Journal of Computational Mathematics, Vol.8, No.3, 1990, 252-260.

SYMPLECTIC COLLOCATION SCHEMES FOR HAMILTONIAN SYSTEMS *

Xie Zhi-yun
(*Zhongshan University, Guangzhou, China*)

Abstract

The symplectic collocation schemes, which are based on the framework established by Feng Kang [1], are proposed for numerical solution of Hamiltonian systems. The sufficient and necessary conditions for various collocation schemes to be symplectic are obtained. Some examples of symplectic collocation schemes are also given.

§1. Introduction

Physics is characterized by conservation laws and symmetry [2]. This point of view suggests that the corresponding numerical method to be designed should preserve these invariants as much as possible, so that the corresponding numerical result could exactly reflect the essence of a real physical process. The Hamiltonian system describes a negligible dissipative physical process which plays an important role in the dynamical system. In recent years, numerical methods for the Hamiltonian system have received extensive attention. Due to the special structures and properties of this system (e.g., the stable phase flow of the system is noncontractible and preserves symplectic structures), the mumerical are naturally expected to be designed in such a way that these characters could be taken into full account.

For the consideration mentioned above, Feng Kang [1] proposed a new approach for the Hamiltonian system, the symplectic numerical method, which appears to be an active and interesting subject. Moreover, Feng and his group have systematically studied the symplectic difference schemes in recent years [1, 4, 5].

The spline function is one of the most useful mathematical tools in numerical analysis, and it is connected inherently with the generalized Hamiltonian system [3]. In this paper, we study the spline collocation method for the Hamiltonian system within the framework established by Feng [1]. The sufficient and necessary conditions for various operator spline collocation schemes to be symplectic are obtained.

§2. The Symplectic Collocation Scheme

Consider the canonical system of equations

$$\frac{dz}{dt} = K^{-1}H_z(z) \tag{2.1}$$

* Received November 14, 1987.

where $z = (z_1, \cdots, z_{2n})^T \in R^{2n}$, K is a given nonsingular anti-symmetric matrix of order $2n$, and $H(z)$ a Gateaux differentiable functional called Hamiltonian of the system.

The 2-form defined on R^{2n} by

$$\omega_k = \sum_{i<j} k_{ij} dz_i \wedge dz_j \tag{2.2}$$

gives a natural symplectic structure on R^{2n}. Let g^t be the stable phase flow of system (2.1). Then g^t is noncontiractible, or generally speaking, g^t preserves the symplectic structure, i.e.,

$$(g^t)^* \omega_k = \omega_k. \tag{2.3}$$

Let $S(t)$ be a collocation solution for canonical equation (2.1) with knots $\{t_i\} : t_0 < t_1 < t_2 < \cdots$. A one-step collocation scheme (e.g., $S(t)$ is a spline function of degree less than three) is said to be symplectic if the transition $S(t_i) \to S(t_{i+1})$ from the i-th time step $S(t_i)$ to the next $(i+1)$-th time step $S(t_{i+1})$ is K-canonical for all i. (A transition is called a K-canonical one if it preserves symplectic structure (2.2)).

§3. Collocation Scheme Using Operator Spline of Order Two

Let $S(t)$ be a second order operator spline with knots $\{t_i\}, S(t) \in C^0$,

$$S(t) = c_i \varphi_1(t) + d_i \varphi_2(t), \quad t_i \le t \le t_{i+1}, \tag{3.1}$$

where $\varphi_1(t), \varphi_2(t)$ is a Tchebysheff system (i.e., φ_1, φ_2 satisfies the Haar condition) and c_i, d_i are constants. The collocation scheme for (2.1) is

$$\dot{S}(t_{i+1/2}) = K^{-1} H_z(S(t_{i+1/2})), \quad i = 0, 1, 2, \cdots \tag{3.2}$$

In the following discussion, we denote $S(t_i)$ by S_i, and $S(t_{i+1/2})$ by $S_{i+1/2}$, etc.

