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PROJECTIVE APPROXIMATION OF DOUBLE LIMIT POINTS FOR NONLINEAR PROBLEMS *1)

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Abstract

In [2], general approximation results for the solutions in a neighborhood of a simple limit point are given. In this paper we give projective approximation results for the solutions in a neighborhood of a double limit point. Application of these results to a nonlinear partial differential equation and numerical results are given.

§1. Introduction

Consider a nonlinear problem of the form

$$F(\lambda, u) = 0 \tag{1.1}$$

where $F : R \times V \to V$ is sufficiently smooth, and V is a Hilbert space. In [2], finite dimensional approximation of branches of solutions of problem (1.1) in a neighborhood of a simple limit point and a simple bifurcation point have been studied. In this paper, we will discuss the projective approximation of branches of solutions of problem (1.1) in a neighborhood of a double limit point (λ_0, u_0) of F, i.e., a point $(\lambda_0, u_0) \in R \times V$ which satisfies the following properties:

1) $F(\lambda_0, u_0) = 0$;
2) $D_u F(\lambda_0, u_0)$ is singular and dim Ker $D_u F(\lambda_0, u_0) =$ codim Range $D_u F(\lambda_0, u_0) = 2$;
3) $D_\lambda F(\lambda_0, u_0) \notin$ Range $D_u F(\lambda_0, u_0)$.

An outline of the paper is as follows. In Section 2, we give a local analysis of a double limit point. In Section 3 we consider the projective approximation problem of (1.1) near the double limit point. Using the method similar to that in [2], we obtain the error estimates and convergence results of the solution sets. In Section 4, we apply our results to a simple example, and give numerical results.

§2. Local Analysis of Double Limit Points

Consider the nonlinear problem

$$F(\lambda, u) \equiv u + TG(\lambda, u) = 0 \tag{2.1}$$

where $T \in \mathcal{L}(V, V)$, and $G \in C^r (r \geq 3) : R \times V \to V; V$ is a Hilbert space.

We assume that $(\lambda_0, u_0) \in R \times V$ is a double limit point of F in the sense that

1) $F^0 \equiv F(\lambda_0, u_0) = 0$; (2.2)

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2) $D_uF^0 \equiv D_uF(\lambda_0, u_0) = I + TD_uG^0 \in \mathcal{L}(V,V), -1$ is an eigenvalue of TD_uG^0 with algebraic multiplicity 2;

3) $D_\lambda F^0 \equiv D_\lambda F(\lambda_0, u_0) \notin$ Range (D_uF^0).

Moreover, we assume that

$$\text{Range } (D_uF^0) \text{ is closed}; \quad D_uF^0 \text{ is self-adjoint.} \tag{2.3}$$

Remark. Under the assumptions that T is compact and F is symmetric in some sense, Raugel [5] has discussed multiple limit point problems. Here discarding the above assumptions, we only assume that (2.3) holds. We notice that condition (2.3) holds if D_uF^0 is a Fredholm operator and self-adjoint. Particularly, (2.3) holds for T compact and D_uF^0 self-adjoint.

From 2) of (2.2) and the properties of self-adjoint operators, it follows that

$$\text{Ker } (D_uF^0) = \text{ Ker } ((D_uF^0)^n), \quad n = 2, 3, \cdots.$$

Hence we can find $\varphi_1, \varphi_2 \in V, (\varphi_i, \varphi_j) = \delta_{ij}, i, j = 1, 2$, such that

$$\text{Ker}(D_uF^0) = \text{span}\{\varphi_1, \varphi_2\}.$$

By the closed range theorem[1], we have

$$\text{Range } (D_uF^0) = \text{Ker}(D_uF^0) = \{v \in V : (v, \varphi_i) = 0, \quad i = 1, 2\}.$$

Set

$$V_1 = \text{ Ker } (D_uF^0), \quad V_2 = \text{ Range } (D_uF^0).$$

Then $V = V_1 + V_2$, and D_uF^0 is an isomorphism of V_2.

From 3) of (2.2), without loss of generality, we assume

$$(D_\lambda F^0, \varphi_1) = (TD_\lambda G^0, \varphi_1) \neq 0.$$

Now we define the projective operator $Q : V \to V_2$ by

$$Qv = v - \sum_{i=1}^{2} (v, \varphi_i)\varphi_i, \quad v \in V.$$

Then equation (2.1) is equivalent to the system

$$\begin{cases} QF(\lambda, u) = 0, \\ (I - Q)F(\lambda, u) = 0. \end{cases} \tag{2.4}$$

Given $u \in V$, there exists a unique decomposition of the form

$$u = u_0 + \sum_{i=1}^{2} \xi_i \varphi_i + v, \quad \xi_i \in R, \quad i = 1, 2, \quad v \in V_2.$$

Setting $\xi = (\xi_1, \xi_2)$, the first equation of (2.4) becomes

$$\mathcal{F}(\lambda, \xi, v) \equiv QF(\lambda, u_0 + \sum_{i=1}^{2} \xi_i \varphi_i + v) = 0. \tag{2.5}$$

Since $\mathcal{F} = (\lambda_0, 0, 0) = 0, D_v\mathcal{F}(\lambda_0, 0, 0) = D_uF^0|_{V_2}$ is an isomorphism of V_2. Hence, by the implicit function theorem, there exist $\delta_0 > 0$ and a unique C^r function $v(\lambda, \xi)$, for all λ and ξ with $|\lambda - \lambda_0| \leq \delta_0, |\xi_i| \leq \delta_0, i = 1, 2$, such that

$$\mathcal{F}(\lambda, \xi, v(\lambda, \xi)) = 0, \quad v(\lambda_0, 0) = 0, \quad \frac{\partial v}{\partial \xi_i}(\lambda_0, 0) = 0, \quad i = 1, 2. \tag{2.6}$$

The last equality can be obtained by differentiating the first one with respect to ξ_i.

The second equation of (2.4) now becomes

$$f_j(\lambda, \xi) \equiv (F(\lambda, u_0 + \sum_{i=1}^{2} \xi_i\varphi_i + v(\lambda, \xi)), \varphi_j) = 0, \quad j = 1, 2. \tag{2.7}$$

Since $f_1(\lambda_0, 0) = 0, \frac{\partial f_1}{\partial \lambda}(\lambda_0, 0) = (TD_\lambda G^0 + D_uF^0\frac{\partial v}{\partial \lambda}(\lambda_0, 0), \varphi_1) = (TD_\lambda G^0, \varphi_1) \neq 0$, by the implicit function theorem, one can find a constant $\alpha_0 > 0$ (let $\alpha_0 \leq \delta_0$). As $|\xi_i| \leq \alpha_0, i = 1, 2$, there exists a unique C^r function $\lambda(\xi)$, such that

$$f_1(\lambda(\xi), \xi) = 0, \quad \lambda(0) = \lambda_0, \quad \frac{\partial \lambda}{\partial \xi_i}(0) = 0, \quad i = 1, 2. \tag{2.8}$$

The last equality can be obtained by differentiating the first one with respect to ξ_i.

