = 0, \cdots, m$. Let

$$\Psi_m = \text{span } \{\psi_0, \cdots, \psi_m\}.$$

Then we have

$$k = \sum_{j=0}^{m} \alpha_j \psi_j(x), \quad \forall \ k \in æ \cap \Psi_m,$$

$\alpha_j \in R, j = 0, \cdots, m$. Define a set

$$D = \{\alpha = (\alpha_0, \cdots, \alpha_m) \in R^{m+1}; \quad \sum_{j=0}^{m} \alpha_j \psi_j(x) \geq k_*, \quad x \in [0,1]$$

If we solve the operator equation (2.8) on the set æ $\cap \Psi_m$, problem (I) is equivalent to

Problem (II). Find $\alpha \in D$ such that

$$Q(\alpha; t) = h(t), \tag{2.9}$$

where

$$Q(\alpha; t) = Q(\alpha_0, \cdots, \alpha_m; t) = Q^0\Big(\sum_{j=0}^{m} \alpha_j \psi_j; t\Big).$$

Thus, for any $\alpha^* \in D$, the Fréchet derivative

$$Q'_\alpha(\alpha^*; t) = \Big(\frac{\partial Q(\alpha^*; t)}{\partial \alpha_0}, \cdots, \frac{\partial Q(\alpha^*; t)}{\partial \alpha_m}\Big)^T \tag{2.10}$$

is an $(m+1)$-dimensional vector function. We will discuss how to find $Q'_\alpha(\alpha^*; t)$ in the next section.

Now, we will give an iterative scheme to solve problem (II). Suppose $r \in R^{m+1}$. We define the functional

$$J_\omega[\alpha^*, r] = \|Q'_\alpha(\alpha^*; t) \cdot r + Q(\alpha^*; t) - h(t)\|_S^2 + \omega \|r\|_2^2,$$

where $S = L^2(0,T), T > 0$ is a given number, and $\|\cdot\|_2$ is the Euclidean norm in R^{m+1}. The following Newton-regularized iterative scheme will be used to find the solution of equation (2.9):

$$\alpha^{(l+1)} = \alpha^{(l)} + r^{(l)}, \quad l = 0, 1, 2, \cdots \tag{2.11}$$

where $r^{(l)} \in R^{m+1}$ is the minimum point of the functional $J_{\omega^{(l)}}[\alpha^{(l)}, r]$, i.e.,

$$J_{\omega^{(l)}}[\alpha^{(l)}, r^{(l)}] = \inf_{r \in R^{m+1}} J_{\omega^{(l)}}[\alpha^{(l)}, r], \tag{2.12}$$

and $\omega^{(l)} > 0$ is a chosen number.

By the variational principle, problem (2.12) is equivalent to the Euler equation

$$(D^{(l)} + \omega^{(l)} E) r^{(l)} = Z^{(l)}, \tag{2.13}$$

where

$$D^{(l)} = \Big[\Big(\frac{\partial Q(\alpha^{(l)})}{\partial \alpha_i}, \frac{\partial Q(\alpha^{(l)})}{\partial \alpha_j}\Big)_S\Big]_{(m+1)\times(m+1)},$$

$$Z^{(l)} = \Big[\Big(h - Q(\alpha^{(l)}), \frac{\partial Q(\alpha^{(l)})}{\partial \alpha_0}\Big)_S, \cdots, \Big(h - Q(\alpha^{(l)}), \frac{\partial Q(\alpha^{(l)})}{\partial \alpha_m}\Big)_S\Big]^T,$$

and E is an identity matrix. So, (2.11)–(2.12) become

$$\begin{cases} \alpha^{(l+1)} = \alpha^{(l)} + r^{(l)}, \\ (D^{(l)} + \omega^{(l)} E) r^{(l)} = Z^{(l)}, \quad l = 0, 1, 2, \cdots. \end{cases} \tag{2.14}$$

2.3. Computation of the Fréchet derivative $Q'_\alpha(\alpha^{(l)};t)$

As discussed above, if we find $\dfrac{\partial Q(\alpha^{(l)};t)}{\partial \alpha_j}$, $j=0,\cdots,m, Q'_\alpha(\alpha^{(l)},t)$ is achieved immediately by (2.10).

For any $\alpha \in D$, let $k=\sum_{j=0}^{M}\alpha_j\psi_j$. Then $k \in æ$. It follows from (2.5) that

$$B_n(k)=\sum_{j=0}^{m}\alpha_j H_j,$$

$$R_n(k)=\sum_{j=0}^{m}\alpha_j Y_j,$$

where

$$H_j=[(\psi_j e'_i, e'_l)]_{n\times n}, \quad j=0,\cdots,m,$$
$$Y_j=[(\psi_j,e'_1),\cdots,(\psi_j,e'_n)]^T, j=0,\cdots,m.$$

(2.6)–(2.7) become

$$\begin{cases} A_n\dfrac{dC}{dt}+\Big[\sum_{j=0}^{m}\alpha_j H_j+V_n(t)\Big]C=F_n(t)+g(t)\sum_{j=0}^{m}\alpha_j Y_j, \\ C(0)=C^0. \end{cases}$$

Let $W^{(i)}=\dfrac{\partial C(\alpha^{(l)};t)}{\partial \alpha_i}, i=0,\cdots,m$. Then $W^{(i)}$ satisfies the following equations

$$\begin{cases} A_n\dfrac{dW^{(i)}}{dt}+[B_n(k^{(l)})+V_n(t)]W^{(i)}=g(t)Y_i-H_iC(\alpha^{(l)};t), & (2.15)\\ W^{(i)}(0)=0, \qquad i=0,\cdots,m & (2.16)\end{cases}$$

where

$$B_n(k^{(l)})=\sum_{j=0}^{m}\alpha_j^{(l)}H_j, \quad k^{(l)}=\sum_{j=0}^{m}\alpha_j^{(l)}\psi_j,$$

and $C(\alpha^{(l)};t)$ is the solution of (2.6)–(2.7) for $k=k^{(l)}$.

From the expression of $Q(\alpha;t)$, we can see

$$\frac{\partial Q(\alpha^{(l)};t)}{\partial \alpha_i}=\sum_{j=0}^{m}W_j^{(i)}(t)Pe_j(\cdot), \quad i=0,\cdots,m. \tag{2.17}$$

Thus, we can get $Q'_\alpha(\alpha^{(l)};t)$ through (2.15)–(2.17).

2.4. The choice of regularized parameter $\omega^{(l)}$

The choice of the regularized parameter $\omega^{(l)}$ plays an important role in the iterative scheme (2.14). It affects the convergence and the rate of convergence of the scheme. However, so far, there is no fixed principle to follow to choose the parameter and there is no direct and convenient method to apply. Next, we will propose a self-adaptive algorithm to choose the parameter $\omega^{(l)}$, so that the iterative process is convergent. By the way, we do not think our method is the best.

Let

$$\varphi(\alpha) = \|Q(\alpha;t) - h(t)\|_S^2.$$

The following is our algorithm:

(1) Pick an initial guess $\alpha^{(0)}$, and control quantities $\varepsilon_1 > 0, \varepsilon_2 > 0$, and $\mu > 1$ (μ can be 2, 5 or 10 etc.);

(2) Compute $C(\alpha^{(0)};t), Q(\alpha^{(0)};t), Q'_\alpha(\alpha^{(0)};t), \varphi(\alpha^{(0)})$, matrix $D^{(0)}$ and vector $Z^{(0)}$. Let $\omega^{(0)} = \varphi(\alpha^{(0)})$. If $\omega^{(0)} < \varepsilon_1$, then $\alpha^{(0)}$ can be considered as the approximate solution and the process can be ended. Else, turn to (3);

(3) Solving the system of equations

$$(D^{(0)} + \omega^{(0)}E)r^{(0)} = Z^{(0)},$$

we obtain $r^{(0)}(\omega^{(0)})$. Compute $\alpha^{(1)} = \alpha^{(0)} + r^{(0)}(\omega^{(0)}), \varphi(\alpha^{(1)})$. If $\varphi(\alpha^{(0)}) < \varphi(\alpha^{(1)})$, then turn to (4); else turn to (5);

(4) Let $\omega^{(0)} = \mu\omega^{(0)}$; turn to (3);

(5) If $\|r^{(0)}\|_2 < \varepsilon_2$, $\alpha^{(1)}$ is the approximate solution we want; else, let $\alpha^{(0)} = \alpha^{(1)}$ and turn to (2).