Theorem 3.1. *Collocation scheme* (3.2) *is symplectic if and only if the following equalities*

$$(\alpha_i' + \beta_i')(\alpha_i' - \beta_i')I = (\alpha_i - \beta_i)(\alpha_i + \beta_i)(K^{-1} H_{zz}(S_{i+1/2}))^2, \quad i = 0, 1, 2, \cdots \tag{3.3}$$

hold, where

$$\alpha_i = \phi(\sqrt{2}), \ \alpha_i' = d\phi(1/2)/dt, \ \beta_i = \psi(1/2), \quad \beta_i' = d\psi(1/2)/dt,$$

$$\phi(t) = [\varphi_2(t_{i+1})\varphi_1(t) - \varphi_1(t_{i+1})\varphi_2(t)]/\det\begin{pmatrix} t_i & t_{i+1} \\ \varphi_1 & \varphi_2 \end{pmatrix},$$

$$\psi(t) = [\varphi_1(t_i)\varphi_2(t) - \varphi_2(t_i)\varphi_1(t)]/\det\begin{pmatrix} t_i & t_{i+1} \\ \varphi_1 & \varphi_2 \end{pmatrix},$$

and $H_{zz}(S_{i+1/2})$ is the Hessian matrix of the function $H(z)$.

Proof. From (3.1),

$$S_i = c_i \varphi_1(t_i) + d_i \varphi_2(t_i),$$

$$S_{i+1} = c_i \varphi_1(t_{i+1}) + d_i \varphi_2(t_{i+1}),$$

one obtains

$$c_i = (\varphi_2(t_{i+1})S_i - \varphi_2(t_i)S_{i+1})/\det\begin{pmatrix} t_i & t_{i+1} \\ \varphi_1 & \varphi_2 \end{pmatrix},$$

$$d_i = (\varphi_1(t_i)S_{i+1} - \varphi_1(t_{i+1})S_i)/\det\begin{pmatrix} t_i & t_{i+1} \\ \varphi_1 & \varphi_2 \end{pmatrix}.$$

Hence

$$S_{i+1/2} = \alpha_i S_i + \beta_i S_{i+1}, \quad S'_{i+1/2} = \alpha'_i S_i + \beta'_i S_{i+1}.$$

In view of (3.1), one has

$$\alpha'_i S_i + \beta'_i S_{i+1} = K^{-1}H_z(\alpha_i S_i + \beta_i S_{i+1}).$$

It follows that

$$\frac{dS_{i+1}}{dS_i} = (\beta'_i I - \beta_i K^{-1}H_{zz}(S_{i+1/2}))^{-1}(-\alpha'_i I + \alpha_i K^{-1}H_{zz}(s_{i+1/2}))$$

which is K-canonical if and only if

$$[(\alpha'_i)^2 - (\beta'_i)^2]I = (\alpha_i^2 - \beta_i^2)(K^{-1}H_{zz}(S_{i+1/2}))^2.$$

It turns out to be (3.3), and this completes the proof.

Corollary 3.1. It is sufficient for scheme (3.2) to be symplectic that

$$(\alpha'_i + \beta'_i)(\alpha'_i - \beta'_i) = (\alpha_i + \beta_i)(\alpha_i - \beta_i) = 0, \quad i = 0, 1, \cdots \tag{3.4}$$

and moreover, when $(K^{-1}H_{zz})$ is not a scalar matrix, then (3.4) is also necessary.

Example 3.1. Take $S(t) = c_i e^{\lambda t} + d_i e^{-\lambda t} (\lambda \neq 0), t_i \leq t \leq t_{i+1}$. In this case, $\varphi_1(t) = e^{\lambda t}, \varphi_2(t) = e^{-\lambda t}$. A straightforward calculation shows that condition (3.4) is satisfied for this case. Therefore the corresponding collocation scheme (3.2) is always symplectic.

Example 3.2. Take $S(t) = c_i \cos(\lambda t) + d_i \sin(\lambda t)$ $(\lambda \neq 0)$, $t_i \leq t \leq t_{i+1}$. A straightforward calculation shows that condition (3.4) is satisfied in this case. Therefore the corresponding collocation scheme (3.2) is always symplectic.

Example 3.3. Take $S(t) = c_i + d_i t, t_i \leq t \leq t_{i+1}$. The corresponding collocation scheme (3.2) is always symplectic.

The above three examples give the three basic kinds of symplectic collocation schemes using the second order operator spline.