Setting

$$g(\xi) = f_2(\lambda(\xi), \xi)$$

we have

$$g(0) = 0,$$
$$\frac{\partial g}{\partial \xi_i}(0) = \Big(D_\lambda F^0\frac{\partial \lambda}{\partial \xi_i}(0) + D_uF^0\Big(\varphi_i + \frac{\partial v}{\partial \lambda}\frac{\partial \lambda}{\partial \xi_i}(0)\Big), \varphi_2\Big) = 0, \quad i = 1, 2,$$
$$\frac{\partial^2 g}{\partial \xi_1^2}(0) = (TD_\lambda G^0\frac{\partial^2 \lambda}{\partial \xi_1^2}(0) + TD_{uu}G^0\varphi_1\varphi_1, \varphi_2) \equiv A_0,$$

$$\frac{\partial^2 g}{\partial \xi_1 \partial \xi_2}(0) = (TD_\lambda G^0\frac{\partial^2 \lambda}{\partial \xi_1 \partial \xi_2}(0) + TD_{uu}G^0\varphi_1\varphi_2, \varphi_2) \equiv B_0,$$
$$\frac{\partial^2 g}{\partial \xi_2^2}(0) = (TD_\lambda G^0\frac{\partial^2 \lambda}{\partial \xi_2^2}(0) + TD_{uu}G^0\varphi_2\varphi_2, \varphi_2) \equiv C_0,$$
$$\frac{\partial^2 \lambda}{\partial \xi_1^2}(0) = -(TD_{uu}G^0\varphi_1\varphi_1, \varphi_1)/(TD_\lambda G^0, \varphi_1) \equiv A_1,$$
$$\frac{\partial^2 \lambda}{\partial \xi_1 \partial \xi_2}(0) = -(TD_{uu}G^0\varphi_1\varphi_2, \varphi_1)/(TD_\lambda G^0, \varphi_1) \equiv B_1,$$
$$\frac{\partial^2 \lambda}{\partial \xi_2^2}(0) = -(TD_{uu}G^0\varphi_2\varphi_2, \varphi_1)/(TD_\lambda G^0, \varphi_1) \equiv C_1.$$

Assume that

$$B_0^2 - A_0C_0 > 0. \tag{2.9}$$

Set $\xi_1 = t\sigma, \xi_2 = ta$. Then

$$g(\xi_1, \xi_2) = \frac{1}{2}t^2(A_0\sigma^2 + 2B_0\sigma a + C_0a^2) + o(t^2), \quad t \to 0.$$

Define

$$H(t, \sigma, a) = (t^{-2}g(t\sigma, ta), \quad \sigma^2 + a^2 - 1).$$

Then $H \in C^{r-2}$, and

$$H(0, \sigma, a) = \left(\frac{1}{2}(A_0\sigma^2 + 2B_0\sigma a + C_0a^2), \sigma^2 + a^2 - 1\right).$$

From (2.9), there exist two distinct pairs $(\sigma_i^0, a_i^0), i = 1, 2$, such that

$$H(0, \sigma_i^0, a_i^0) = 0.$$

Moreover,

$$\det D_{(\sigma,a)}H(0, \sigma_i^0, a_i^0) = \det \begin{pmatrix} A_0\sigma_i^0 + B_0a_i^0 & 2\sigma_i^0 \\ B_0\sigma_i^0 + C_0a_i^0 & 2a_i^0 \end{pmatrix}$$

$$= 2B_0((a_i^0)^2 - (\sigma_i^0)^2) + 2(A_0 - C_0)\sigma_i^0a_i^0 \neq 0.$$

Hence we may apply the implicit function theorem to the function H at each point $(0, \sigma_i^0, a_i^0)$ for $i = 1, 2$. There exists a unique pair of C^{r-2} functions $(\sigma_i(t), a_i(t)), i = 1, 2$, defined for $|t| \leq t_0$, such that

$$\begin{aligned} &H(t, \sigma_i(t), a_i(t)) = 0, \\ &\sigma_i(0) = \sigma_i^0, a_i(0) = a_i^0, \quad i = 1, 2. \end{aligned} \tag{2.10}$$

Let

$$\xi^i(t) = (\xi_1^i(t), \xi_2^i(t)) = (t\sigma_i(t), ta_i(t)), \quad i = 1, 2.$$

Then problem (2.1) has two C^{r-2} branches of solutions in the neighborhood of (λ_0, u_0), which are of the form

$$\begin{cases} \lambda_i(t) = \lambda(\xi^i(t)), \\ u_i(t) = u_0 + \xi_1^i(t)\varphi_1 + \xi_2^i(t)\varphi_2 + v(\lambda_i(t), \xi^i(t)), \quad i = 1, 2, \quad |t| \leq t_0. \end{cases}$$

From above we have

$$\frac{d\lambda_i}{dt} = \frac{\partial\lambda}{\partial\xi_1}\frac{d\xi_1^i}{dt} + \frac{\partial\lambda}{\partial\xi_2}\frac{d\xi_2^i}{dt}.$$

Using (2.8) we get

$$\frac{d\lambda_i}{dt}(0) = 0, \quad i = 1, 2. \tag{2.11}$$

Moreover,

$$\frac{d^2\lambda_i}{dt^2} = \frac{\partial^2\lambda}{\partial\xi_1^2}\left(\frac{d\xi_1^i}{dt}\right)^2 + 2\frac{\partial^2\lambda}{\partial\xi_1\partial\xi_2}\frac{d\xi_1^i}{dt}\frac{d\xi_2^i}{dt} + \frac{\partial^2\lambda}{\partial\xi_2^2}\left(\frac{d\xi_2^i}{dt}\right)^2 + \frac{\partial\lambda}{\partial\xi_1}\frac{d^2\xi_1^i}{dt^2} + \frac{\partial\lambda}{\partial\xi_2}\frac{d^2\xi_2^i}{dt^2},$$

$$\frac{d^2\lambda_i}{dt^2}(0) = A_1(\sigma_i^0)^2 + 2B_1\sigma_i^0a_i^0 + C_1(a_i^0)^2.$$

Therefore, the graph of $\lambda_i(t)$ can be divided into several cases as follows.

1) $B_1^2 - A_1C_1 < 0$.

If $A_1 > 0$, then $\frac{d^2\lambda_i}{dt^2}(0) > 0, i = 1, 2$ (Fig. 1).

If $A_1 < 0$, then $\frac{d^2\lambda_i}{dt^2}(0) < 0, \quad i = 1, 2$ (Fig. 2).

2) $B_1^2 - A_1C_1 \geq 0$.

If vectors (A_1, B_1, C_1) and (A_0, B_0, C_0) are linearly dependent, then $\frac{d^2\lambda_i}{dt^2}(0) \neq 0, i = 1, 2$.

When $\frac{d^2\lambda_i}{dt^2}(0) \neq 0, i = 1, 2$, the graphs of $\lambda_1(t)$ and $\lambda_2(t)$ are tangent at $t = 0$. If $\frac{d^2\lambda_1}{dt^2}(0)$ and $\frac{d^2\lambda_2}{dt^2}(0)$ have the same sign, their graphs are as in Fig. 1 or Fig. 2. If they have different signs, their graphs are as in Fig. 3.