Remark. Step (4) is based on the work of Wang [8].

§3. Numerical Examples and Discussion

We have not yet demonstrated the convergence of the above algorithm. In order to test the feasibility of the scheme, some examples are given. See Figs. 1–15. These numerical results have fully proved that our algorithm is effective in practice.

In Figs. 1–5, the operator P takes the form of (1.6), i.e. $Pu(\cdot,t) = \partial u/\partial x(0,t)$; in Figs. 6–15, P takes the form of (1.5), i.e. $Pu(\cdot,t) = u(x_0,t), x_0 \in (0,1)$. In Figs. 1–9, the values of the unknown function $k(x)$ at the ends of the interval [0, 1], $k(0)$ and $k(1)$, are priori known, while in the remaining figures, we know nothing about function $k(x)$. We have the following experiences in numerical experiments: Whether $k(0)$ and $k(1)$ are known or not will not affect the convergence of the algorithm. But, it will produce a more or less impact on the rate of convergence, especially for the case that the operator P takes the form of (1.6). In addition, the choice of the operator P, that is, the nature of measurement data, will affect considerably the convergence and the rate of convergence of the algorithm.

In light of the main idea of our method, it is clear that the algorithm can be extended to the coefficient inverse problems of all the linear differential equations and linear operator equations, including the determination of several unknown coefficient functions. Some results will be reported in the near future.

Compared with some algorithm presented already, ours avoids finding Green's function (PST method) and solving the adjoint equation (perturbation method). However, a large amount of computation has not been reduced yet. The shortcoming can be overcome through parallel computation or a kind of discrete technique. This point will be shown in another paper.

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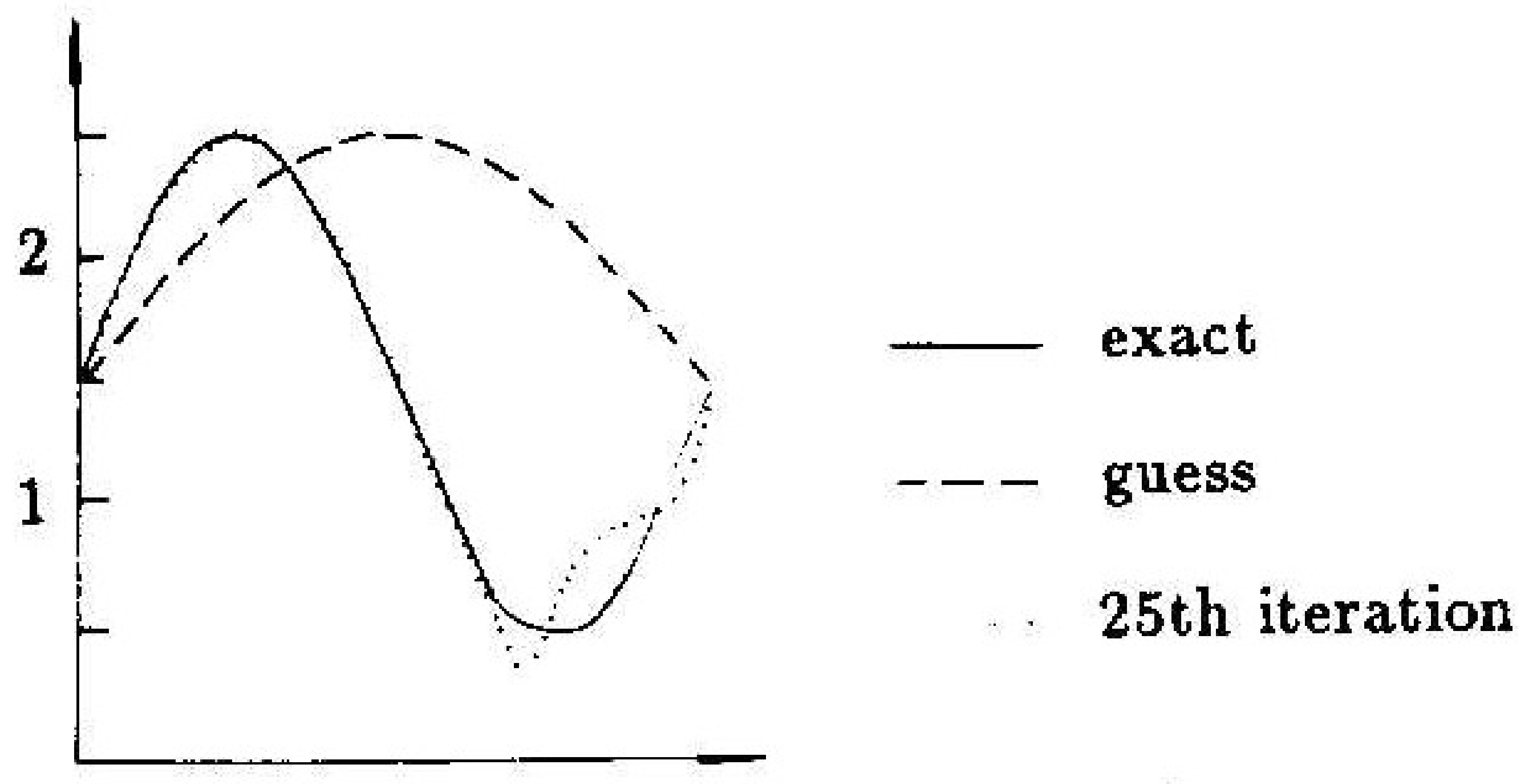