§4. Collocation Schemes Using Operator Splines of Order Three: for Linear Equations

Consider the linear canonical equations

$$\frac{dz(t)}{dt} = K^{-1}Bz(t) \tag{4.1}$$

with the Hamiltonian $H(z) = \frac{1}{2}z(t)Bz(t)$, where B is a constant matrix.

Let $S(t)$ be an operator spline of order three with knots $\{t_i\}$, defined by a differential operator $L(D) = (D-\lambda_1)(D-\lambda_2)(D-\lambda_3)$. $S(t) \in C^1$, where $\lambda_i (i = 1,2,3)$ are constants. Denote $h_i = t_{i+1} - t_i$, or simply $h = t_{i+1} - t_i$.

The collocation scheme for (4.1) is

$$S'(t_i) = K^{-1}BS(t_i), \quad i = 0, 1, 2, \cdots. \tag{4.2}$$

For constants $\lambda_i (i = 1,2,3)$, there exist four cases:

(I) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1, \lambda_i$ are real numbers, $i = 1,2,3$,

(II) $\lambda_1 = \lambda_2 = \lambda_3$, λ_i are real numbers, $i = 1,2,3$,

(III) $\lambda_1 = \bar{\lambda}_2$, λ_3 is a real number, and λ_i are complex numbers, $i = 1,2$,

(IV) $\lambda_1 = \lambda_2 \neq \lambda_3$, λ_i are real numbers, $i = 1,2,3$.

The corresponding operator splines will be called splines of class (I), (II), (III) and (IV), respectively.

4.1. Collocation using splines of class (I)

Theorem 4.1. *For splines of class* (I), *collocation scheme* (4.2) *is symplectic if and only if*

$$(\alpha_+\lambda_2 - \beta_+\lambda_3)(\alpha_-\lambda_2 - \beta_-\lambda_3)I = (\alpha_+ - \beta_+)(\alpha_- - \beta_-)(K^{-1}B)^2 \tag{4.3}$$

holds for $i = 0, 1, 2, \cdots$, *where*

$$\alpha_\pm = (\lambda_3 - \lambda_1)(e^{\lambda_2 h} - e^{\lambda_1 h})(1 \pm e^{\lambda_3 h}),$$
$$\beta_\pm = (\lambda_3 - \lambda_1)(e^{\lambda_3 h} - e^{\lambda_1 h})(1 \pm e^{\lambda_2 h}).$$

Proof. The spline of class (I) can be expressed by

$$S(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}, \quad t_i \le t \le t_{i+1},$$

where c_1, c_2, c_3 are constants. This implies

$$S_i = c_1 e^{\lambda_1 t_i} + c_2 e^{\lambda_2 t_i} + c_3 e^{\lambda_3 t_i}, \tag{4.4}$$
$$S_{i+1} = c_1 e^{\lambda_1 t_{i+1}} + c_2 e^{\lambda_2 t_{i+1}} + c_3 e^{\lambda_3 t_{i+1}}, \tag{4.5}$$
$$S_i' = c_1\lambda_1 e^{\lambda_1 t_i} + c_2\lambda_2 e^{\lambda_2 t_i} + c_3\lambda_3 e^{\lambda_3 t_i}, \tag{4.6}$$
$$S_{i+1}' = c_1\lambda_1 e^{\lambda_1 t_{i+1}} + c_2\lambda_2 e^{\lambda_2 t_{i+1}} + c_3\lambda_3 e^{\lambda_3 t_{i+1}}. \tag{4.7}$$

From (4.6) and (4.7), one obtains

$$S_i' e^{\lambda_3 t_{i+1}} - S_{i+1}' e^{\lambda_3 t_i} = c_1\lambda_1 e^{\lambda_1 t_i + \lambda_3 t_{i+1}}(1 - e^{(\lambda_1-\lambda_3)h}) + c_2\lambda_2 e^{\lambda_2 t_i + \lambda_3 t_{i+1}}(1 - e^{(\lambda_2-\lambda_3)h}).$$

From (4.4) and (4.5)

$$S_i e^{\lambda_3 t_{i+1}} - S_{i+1} e^{\lambda_3 t_i} = c_1 e^{\lambda_1 t_i + \lambda_3 t_{i+1}}(1 - e^{(\lambda_1-\lambda_3)h}) + c_2 e^{\lambda_2 t_i + \lambda_3 t_{i+1}}(1 - e^{(\lambda_2-\lambda_3)h}).$$