Remark. If $B_0^2 - A_0C_0 = 0$, then problem (2.1) has only one branch of solution near (λ_0, u_0). If $B_0^2 - A_0C_0 < 0$, the solution set of problem (2.1) near (λ_0, u_0) consists of an isolated point (λ_0, u_0).

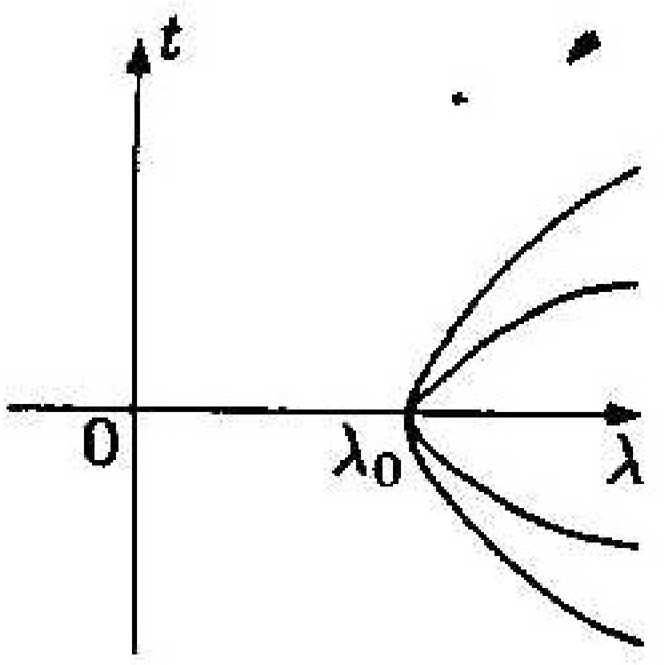

Fig. 1

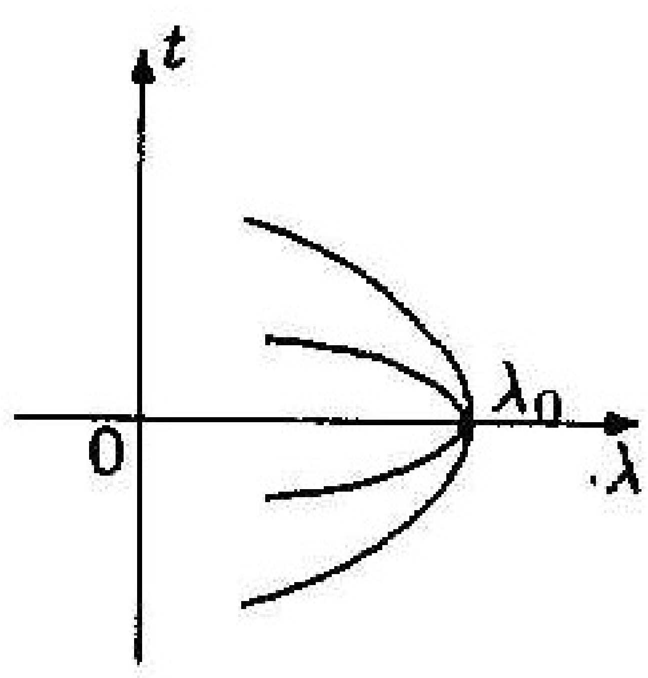

Fig. 2

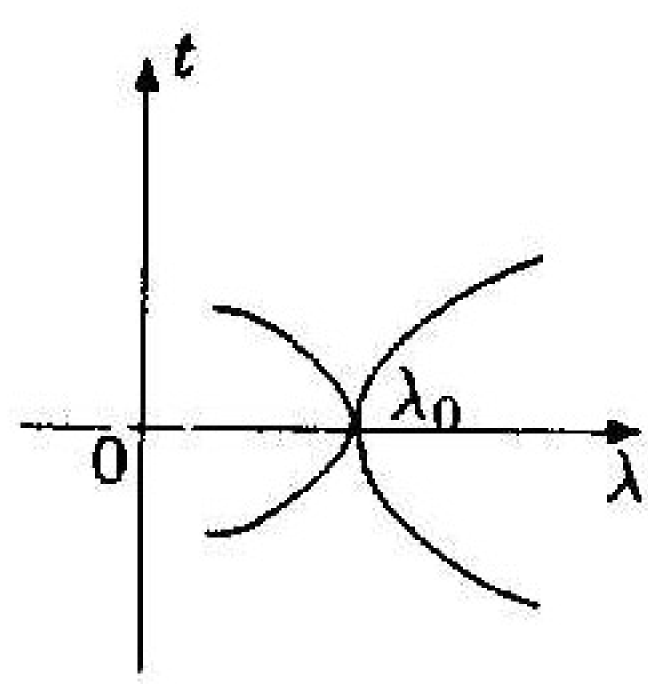

Fig. 3

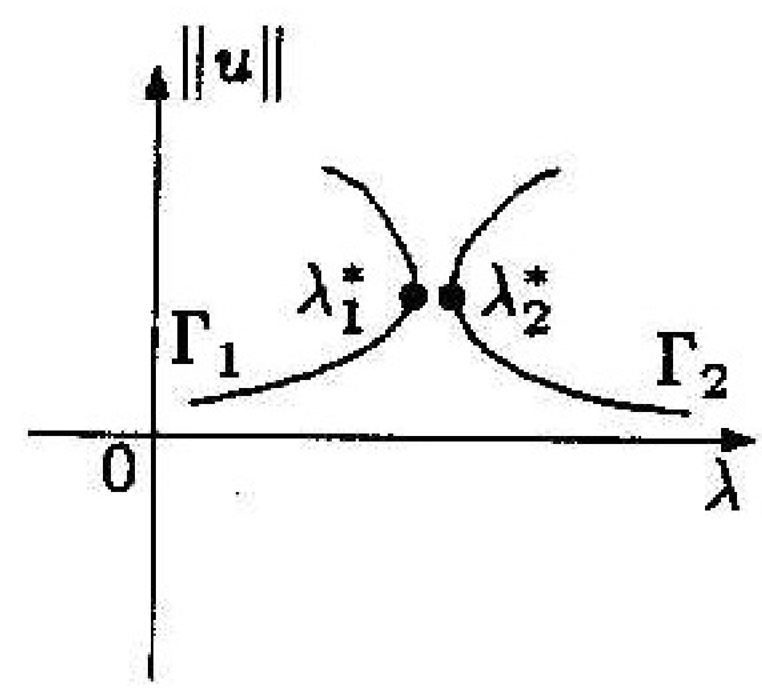

Fig. 4

§3. Projective Approximation

Let us first introduce a result of [2]. Here we slightly weaken the conditions, but the proof is the same.

Let X, Y, Z be three Banach spaces and Φ be a C^r mapping $(r \geq 2)$ from $B \times Y$ into Z where B is a bounded open subset of X. We shall denote by $D\Phi(x, y) \in \mathcal{L}(X \times Y; Z)$ the total derivative of Φ at the point (x, y) and by $D^l\Phi(x, y) \in \mathcal{L}_l(X \times Y; Z), 2 \leq l \leq r$, the total derivative of Φ where $\mathcal{L}_l(X \times Y; Z)$ is the space of all l-linear mappings from $X \times Y$ into Z.