Fig. 1

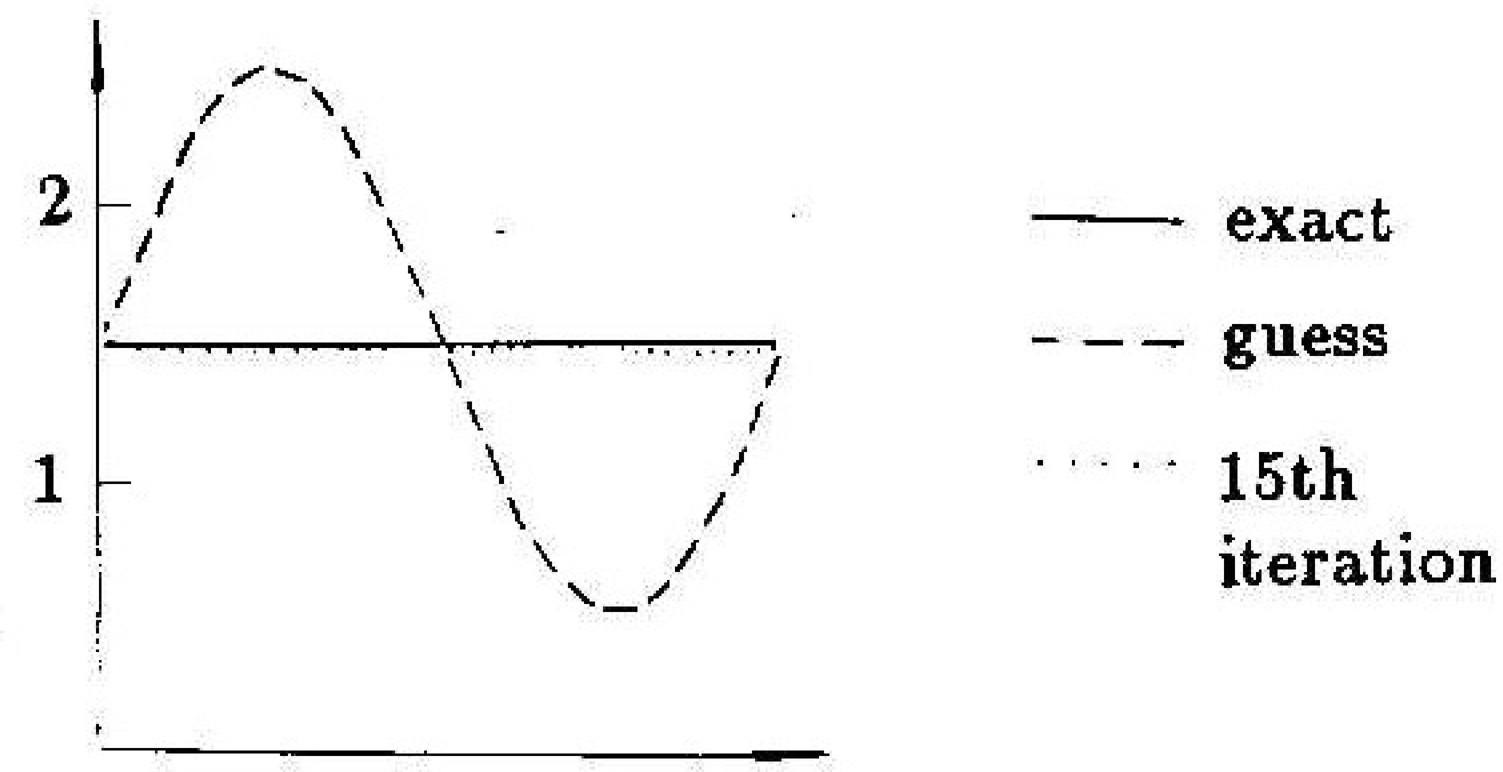

Fig. 2

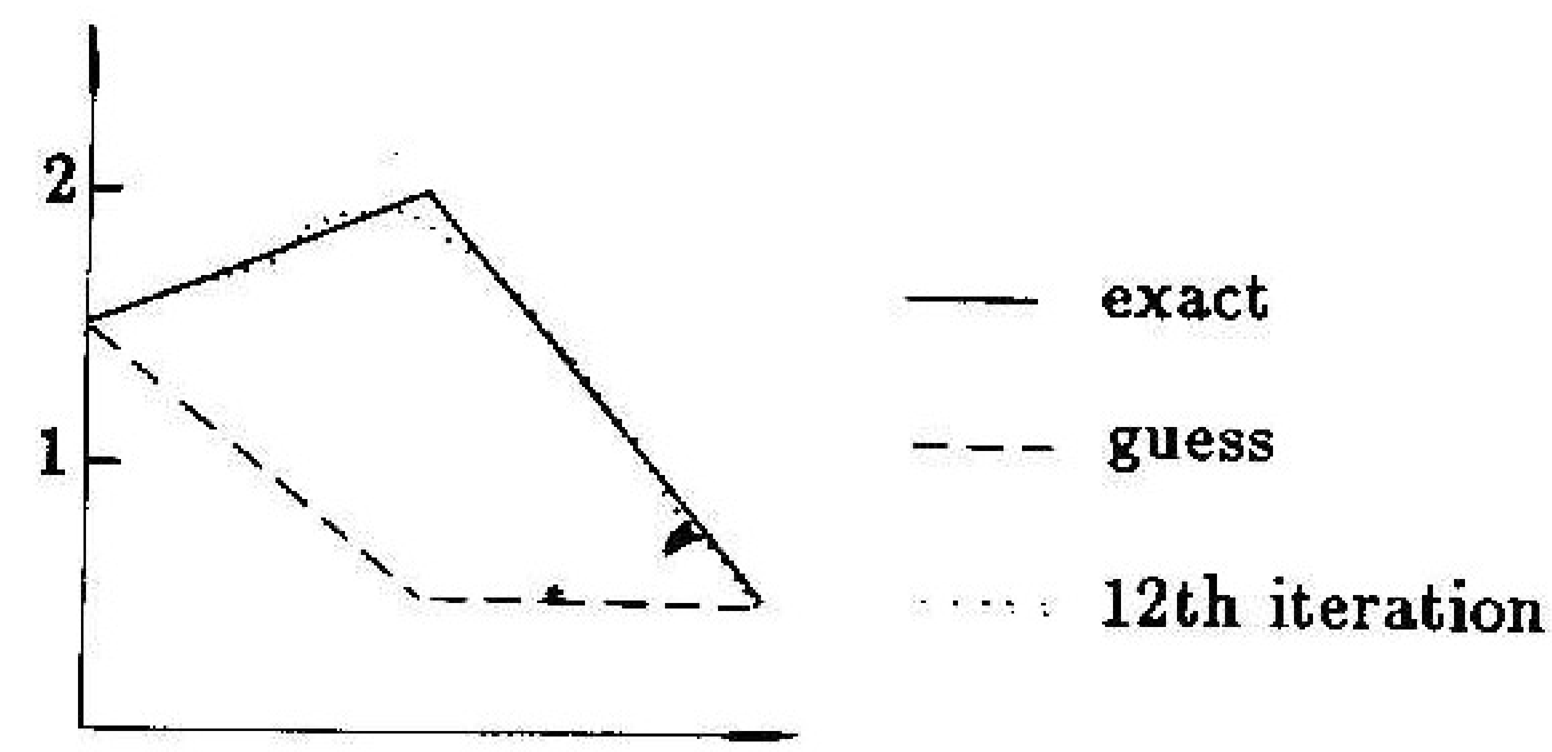

Fig. 3