Since $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, the coefficient matrix for c_1 and c_2 is nonsingular. Hence

$$c_1 = [(S_i e^{\lambda_3 h} - S_{i+1})\lambda_2 - (S_i' e^{\lambda_3 h} - S_{i+1}')]/[(\lambda_2 - \lambda_1)(e^{\lambda_3 h} - e^{\lambda_1 h})e^{\lambda_1 t_i}]. \tag{4.8}$$

From (4.5) and (4.7),

$$\lambda_3 S_{i+1} - S_{i+1}' = (\lambda_3 - \lambda_1)c_1 e^{\lambda_1 t_{i+1}} + (\lambda_3 - \lambda_2)c_2 e^{\lambda_2 t_{i+1}}.$$

From (4.4) and (4.6)

$$\lambda_3 S_i - S_i' = (\lambda_3 - \lambda_1) c_1 e^{\lambda_1 t_i} + (\lambda_3 - \lambda_2) c_2 e^{\lambda_2 t_i}.$$

The above two equalities give

$$c_1 = [(\lambda_3 S_i - S_i') e^{\lambda_2 h} - (\lambda_3 S_{i+1} - S_{i+1}')]/[(\lambda_3 - \lambda_1)(e^{\lambda_2 h} - e^{\lambda_1 h}) e^{\lambda_1 t_i}]. \tag{4.9}$$

Comparing (4.8) with (4.9) yields

$$(\alpha - \beta) S_{i+1}' - (\alpha e^{\lambda_3 h} - \beta e^{\lambda_2 h}) S_i' = (\alpha\lambda_2 - \beta\lambda_3) S_{i+1} - (\alpha\lambda_2 e^{\lambda_3 h} - \beta\lambda_3 e^{\lambda_2 h}) S_i$$

where

$$\alpha = (\lambda_3 - \lambda_1)(e^{\lambda_2 h} - e^{\lambda_1 h}), \quad \beta = (\lambda_2 - \lambda_1)(e^{\lambda_3 h} - e^{\lambda_1 h}).$$

In view of $\dot{S}_i = K^{-1} B S_i$ and $\dot{S}_{i+1} = K^{-1} B S_{i+1}$, one obtains

$$S_{i+1} = F S_i$$

with $F : S_i \to S_{i+1}$,

$$F = [(\alpha - \beta) K^{-1} B - (\alpha\lambda_2 - \beta\lambda_3) I]^{-1} [(\alpha e^{\lambda_3 h} - \beta e^{\lambda_2 h}) K^{-1} B - (\alpha\lambda_2 e^{\lambda_3 h} - \beta\lambda_3 e^{\lambda_2 h}) I],$$

which is K-canonical if and only if

$$[(\alpha\lambda_2 - \beta\lambda_3)^2 - (\alpha\lambda_2 e^{\lambda_3 h} - \beta\lambda_3 e^{\lambda_2 h})^2] I = [(\alpha - \beta)^2 - (\alpha e^{\lambda_3 h} - \beta e^{\lambda_2 h})^2](K^{-1} B)^2.$$

By setting $\alpha_\pm = \alpha(1 \pm e^{\lambda_3 h}), \beta_\pm = \beta(1 \pm e^{\lambda_2 h})$, we get (4.3).

Theorem 4.2. *The collocation scheme* (4.2) *using splines of class* (I) *is symplectic for any linear cononical equations, if and only if the following equalities hold:*

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{4.10}$$

Proof. If $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0, \lambda_1 + \lambda_2 + \lambda_3 = 0$, a straitforward calculation yields

$$\alpha_- \lambda_2 - \beta_- \lambda_3 = 0, \quad \alpha_+ - \beta_+ = 0.$$

It is known from Theorem 4.1 that in this case the corresponding collocation scheme (4.2) is symplectic for any linear canonical equations (4.1).