Lemma 1. *We assume that the mapping $D^r\Phi$ is bounded on all bounded subsets of $B \times Y$. Let g be a bounded C^r function from B into Y such that, for all $x \in B$, the following two properties hold:*

$$\Phi(x, g(x)) = 0, \tag{3.1}$$

and $D_y\Phi(x, g(x))$ *is an isomorphism from* Y *onto* Z *with*

$$\|D_y\Phi(x, g(x))^{-1}\|_{\mathcal{L}(Z;Y)} \le c. \tag{3.2}$$

For each value of a parameter $h > 0$, let Φ_h be a C^r mapping from $B \times Y$ into Z such that

$$\begin{aligned}&1)\ \lim_{h\to 0}\sup_{(x,y)\in \mathcal{B}} \|D^l\Phi(x,y) - D^l\Phi_h(x,y)\|_{\mathcal{L}_l(X\times Y;Z)} = 0, \quad l = 0,1,\\ &2)\ \sup_{(x,y)\in\mathcal{B}} \|D^r\Phi_h\|_{\mathcal{L}_r(X\times Y;Z)} \le c \text{ (c independent of } h)\end{aligned} \tag{3.3}$$

for all bounded subsets $\mathcal{B} \subset B \times Y$.

Then there exist a constant $h_0 > 0$ and, for $h \le h_0$, a unique C^r mapping g_h from B into Y such that we have for all $x \in B$

$$\Phi_h(x, g_h(x)) = 0. \tag{3.4}$$

Moreover, we have for all $x, x^* \in B$, and all integers m with $0 \le m \le r-1$ the following error bound:

$$\begin{aligned}&1)\ \|D^m g_h(x^*) - D^m g(x)\|_{\mathcal{L}_m(X;Y)}\\ &\qquad \le K\Big\{\|x^* - x\|_X + \sum_{l=0}^{m}\Big\|\frac{d^l}{dx^l}(\Phi(x,g(x)) - \Phi_h(x,g(x)))\Big\|_{\mathcal{L}_l(X;Z)}\Big\},\\ &2)\ \sup_{x\in B}\|D^r g_h\|_{\mathcal{L}_r(X;Y)} \le K,\end{aligned} \tag{3.5}$$

where $D^m g_h$ and $D^m g$ are the mth derivatives of g_h and g respectively and $K > 0$ is a constant independent of h.

Let us now consider the discrete problem of (2.1)

$$u_h + P_h TG(\lambda, u_h) = 0, \quad (\lambda, u_h) \in R \times V_h \tag{3.6}$$

where P_h is a linear projective operator from V into V_h, and $\|P_h x - x\| \to 0$ $(h \to 0)$ holds for all $x \in V$. $V_h \subset V$ is a Hilbert space of finite dimension. Define $F_h : R \times V \to V$ by

$$F_h(\lambda, u) = u + P_h TG(\lambda, u).$$

Let us note that we can equivalently solve equation (3.6) in $R \times V$.

As in the previous section, equation (3.6) is equivalent to the system

$$\begin{cases} QF_h(\lambda, u_h) = 0,\\ (I - Q)F_h(\lambda, u_h) = 0.\end{cases} \tag{3.7}$$

Set

$$u_h = u_0 + \sum_{i=1}^{2}\xi_i\varphi_i + v_h, \quad \xi_i \in R^1, \quad v_h \in V_2, \quad i = 1,2.$$

The first equation of (3.7) now becomes

$$QF_h(\lambda, u_0 + \sum_{i=1}^{2}\xi_i\varphi_i + v_h) = 0. \tag{3.8}$$

Set $\xi = (\xi_1, \xi_2)$, and

$$\mathcal{F}_h(\lambda, \xi, v) = QF_h(\lambda, u_0 + \sum_{i=1}^{2} \xi_i \varphi_i + v).$$

For any bounded subset B of $R \times V$, we assume

$$\sup_{(\lambda,u)\in B} \|(P_h - I)TD^i G(\lambda, u)\| \to 0, \quad i = 0, 1, \cdots, i_0, \quad h \to 0,$$
$$\sup_{(\lambda,u)\in B} \|D^r G(\lambda, u)\| \leq c_0(B). \tag{3.9}$$

Set $J(\lambda, \xi) = G(\lambda, u_0 + \sum_{i=1}^{2} \xi_i \varphi_i + v(\lambda, \xi))$. In this section, K, C, K_i and $C_i (i = 1, 2, \cdots)$ are used to denote positive constants independent of h.

Theorem 1. *Under the assumptions of (2.2), (2.3), and (3.9) for $i_0 = 1$, one can find $h_0, \delta_0 > 0$ such that for $h \leq h_0$, there exists a unique C^r function $v_h(\lambda, \xi)$, which satisfies, for all λ, λ^*, ξ with $|\lambda - \lambda_0| \leq \delta_0, |\lambda^* - \lambda_0| \leq \delta_0, |\xi_i| \leq \delta_0, i = 1, 2$, and for any m with $0 \leq m \leq r - 1$,*

$$\mathcal{F}_h(\lambda, \xi, v_h(\lambda, \xi)) = 0, \quad \|D^r v_h(\lambda, \xi)\| \leq K, \tag{3.10}$$
$$\|D^m v_h(\lambda^*, \xi) - D^m v(\lambda, \xi)\| \leq K(|\lambda^* - \lambda| + \sum_{i=0}^{m} \|(P_h - I)TD^i J(\lambda, \xi)\|), \tag{3.11}$$

where K is independent of λ, λ^*, ξ.

Proof. We intend to verify that the conditions of Lemma 1 can be satisfied for $\mathcal{F}$ and $\mathcal{F}_h$. Since $D_v \mathcal{F}(\lambda_0, 0, 0)$ is an isomorphism of V_2, we may choose δ_0 small enough such that

$$\|(D_v \mathcal{F}(\lambda, \xi, v(\lambda, \xi)))^{-1}\| \leq c_1, \quad |\lambda - \lambda_0| \leq \delta_0, \quad |\xi_i| \leq \delta_0, \quad i = 1, 2.$$

Furthermore,

$$\mathcal{F}_h(\lambda, \xi, v(\lambda, \xi)) = \mathcal{F}_h(\lambda, \xi, v(\lambda, \xi) - \mathcal{F}(\lambda, \xi, v(\lambda, \xi)) = Q(P_h - I)TJ(\lambda, \xi),$$

$$D_V \mathcal{F}_h(\lambda, \xi, v(\lambda, \xi)) - D_v \mathcal{F}(\lambda, \xi, v(\lambda, \xi)) = Q(P_h - I)TD_u G(\lambda, u_0 + \sum_{i=1}^{2} \xi_i \varphi_i + v(\lambda, \xi)).$$