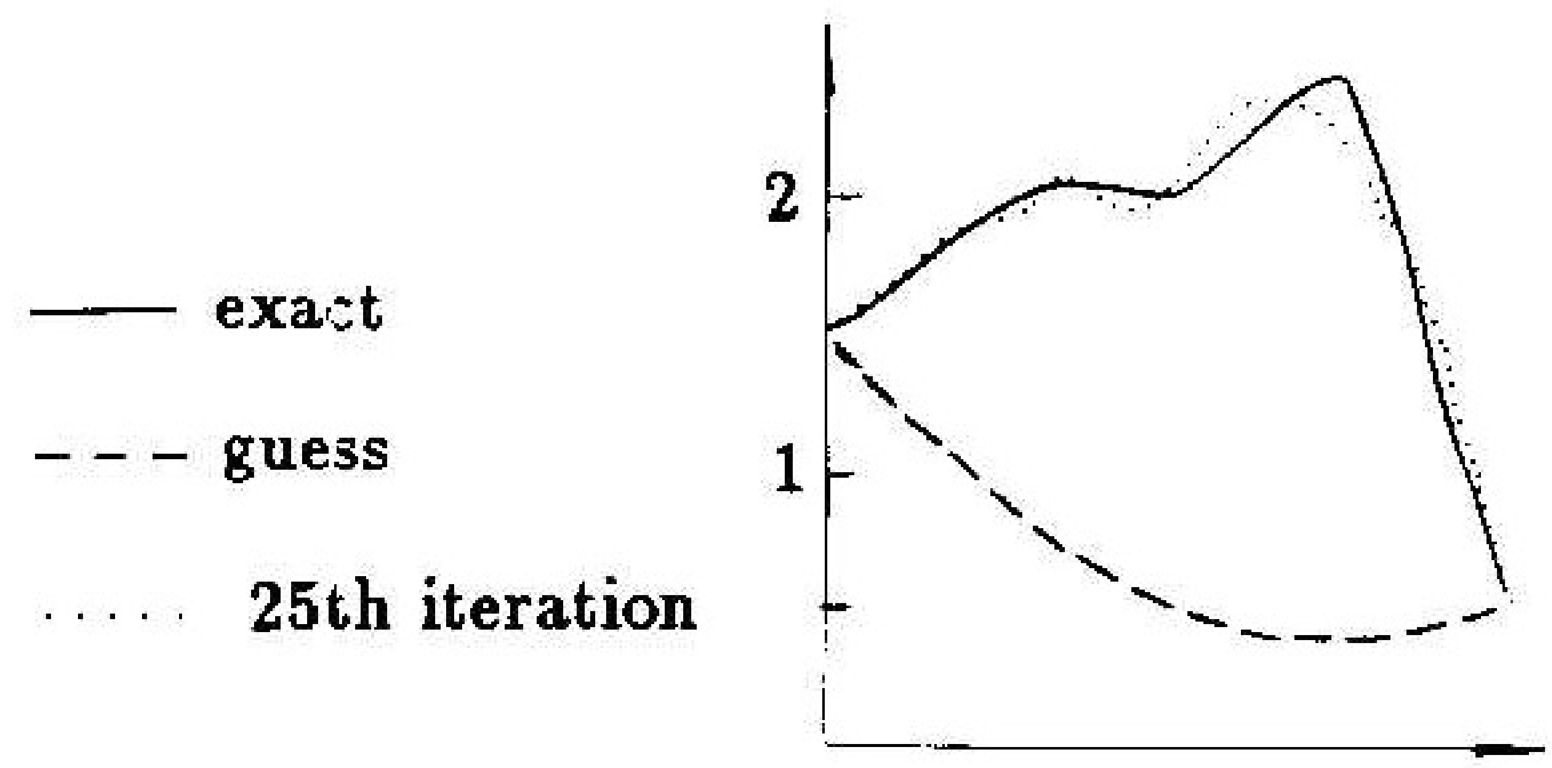

Fig. 4

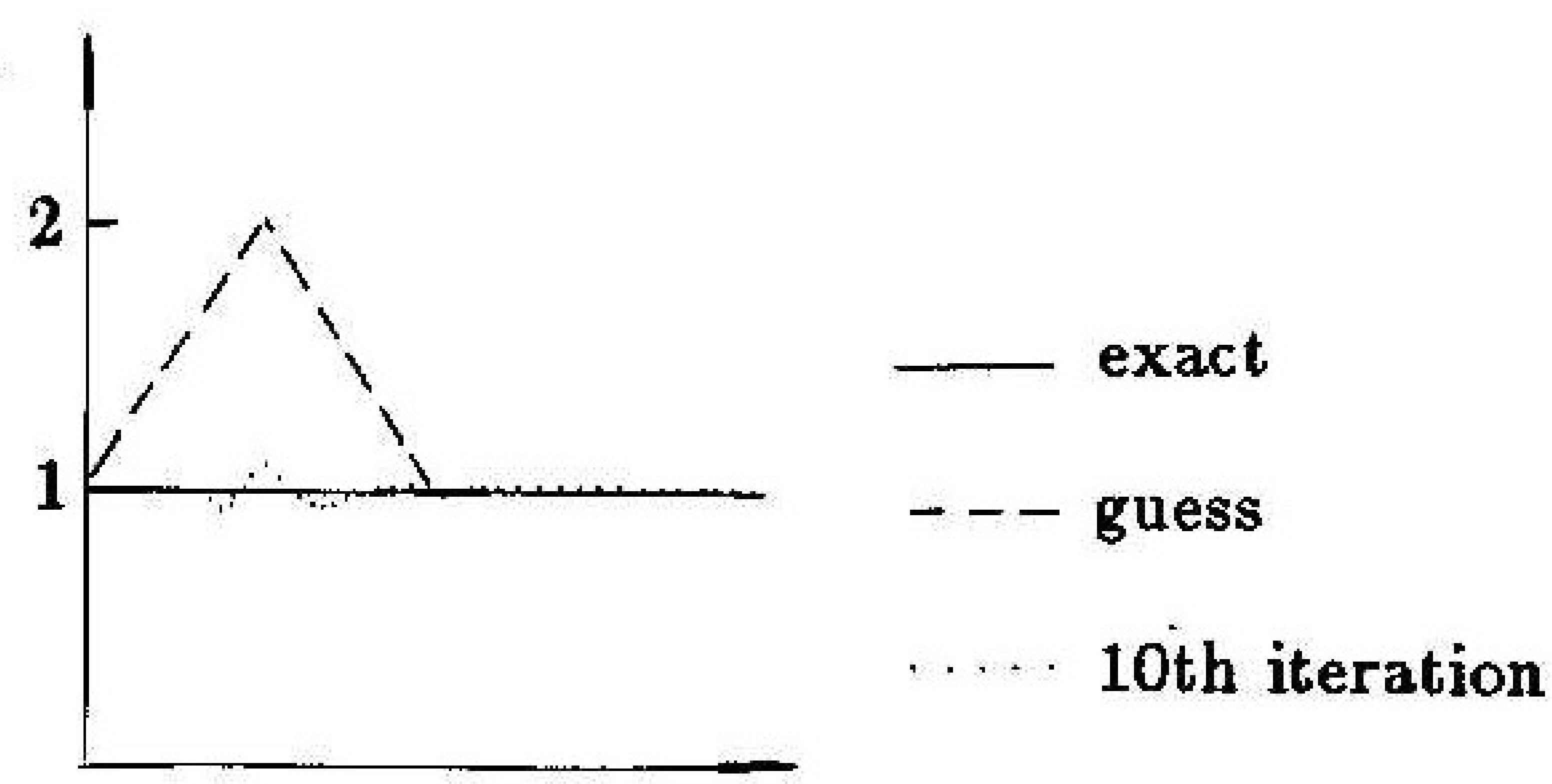

Fig. 5

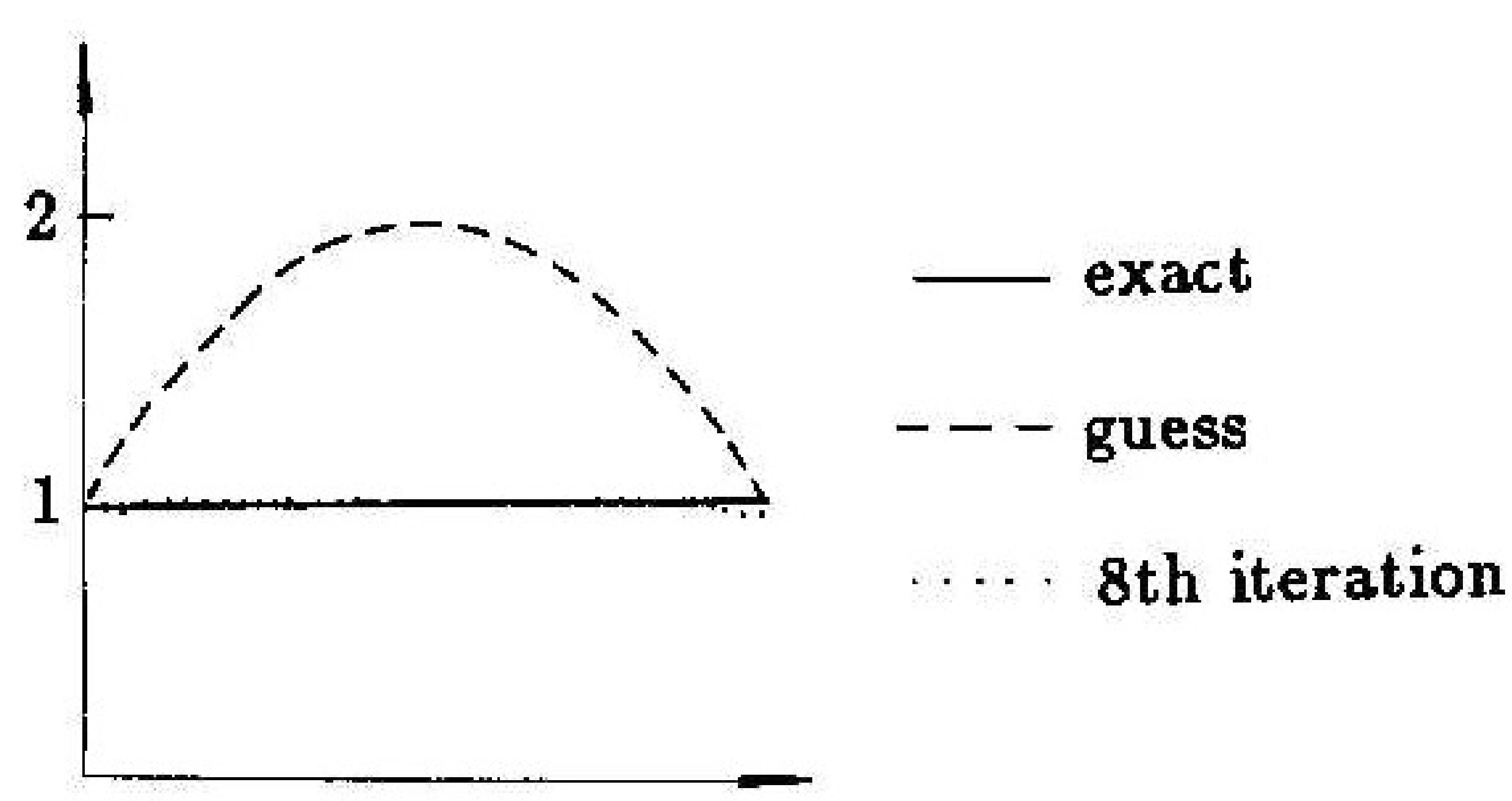

Fig. 6