From (3.9) we have for $h \to 0$

$$\|\mathcal{F}_h(\lambda, \xi, v(\lambda, \xi))\| \to 0,$$
$$\|D\mathcal{F}_h(\lambda, \xi, v(\lambda, \xi)) - D\mathcal{F}(\lambda, \xi, v(\lambda, \xi))\| \to 0,$$

in which λ, ξ satisfy $|\lambda - \lambda_0| \leq \delta_0, |\xi_i| \leq \delta_0, i = 1, 2$. Moreover, it follows from (3.9) that $D^r \mathcal{F}_h(\lambda, \xi, v)$ and $D^r \mathcal{F}(\lambda, \xi, v)$ are bounded on any bounded sets. Hence, by Lemma 1, one can find $h_0 > 0$ such that for $h \leq h_0$, there exists a unique C^r function $v_h(\lambda, \xi)$ that satisfies, for all λ, λ^*, ξ with $|\lambda - \lambda_0| \leq \delta_0$, $|\lambda^* - \lambda_0| \leq \delta_0, |\xi_i| \leq \delta_0, i = 1, 2$,

$$\|D^m v_h(\lambda^*, \xi) - D^m v(\lambda, \xi)\| \leq \tilde{K}\Big(|\lambda^* - \lambda| + \sum_{i=0}^{m} \|D^i_{(\lambda,\xi)} \mathcal{F}_h(\lambda, \xi, v(\lambda, \xi))\|\Big).$$

Therefore, (3.11) holds since

$$D^i_{(\lambda,\xi)}\mathcal{F}_h(\lambda,\xi,v(\lambda,\xi)) = Q(P_h - I)TD^iJ(\lambda,\xi).$$

(3.10) is a direct result of Lemma 1. This completes the proof.

Thus, the second equation of (3.7) now becomes

$$f_h^j(\lambda,\xi) = \Big(F_h\Big(\lambda, u_0 + \sum_{i=1}^{2}\xi_i\varphi_i + v_h(\lambda,\xi)\Big),\varphi_j\Big) = 0, \quad j = 1,2. \tag{3.12}$$

Theorem 2. *Under the assumptions of Theorem 1, there exist constants $h_0, \delta_0 > 0$ and, for $h \le h_0$, a unique C^r function $\lambda_h(\xi)$ such that for all ξ with $|\xi_i| \le \alpha_0, i = 1,2$ and $0 \le m \le r-1$,*

$$f_h^1(\lambda_h(\xi),\xi) = 0, \qquad \sup_{|\xi_i|\le\alpha_0}\|D^r\lambda_h(\xi)\| \le K; \tag{3.13}$$

$$\|D^m\lambda_h(\xi) - D^m\lambda(\xi)\| \le K\sum_{i=0}^{m}\|(P_h - I)TD^iJ(\lambda(\xi),\xi)\| \tag{3.14}$$

where K is independent of ξ.

Proof. Since $D_\lambda f_1(\lambda_0,0) = (TD_\lambda G^0,\varphi_1) \ne 0$, we can choose α_0 so small that for ξ with $|\xi_i| \le \alpha_0$, $i = 1,2$,

$$D_\lambda f_1(\lambda(\xi),\xi) \ne 0.$$

Next

$$\begin{aligned} f_h^1(\lambda(\xi),\xi) &= f_h^1(\lambda(\xi),\xi) - f_1(\lambda(\xi),\xi) \\ &= (v_h(\lambda(\xi),\xi) - v(\lambda(\xi),\xi) + (P_h - I)TG(\lambda(\xi),u(\xi)),\varphi_1) \\ &\quad + \Big(P_hT\Big\{G\Big(\lambda(\xi),u_0 + \sum_{i=1}^{2}\xi_i\varphi_i + v_h(\lambda(\xi),\xi)\Big) - G(\lambda(\xi),u(\xi))\Big\},\varphi_1\Big) \end{aligned}$$

where

$$u(\xi) = u_0 + \sum_{i=1}^{2}\xi_i\varphi_i + v(\lambda(\xi),\xi),$$

$$\begin{aligned} Df_h^1(\lambda(\xi),\xi) - Df_1(\lambda(\xi),\xi) &= (Dv_h(\lambda(\xi),\xi), -Dv(\lambda(\xi),\xi) \\ &\quad +(P_h - I)TDG(\lambda(\xi),u(\xi)),\varphi_1) + \Big(P_hT\{DG(\lambda(\xi),u_0 \\ &\quad +\sum_{i=1}^{2}\xi_i\varphi_i + v_h(\lambda(\xi),\xi) - DG(\lambda(\xi),u(\xi))\},\varphi_1\Big). \end{aligned}$$

By the assumptions of the theorem, it is easy to check that for $h \to 0$

$$\sup_{|\xi_i|\le\alpha_0}|f_h^1(\lambda(\xi),\xi)| \to 0,$$

$$\sup_{|\xi_i|\le\alpha_0}|Df_h^1(\lambda(\xi),\xi) - Df_1(\lambda(\xi),\xi)| \to 0.$$

Furthermore, it follows from condition (3.9) that $D^r f_h^1(\lambda,\xi)$ and $D^r f_1(\lambda,\xi)$ are bounded on any bounded set of R^3. Hence, by Lemma 1, there exist an $h_0>0$ and, for $h\le h_0$, a unique C^r function $\lambda_h(\xi)$, such that for $|\xi_i|\le\alpha_0, i=1,2$, and for $0\le m\le r-1$, (3.13) holds. Furthermore,

$$\|D^m\lambda_h(\xi)-D^m\lambda(\xi)\|\le c_1\sum_{i=0}^m\|D_\xi^i f_h^1(\lambda(\xi),\xi)\|\le c_2\sum_{i=0}^m\|D^i f_h^1(\lambda(\xi),\xi)\|,$$

$$D^m f_h^1(\lambda(\xi),\xi)=\Big(D^m v_h(\lambda(\xi),\xi)-D^m v(\lambda(\xi),\xi)+P_hTD^m\Big(G(\lambda(\xi),u_0$$
$$+\sum\xi_i\varphi_i+v_h(\lambda(\xi),\xi))-G(\lambda(\xi),u(\xi))\Big)+(P_h-I)TD^mG(\lambda(\xi),u(\xi)),\varphi_1\Big),$$

$$\|D^m f_h^1(\lambda(\xi),\xi)\|\le c_3\Big\{\|D^m v_h(\lambda(\xi),\xi)-D^m v(\lambda(\xi),\xi)\|$$
$$+\|(P_h-I)TD^mG(\lambda(\xi),u(\xi))\|+\sum_{i=0}^m\|D^i v_h(\lambda(\xi),\xi)-D^i v(\lambda(\xi),\xi)\|\Big\}.$$

By Theorem 1 we obtain

$$\|D^m f_h^1(\lambda(\xi),\xi)\|\le c_4\sum_{i=0}^m\|(P_h-I)TD^mJ(\lambda(\xi),\xi)\|.$$

Hence (3.14) holds. This completes the proof.

Set $g_h(\xi)=f_h^2(\lambda_h(\xi),\xi)$, with $g(\xi)$ defined as in §2.

Lemma 2. *Under the assumptions of Theorem 1, if h_0,α_0 in Theorem 2 are chosen small enough, then for $h\le h_0, |\xi_i|\le\alpha_0$, $|\xi_i^*|\le\alpha_0, i=1,2$, and $0\le m\le r-1$, we have*

$$\sup_{|\xi_i|\le\alpha_0}\|D^r g_h(\xi)\|\le K_r, \tag{3.15}$$

$$\|D^m g_h(\xi^*)-D^m g(\xi)\|\le K_m\Big(\sum_{i=1}^2|\xi_i^*-\xi_i|+\sum_{i=0}^m\|(P_h-I)TD^iG(\lambda(\xi),u(\xi))\|\Big) \tag{3.16}$$

where $K_i(i=0,1,\cdots,r)$ are independent of ξ,ξ^*.

Proof.

$$g_h(\xi)-g(\xi)=\Big(F_h(\lambda_h(\xi),u_0+\sum\xi_i\varphi_i+v_h(\lambda_h(\xi),\xi))-F(\lambda(\xi),u(\xi)),\varphi_2\Big)$$
$$=\Big(v_h(\lambda_h(\xi),\xi)-v(\lambda(\xi),\xi)+(P_h-I)TG(\lambda(\xi),u(\xi))$$
$$+P_hT\Big\{G(\lambda_h(\xi),u_0+\sum\xi_i\varphi_i+v_h(\lambda_h(\xi),\xi)-D^mG(\lambda(\xi),u(\xi))\Big\},\varphi_2\Big),$$

$$D^m g_h(\xi)-D^m g(\xi)=(D^m v_h(\lambda_h(\xi),\xi)-D^m v(\lambda(\xi),\xi)+(P_h-I)TD^mG(\lambda(\xi),u(\xi))$$
$$+P_hT\{D^mG(\lambda_h(\xi),u_0+\sum\xi_i\varphi_i+v_h(\lambda_h(\xi),\xi)-D^mG(\lambda(\xi),u(\xi))\},\varphi_2),$$

$$\|D^m g_h(\xi) - D^m g(\xi)\| \le c_1\{\|D^m v_h(\lambda_h(\xi),\xi) - D^m v(\lambda(\xi),\xi)\|$$
$$+\|(P_h - I)TD^m G(\lambda(\xi),u(\xi))\| + \|D^m G(\lambda_h(\xi), u_0 + \sum \xi_i\varphi_i + v_h(\lambda_h(\xi),\xi)$$
$$-D^m G(\lambda(\xi),u(\xi))\|\} \le c_1\{\|D^m v_h(\lambda_h(\xi),\xi) - D^m v(\lambda(\xi),\xi)\|$$
$$+\|(P_h - I)TD^m G(\lambda(\xi),u(\xi))\| + c_2\sum_{i=0}^{m}(\|D^i\lambda_h(\xi) - D^i\lambda(\xi)\|$$
$$+\|D^i v_h(\lambda_h(\xi),\xi) - D^i v(\lambda(\xi),\xi)\|\}.$$

By Theorem 2, we have

$$\|D^m\lambda_h(\xi) - D^m\lambda(\xi)\| \le K\sum_{i=0}^{m}\|(P_h - I)TD^iG(\lambda(\xi),u(\xi))\|.$$

Particularly, for all ξ with $|\xi_i| \le \alpha_0, i = 1,2$, we have

$$|\lambda_h(\xi) - \lambda(\xi)| \le c_3\|(P_h - I)TG(\lambda(\xi),u(\xi))\|.$$

Hence, if we choose h_0, δ_0 small enough, we can get for $h \le h_0$

$$|\lambda_h(\xi) - \lambda_0| \le \delta_0, \quad |\lambda(\xi) - \lambda_0| \le \delta_0, \text{ if } |\xi_i| \le \alpha_0, \quad i = 1,2,$$

in which δ_0 is as in Theorem 1. Thus by (3.11) we obtain

$$\|D^m v_h(\lambda_h(\xi),\xi) - D^m v(\lambda(\xi),\xi)\|$$
$$\le c_4\Big\{|\lambda_h(\xi) - \lambda(\xi)| + \sum_{i=0}^{m}\|(P_h - I)TD^iG(\lambda(\xi),u(\xi))\|\Big\}.$$

Therefore, we have for $h \le h_0$ and $|\xi_i| \le \alpha_0, i = 1,2$,

$$\|D^m g_h(\xi) - D^m g(\xi)\| \le c_5\sum_{i=0}^{m}\|(P_h - I)TD^iG(\lambda(\xi),u(\xi))\|.$$

From condition (3.9) we know that (3.15) holds. Hence for $h \le h_0$,

$$\|D^m g_h(\xi^*) - D^m g(\xi)\| \le c_6\|\xi^* - \xi\|, \quad |\xi_i| \le \alpha_0, \quad |\xi_i^*| \le \alpha_0.$$

It follows that (3.16) holds. This completes the proof.

Lemma 3. *Under the assumptions of* (2.2), (2.3) *and* (3.9) *for* $i_0 = 2$, *there exist an* $h_0 > 0$ *and for* $h \le h_0$, *a unique point* $(\xi_{1,h}^0, \xi_{2,h}^0)$ *satisfying*

$$Dg_h(\xi_{1,h}^0, \xi_{2,h}^0) = 0, \tag{3.17}$$

$$|\xi_{1,h}^0| + |\xi_{2,h}^0| \le c\sum_{i=0}^{1}\|(P_h - I)TD^iG(\lambda_0,u_0)\|. \tag{3.18}$$

Proof. By Lemma 2, we have for $h \to 0$,

$$\|Dg_h(0,0) - Dg(0,0)\| \to 0, \quad \|D^2 g_h(0,0) - D^2 g(0,0)\| \to 0.$$

Next,

$$Dg(0,0) = 0,$$

$$\det D^2 g(0,0) = \det \begin{pmatrix} A_0 & B_0 \\ B_0 & C_0 \end{pmatrix} = A_0 C_0 - B_0^2 < 0.$$

Thus $D^2 g(0,0)$ is invertible. Therefore, we can apply Lemma 1 in the following situation:

$$\Phi(x,\xi) = Dg(\xi), \quad \Phi_h(x,\xi) = Dg_h(\xi), \quad \xi(x) = 0, \text{ for } x \in R^1.$$

There exists a unique point $(\xi_{1,h}^0, \xi_{2,h}^0)$ such that (3.17) holds and

$$|\xi_{1,h}^0| + |\xi_{2,h}^0| \le K\|Dg_h(0,0)\|.$$

From (3.16) we can get (3.18). This completes the proof.