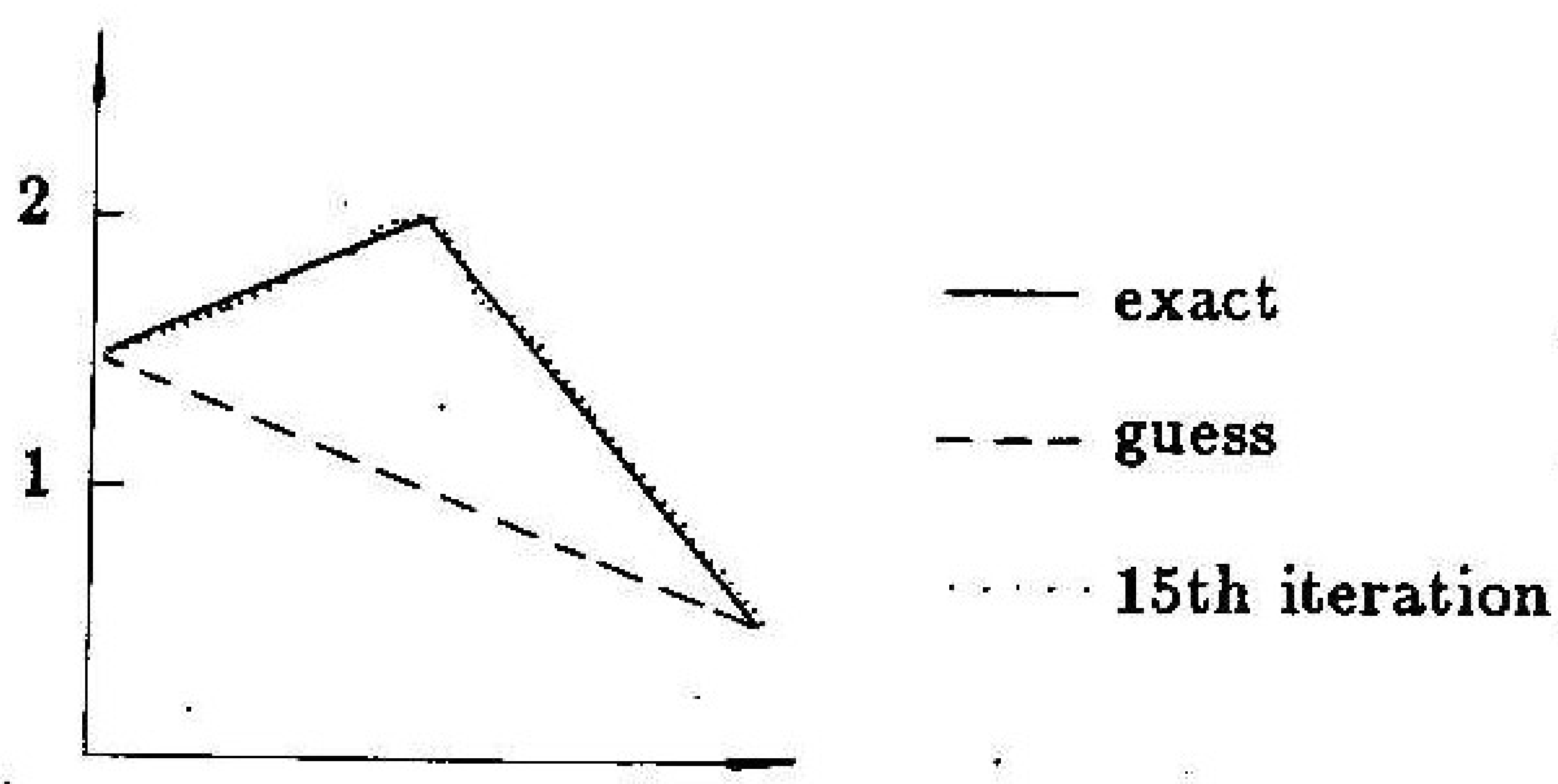

Fig. 7

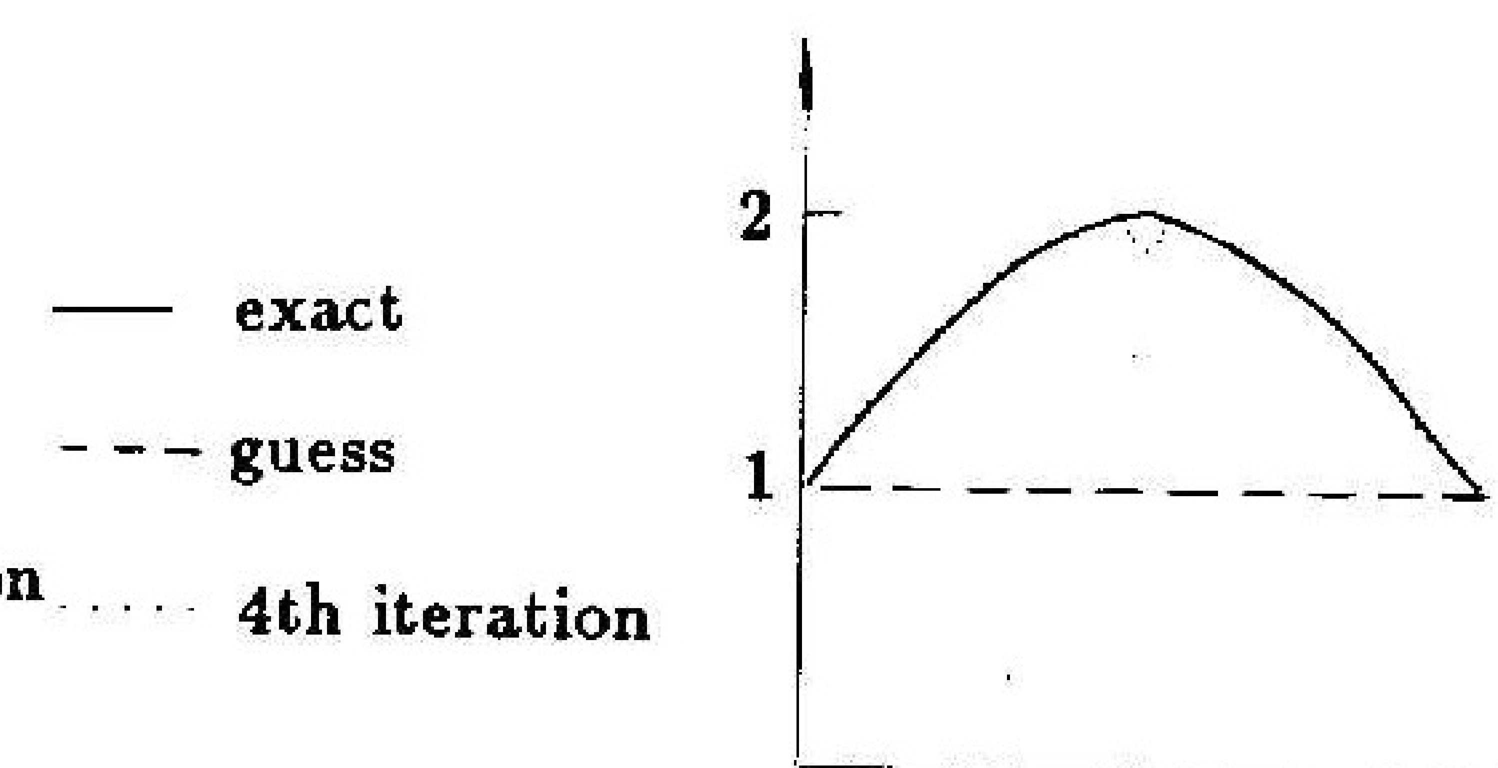

Fig. 8

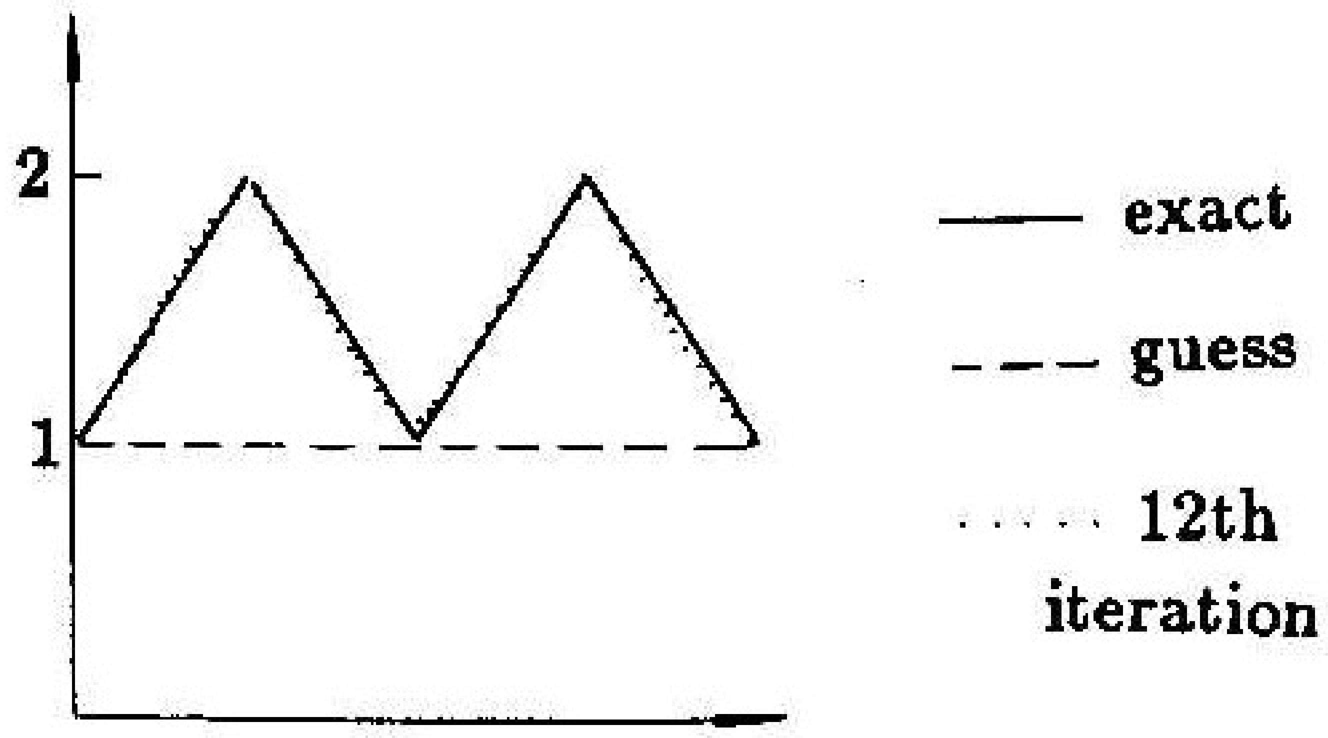