Now we set $g_h^0 = g_h(\xi_{1,h}^0, \xi_{2,h}^0)$. Using the Taylor expansion and Lemma 2, we have

$$|g_h^0| \le |g_h(0,0)| + \|Dg_h(0,0)\|(|\xi_{1,h}^0| + |\xi_{2,h}^0|) + c(|\xi_{1,h}^0|^2 + |\xi_{2,h}^0|^2).$$

By (3.19), we have

$$|g_h^0| \le |g_h(0,0)| + c_1\|Dg_h(0,0)\|^2,$$

$$|g_h^0| \le c_2\Big\{\|(P_h - I)TG(\lambda_0,u_0)\| + \Big[\sum_{i=0}^{1}\|(P_h - I)TD^iG(\lambda_0,u_0)\|\Big]^2\Big\}. \tag{3.20}$$

Define a function g_h by

$$\tilde{g}_h(\xi) = g_h(\xi) - g_h^0.$$

Then we have $\tilde{g}_h(\xi_h^0) = 0, D\tilde{g}_h(\xi_h^0) = 0$. Set

$$A_h = \partial^2 g_h^0/\partial\xi_1^2, \quad B_h = \partial^2 g_h^0/\partial\xi_1\partial\xi_2, \quad C_h = \partial^2 g_h^0/\partial\xi_2^2.$$

It follows from Lemma 2 and Lemma 3 that for $h \to 0$

$$\|D^2 g_h(\xi_h^0) - D^2 g(0)\| \to 0, \quad B_h^2 - A_h C_h > 0.$$

Now we can prove the following lemma.

Lemma 4. *Under the assumptions of* (2.2), (2.3), (2.9) *and* (3.9) *for* $i_0 = 3$, *one can find* $h_0, t_0 > 0$ *such that for* $h \le h_0$ *the branches of solutions of* $\tilde{g}_h(\xi) = 0$ *may be parametrized in the form* $\{(\tilde{\xi}_{1,h}^i(t), \tilde{\xi}_{2,h}^i(t)) : |t| \le t_0\}$, *in which the* C^{r-2} *functions* $\tilde{\xi}_{1,h}^i(t), \tilde{\xi}_{2,h}^i(t)$ *satisfy*

$$\tilde{\xi}_{1,h}^i(0) = \xi_{1,h}^0, \quad \tilde{\xi}_{2,h}^i(0) = \xi_{2,h}^0, \quad i = 1,2.$$

Moreover, for all integers m *with* $0 \le m \le r-3$, *there exists a constant* K_m *such that*

$$\sup_{|t|\le t_0}\Big\{\Big|\frac{d^m}{dt^m}(\tilde{\xi}_{1,h}^i(t) - \xi_1^i(t))\Big| + \Big|\frac{d^m}{dt^m}(\tilde{\xi}_{2,h}^i(t) - \xi_2^i(t))\Big|\Big\}$$

$$\le K_m\Big\{\sum_{i=0}^{1}\|(P_h - I)TD^iG(\lambda_0,u_0)\| + \sup_{|t|\le t_0}\sum_{i=0}^{m+1}\|(P_h - I)T\frac{d^i}{dt^i}G(\lambda_i(t),u_i(t))\|\Big\}. \tag{3.21}$$

Proof. Define the function $H_h : R^3 \to R^2$ by

$$H_h(t,\sigma,a) = (t^{-2}g_h(\xi^0_{1,h}+t\sigma, \xi^0_{2,h}+ta), \sigma^2+a^2-1).$$

Since $(\xi^0_{1,h}, \xi^0_{2,h})$ is a critical point of the function g_h, we have

$$H_h(t,\sigma,a) = \Big(\int_0^1 (1-s)D^2g_h(\xi^0_{1,h}+st\sigma, \xi^0_{2,h}+sta)(\sigma,a)^2 ds, \sigma^2+a^2-1\Big).$$

On the other hand, we have

$$H(t,\sigma,a) = \Big(\int_0^1 (1-s)D^2g(st\sigma, sta)\cdot(\sigma,a)^2 ds, \sigma^2+a^2-1\Big).$$

Using Lemma 2 and 3, we have for $h \to 0$

$$\|D^m g_h(\xi^0_{1,h}+st\sigma, \xi^0_{2,h}+sta) - D^m g(st\sigma, sta)\| \to 0, \quad m=0,1,2,3.$$

This limit is uniformly convergent for $(s,t,\sigma,a) \in B \subset R^4$, in which B is any given bounded closed set. Hence for $h \to 0, H_h$ converges uniformly to H together with its first derivative. Moreover,

$$\det D_{(\sigma,a)}H(0,\sigma_i(0),a_i(0)) \neq 0, \quad i=1,2.$$

Therefore, by choosing t_0 small enough, we can get for $|t| \le t_0$

$$\|D_{(\sigma,a)}H(t,\sigma_i(t),a_i(t))^{-1}\| \le c_1,$$

and it can be derived from (3.9) that $D^{r-2}H_h(t,\sigma,a)$ and $D^{r-2}H(t,\sigma,a)$ are bounded on any bounded set of R^3. Thus by Lemma 1, there exist an $h_0 > 0$ and for $h \le h_0$, two pairs of C^{r-2} functions $(\sigma^i_h(t), a^i_h(t)), i=1,2$, such that for all t with $|t| \le t_0$,

$$H_h(t,\sigma^i_h(t),a^i_h(t)) = 0, \quad i=1,2.$$

Furthermore, we have for all integers m with $0 \le m \le r-3$ and all $|t| \le t_0$

$$\Big|\frac{d^m}{dt^m}(\sigma^i_h(t)-\sigma_i(t))\Big| + \Big|\frac{d^m}{dt^m}(a^i_h(t)-a_i(t))\Big| \le c_2\sum_{i=0}^m \Big\|\frac{d^i}{dt^i}H_h(t,\sigma_i(t),a_i(t))\Big\|.$$

Set

$$\tilde\xi^i_{1,h}(t) = \xi^0_{1,h} + t\sigma^i_h(t), \quad \tilde\xi^i_{2,h}(t) = \xi^0_{2,h} + ta^i_h(t).$$

Then $\{(\tilde\xi^i_{1,h}(t), \tilde\xi^i_{2,h}(t)) : |t| \le t_0\}, i=1,2$, are the solutions of $\tilde g_h(\xi)=0$.

The proof of (3.21) is very similar to the proof of Lemma 6 in [2] (Part III), so it is omitted here. This completes the proof.

If $g^0_h = 0$, then $\tilde g_h(\xi) = g_h(\xi)$, and there is no extra work to do. Let us now consider the general case that $g^0_h \neq 0$.

Let $\alpha > 0$. Denote by $S(0,\alpha)$ the neighborhood of $(\xi_1,\xi_2) = (0,0)$, by S_h the set of solutions of $g_h(\xi) = 0$ contained in $\bar S(0,\alpha)$, and by $\tilde S_h$ the set of solutions of $\tilde g_h(\xi) = 0$ contained in $\bar S(0,\alpha)$. Define the distance $d(A,B)$ of two closed sets A and B in a normed space by

$$d(A,B) = \max\Big(\sup_{x\in A}\inf_{y\in B}\|x-y\|, \quad \sup_{y\in B}\inf_{x\in A}\|x-y\|\Big).$$

Similarly to the proof of Lemma 7 in [2] (Part III), we can prove

Lemma 5. *Assume the hypotheses of Lemma 4 and $r \geq 4$. Then the set S_h is C^{r-2}-diffeomorphic to (a part of) a nondegenerate hyperbola. Moreover,*

$$d(S_h, S_h) \leq c\sqrt{|g_h^0|}.$$

Concluding all the above results, we have

Theorem 3. *Under the assumptions of (2.2), (2.3), (2.9) and $r \geq 4$, and if condition (3.9) holds for $i_0 = 3$, then, there exists a neighborhood $\mathcal{N}$ of the point (λ_0, u_0) and a positive constant h_0 such that for $h \leq h_0$, the set φ_h of the solutions of (3.6) contained in $\mathcal{N}$ consists of two C^{r-2} branches.*