Fig. 9

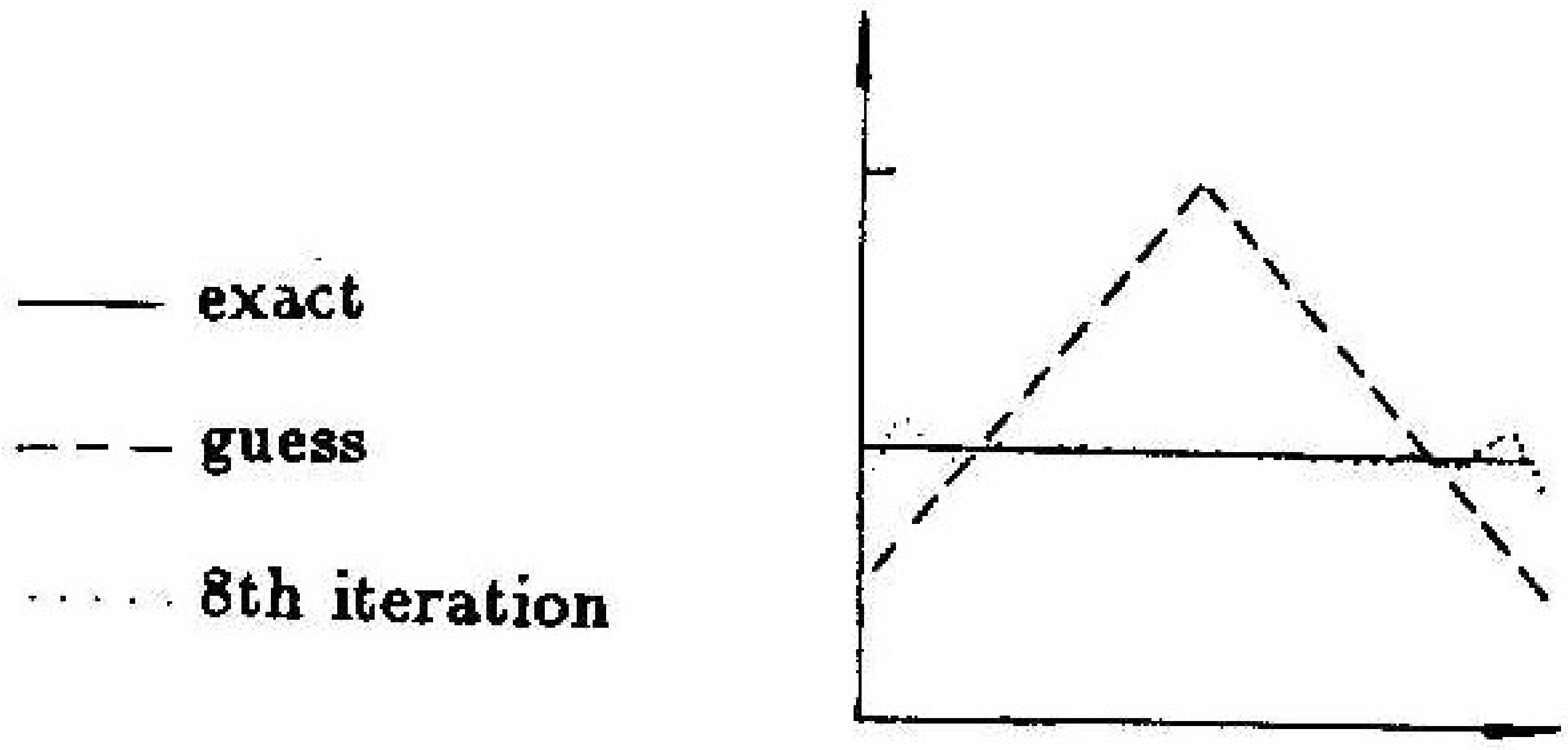

Fig. 10

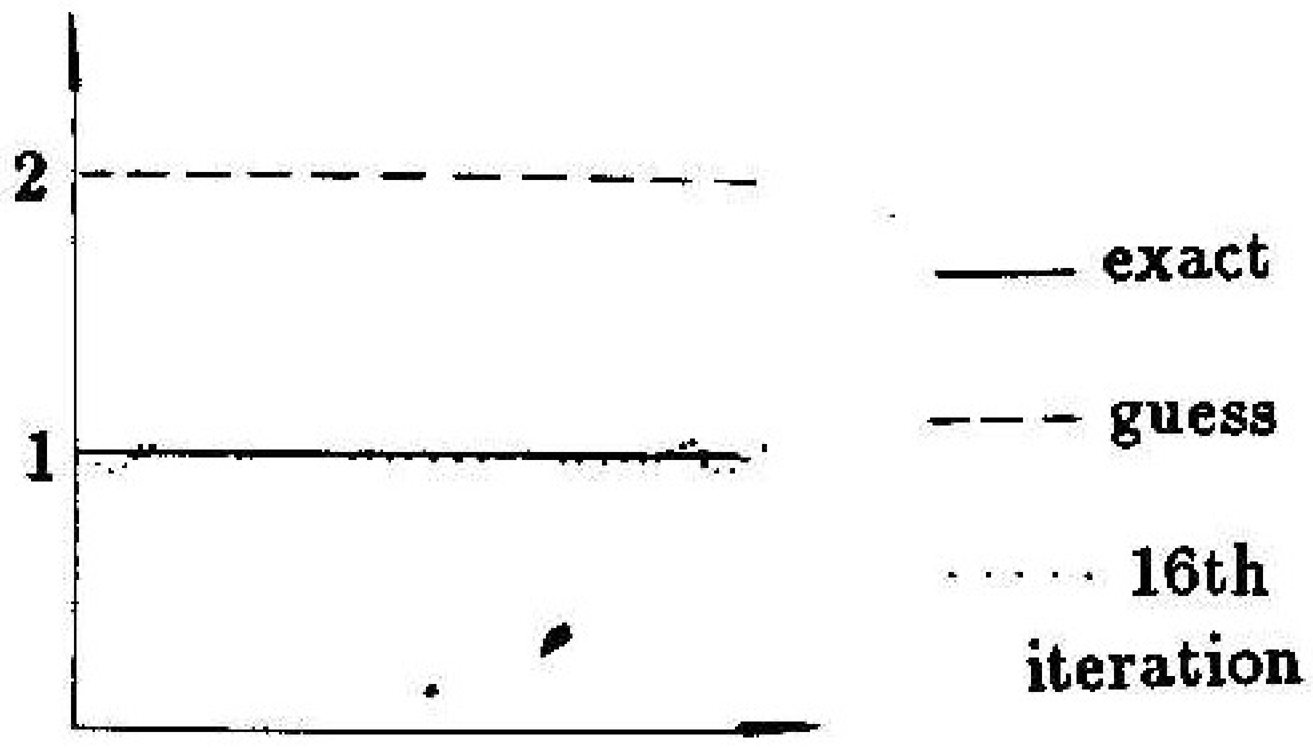

Fig. 11