If these two branches intersect at a point $(\lambda_h^0, u_h^0) \in \mathcal{N}$, they can be parametrized in the form $\{(\lambda_h^i(t), u_h^i(t)) : |t| \leq t_0\}, i = 1, 2$, which satisfies $\lambda_h^i(0) = \lambda_h^0, u_h^i(0) = u_h^0, i = 1, 2$, and moreover, for all $0 \leq m \leq r-3$,

$$\sup_{|t|\leq t_0} \left\{\left|\frac{d^m}{dt^m}(\lambda_h^i(t) - \lambda_i(t))\right| + \left\|\frac{d^m}{dt^m}(u_h^i(t) - u_i(t))\right\|\right.$$
$$\leq K_m\left\{\sum_{i=0}^{1} \|(P_h - I)TD^iG(\lambda_0, u_0)\| + \sup_{|t|\leq t_0} \sum_{i=0}^{m+1} \|(P_h - I)T\frac{d^i}{dt^i}G(\lambda_i(t), u_i(t))\|\right\}. \tag{3.23}$$

Otherwise, the distance between the set φ_h and the set φ of solutions contained in $\mathcal{N}$ may be estimated by

$$d(\varphi_h, \varphi) \leq c\sqrt{|g_h^0|} + \sum_{i=0}^{1} \|(P_h - I)TD^iG(\lambda_0, u_0)\|$$
$$+ \sup_{|t|\leq t_0} \sum_{j=1}^{2}\sum_{i=0}^{1} \|(P_h - I)T\frac{d^i}{dt^i}G(\lambda_j(t), u_j(t))\|\}. \tag{3.24}$$

§4. A Numerical Example

Consider a two-point boundary value problem

$$\begin{cases} u'' + 4\pi^2\lambda u + \cos(\pi t)(u - \lambda\sin(2\pi t))^2 = 0, & 0 < t < 1, \\ u(0) = u(1), u'(0) = u'(1). \end{cases} \tag{4.1}$$

Let $V = \{u \in H^1, u(0) = u(1), u'(0) = u'(1)\}$,

$$(u, v)_V = (\nabla u, \nabla v)_0, \quad \|u\|_V = \|\nabla u\|_0,$$

where $(\cdot, \cdot)_0$ and $\|\cdot\|_0$ are the inner product and norm in $L^2(0, 1)$, and u is the gradient of u. Thus V is a Hilbert space.

By Friedrichs' inequality

$$\|u\|_0 \le c\|\nabla u\|_0 = c\|u\|_V, \quad \forall u \in V, \tag{4.2}$$

$$|(u,v)_0| \le \|u\|_0\|v\|_0 \le c^2\|u\|_V\|v\|_V, \quad \forall u,v \in V.$$

By Riesz' representation theorem, there exists a continuous linear operator $T: V \to V$ such that

$$(u,v)_0 = (Tu,v)_V, \quad \forall u,v \in V. \tag{4.3}$$

Since $H^1(0,1)$ inserts compactly into $L^2(0,1)$, it is easy to check that T is a compact self-adjoint operator.

Define the nonlinear operator $G: R \times V \to V$ by

$$(G(\lambda,u),v) = (-4\pi^2\lambda u - \cos(\pi t)(u - \lambda\sin(2\pi t))^2, v), \tag{4.4}$$

where $u, v \in V$. Thus, problem (4.1) is equivalent to

$$F(\lambda,u) = u + TG(\lambda,u) = 0. \tag{4.5}$$

Set $\lambda_0 = 1, u_0 = \sin 2\pi t$. It is easy to check that

1) $F(\lambda_0,u_0) = 0$;

2) $D_uF^0 = I + TD_uG^0 = I - 4\pi^2T \in \mathcal{L}(V,V)$, 1 is the double eigenvalue of $4\pi^2T$, and

$$\mathrm{Ker}(D_uF^0) \text{ span } \{\sin 2\pi t, \cos 2\pi t\};$$

3) $D_\lambda F^0 = TD_\lambda G^0 = -8\pi^2Tu_0$,

$$(D_\lambda F^0, \sin 2\pi t)_V \ne 0, \quad (D_\lambda F^0, \cos 2\pi t)_V = 0.$$

Since T is compact and self-adjoint, we know that D_uF^0 is self-adjoint, and Range (D_uF^0) is closed.

Setting

$$\varphi_1 = \sin 2t, \quad \varphi_2 = \cos 2t,$$

we have

$$\mathrm{Range}(D_uF^0) = \{v \in V : (v,\varphi_i)_V = 0, \quad i = 1,2\}.$$

From 3) we know that $D_\lambda F^0 \notin \mathrm{Range}(D_uF^0)$, and

$$D_{uu}F^0 = TD_{uu}G^0 = -2T\cos\pi t \cdot I.$$

Hence we have

$$\begin{aligned}
A_0 &= (D_{uu}F^0\varphi_1\varphi_1, \varphi_2)_V \\
&= (-2T\cos\pi t(\sin 2\pi t)^2, \cos\pi t)_V = (-2\cos\pi t(\sin 2\pi t)^2, \cos 2\pi t)_0 = 0, \\
B_0 &= (D_{uu}F^0\varphi_1\varphi_2, \varphi_2)_V = (-2T\cos\pi t\sin 2\pi t\cos 2\pi t, \cos 2\pi t)_V \\
&= (-2\cos\pi t\sin 2\pi t\cos 2\pi t, \cos 2\pi t)_0 = -88/105, \\
C_0 &= (D_{uu}F^0\varphi_2\varphi_2, \varphi_2)_V \\
&= (-2T\cos\pi t(\cos 2\pi t)^2, \cos 2\pi t)_V = (-2\cos\pi t(\cos(2\pi t)^2, \cos 2\pi t)_0 = 0.
\end{aligned}$$

Thus we obtain

$$B_0^2 - A_0C_0 > 0.$$

Therefore, $(\lambda_0, u_0) = (1, \sin 2\pi t)$ is a double limit point of problem (4.1), and there exist two smooth branches of solutions of (4.1), which are tangent or have a common tangent plane at (λ_0, u_0).

We discretize (4.1) by the Galerkin method, in which the cardinal functions are chosen as piecewise polynomials of order 1 and $h = 1/10$. Taking $(0,0)$ as the initial point of the continuation procedure[3], we get a solution arc Γ_1. By taking the step-length of the continuation procedure large enough near the double limit point (λ_h^0, u_h^0), we make the continuation procedure go on the other branch Γ_2 of solutions. Due to the error resulting from discretization and computation, Γ_1 does not intersect Γ_2. Their turning points appear at $\lambda_1^* = 1.03310$ and $\lambda_2^* = 1.09860$ respectively (Fig. 4).

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