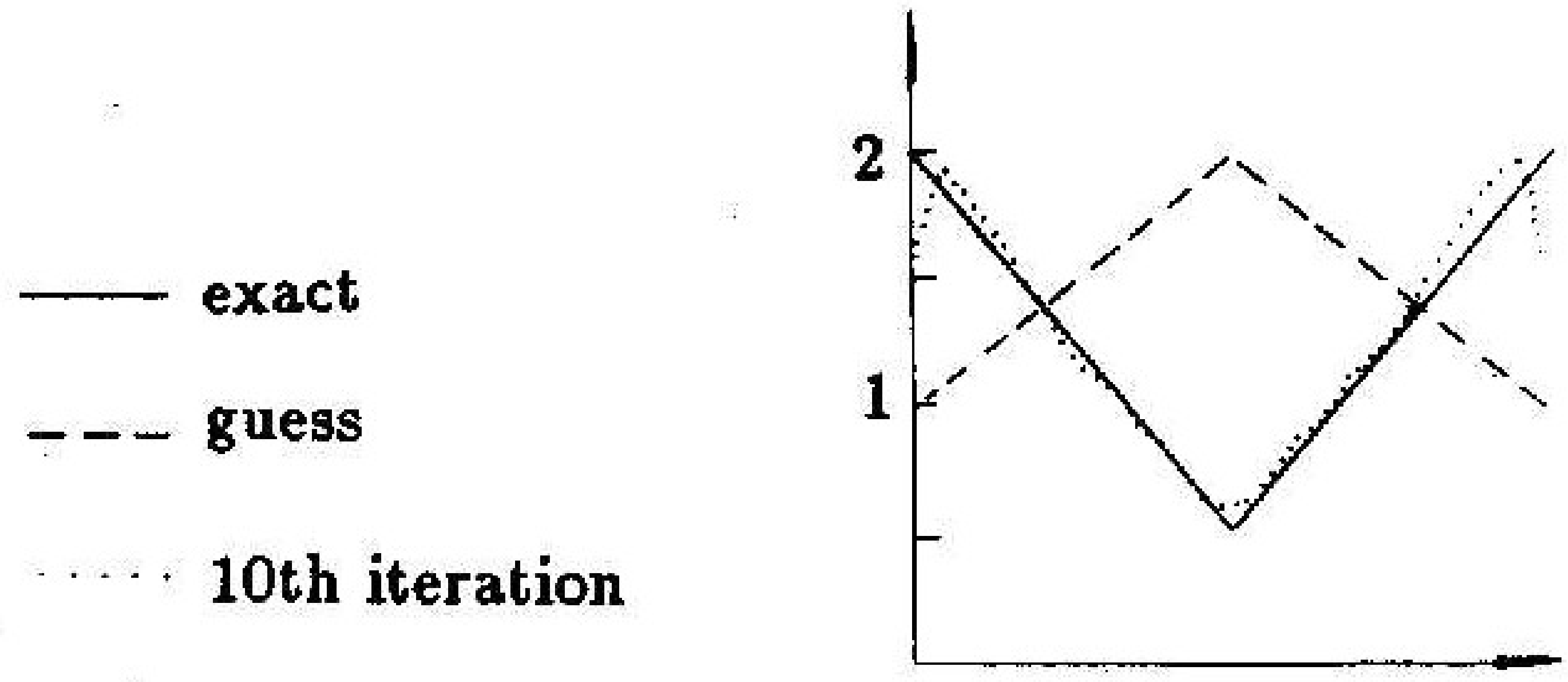

Fig. 12

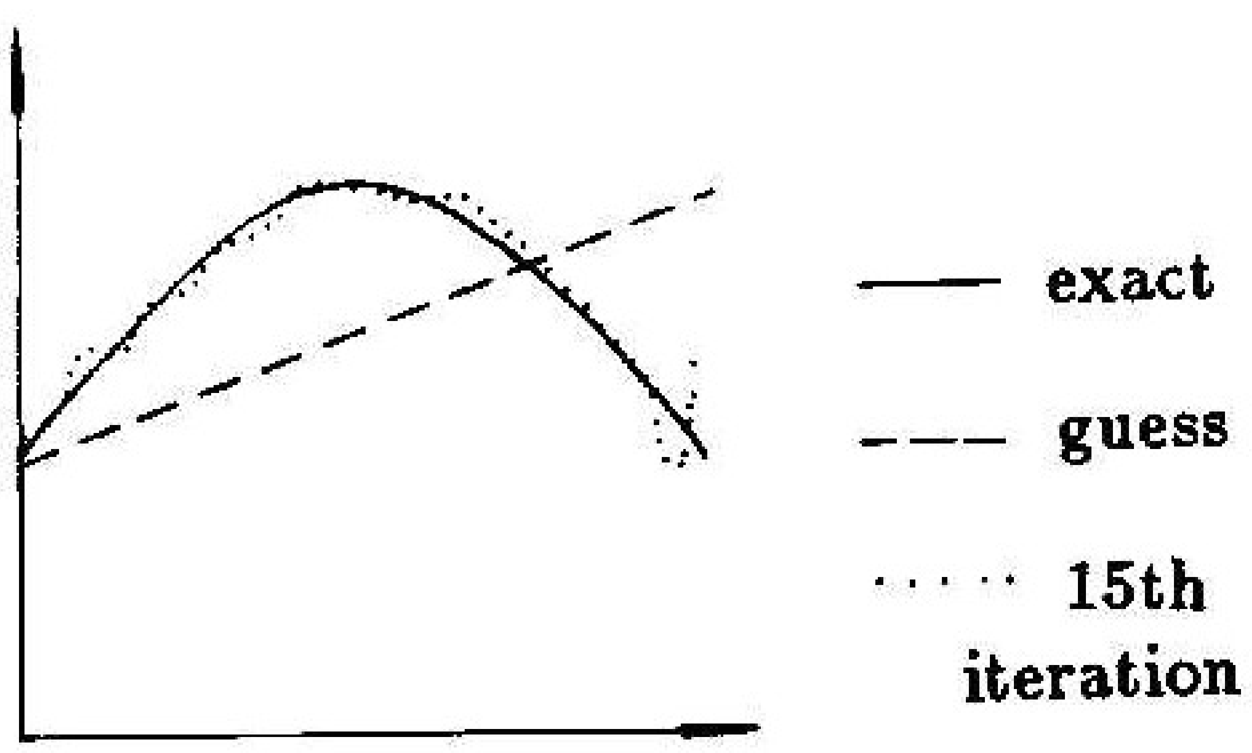

Fig. 13

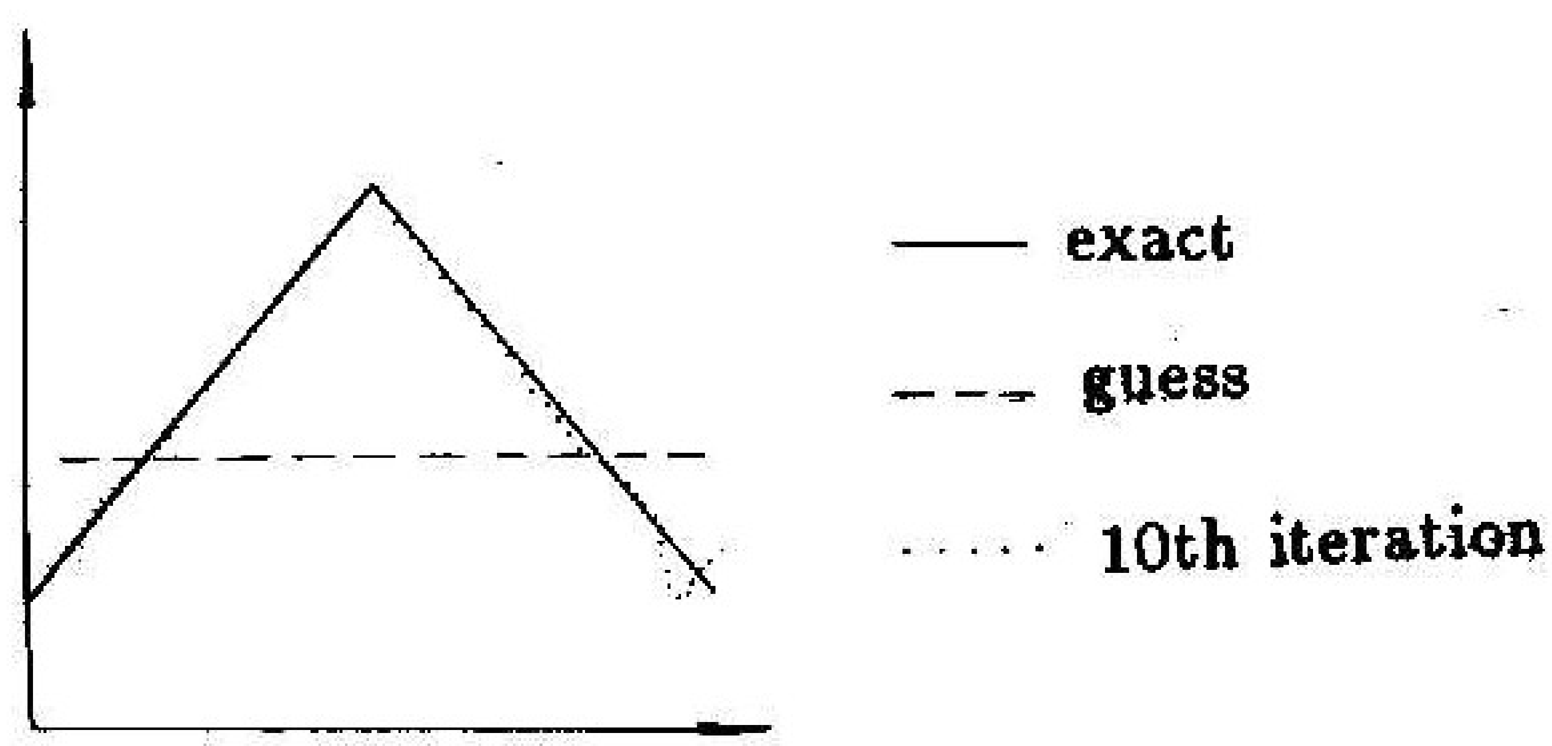

Fig. 14

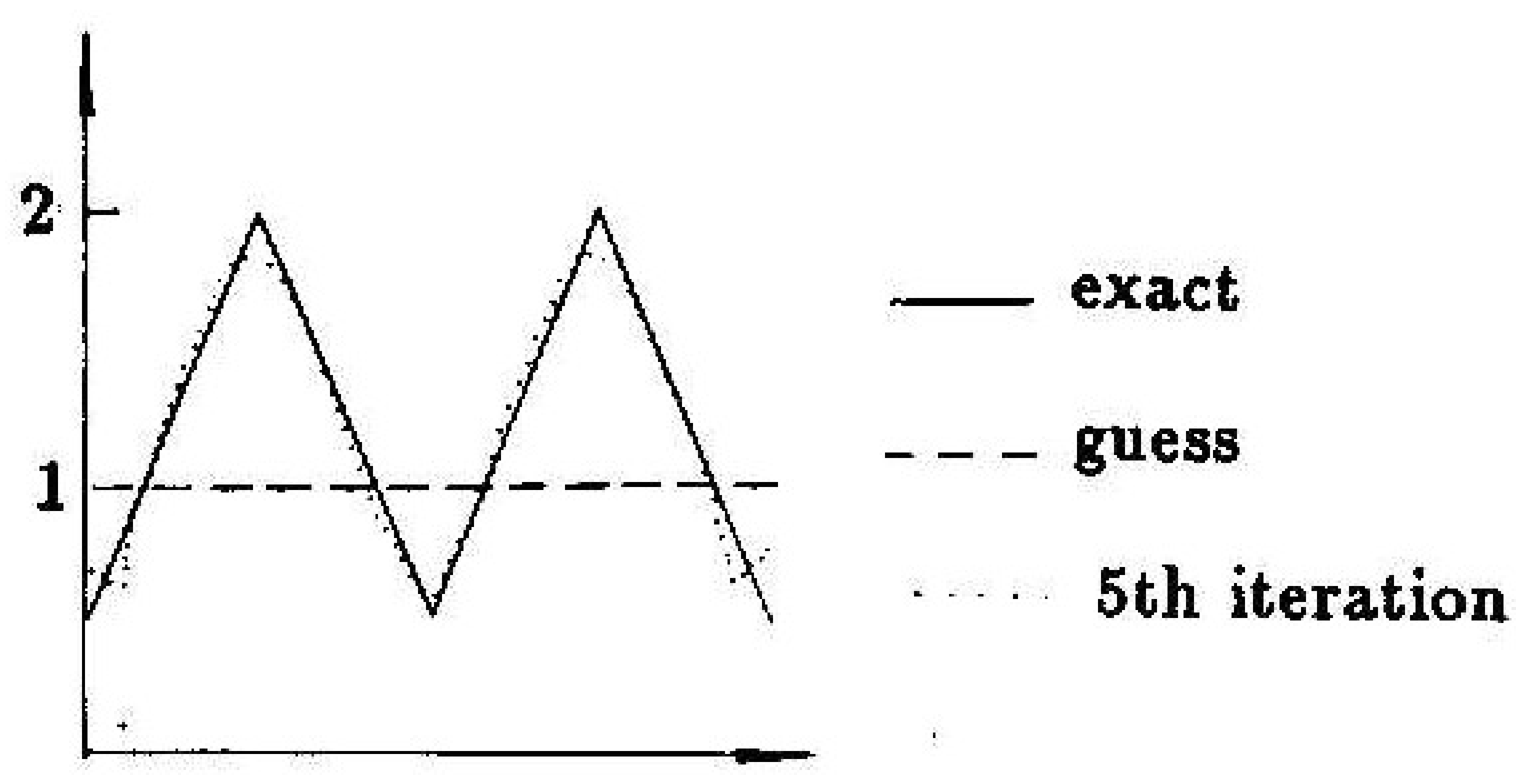

Fig. 15

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