On the other hand, Theorem 4.1 shows that the collocation scheme (4.2) is symplectic for any linear canonical equations (4.1) if and only if the following identities hold:

$$(\alpha_+ \lambda_2 - \beta_+ \lambda_3)(\alpha_- \lambda_2 - \beta_- \lambda_3) \equiv 0, \quad \text{for any } h > 0,$$
$$(\alpha_+ - \beta_+)(\alpha_- - \beta_-) \equiv 0, \quad \text{for any } h > 0.$$

By virtue of the following Taylor expansions

$$\alpha_+ \lambda_2 - \beta_+ \lambda_3 = 2(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3) h + O(h^2), \tag{4.11}$$

$$\alpha_- \lambda_2 - \beta_- \lambda_3 = -\frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}{12} (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) h^4 + O(h^5), \tag{4.12}$$

$$\alpha_+ - \beta_+ = \frac{\lambda_1 + \lambda_2 + \lambda_3}{6} (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) h^3 + O(h^4), \tag{4.13}$$

$$\alpha_- - \beta_- = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1) h^2 + O(h^3), \tag{4.14}$$

we have

$$\alpha_- - \beta_- \neq 0, \quad \alpha_+\lambda_2 - \beta_+\lambda_3 \neq 0$$

for sufficiently small h, since $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Therefore, collocation scheme (4.2) is always symplectic only if

$$\alpha_-\lambda_2 - \beta_-\lambda_3 = 0, \alpha_+ - \beta_+ = 0, \quad \forall h > 0.$$

By expensions (4.12)–(4.13), the above two identities hold only if

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0.$$

This completes the proof.

Example 4.1. Theorem 4.2 shows that, for splines of class (I), the collocation scheme (4.2) is symplectic for any canonical equations (4.1) if and only if the differential operator $L(D)$ takes the form

$$L(D) = D(D-\lambda)(D+\lambda), \quad \lambda \neq 0,$$

i.e., the collocation spline takes the form

$$S(t) = c_1 + c_2 e^{\lambda t} + c_3 e^{-\lambda t}, \quad t_i \leq t \leq t_{i+1},$$

where c_1, c_2, c_3 are constants.

4.2. Collocation using splines of class (II)

By a similar argument, we have

Theorem 4.3. *Collocation scheme* (4.2) *using splines of class* (II) *is symplectic if and only if the following identities hold* :

$$[(\lambda h + 2)^2 - (\lambda h - 2)^2 e^{2\lambda h}] I \equiv [1 - e^{2\lambda h}] h^2 (K^{-1}B)^2.$$

Corollary 4.1. The collocation scheme (4.2) using splines of class (II) is symplectic for any linear canonical equations (4.1) if and only if the collocation spline $S(t)$ is a polynomial spline of order three, i.e., $\lambda = 0$, i.e.,

$$S(t) = c_i + d_i t + b_i t^2, \quad t_i \leq t \leq t_{i+1},$$

where c_i, d_i, b_i are constants, and $S(t) \in C^1$.

4.3. Collocation using splines of class (III)

A similar argument gives the following results:

Theorem 4.4. *For the collocation splines of class* (III) *associated with differential operator* $L(D) = (D-\lambda_3)(D-\lambda)(D-\bar{\lambda})$, *the collocation scheme* (4.2) *is symplectic if and only if*

$$(\alpha_+\sigma_2 - \beta_+\sigma_1)(\alpha_-\sigma_2 - \beta_-\sigma_1) I = \beta_+\beta_-(K^{-1}B)^2,$$

where λ_3 *is a real number, and* $\lambda = \sigma_1 + \sigma_2\, i$ *is a complex number,*

$$\alpha_\pm = \sigma_2(-1 \mp e^{\lambda_3 h}) \sin \sigma_2 h + (\lambda_3 - \sigma_1)(1 \mp e^{\lambda_3 h}) \cos \sigma_2 h - (\lambda_3 - \sigma_1)(e^{(\lambda_3 - \sigma_1)h} \mp e^{\sigma_1 h}),$$

$$\beta_\pm = (\lambda_3 - \sigma_1)(-1 \mp e^{\lambda_3 h}) \sin \sigma_2 h + \sigma_2(-1 \pm e^{\lambda_3 h}) \cos \sigma_2 h + \sigma_2(e^{(\lambda_3 - \sigma_1)h} \mp e^{\sigma_1 h}).$$

Corollary 4.2. The collocation scheme (4.2) using splines of class (III) is symplectic for any canonical equations (4.1) if and only if the collocation spline $S(t)$ takes the following form

$$S(t) = c_1 + c_2 \sin \lambda t + c_3 \cos \lambda t, \quad \lambda \neq 0, \quad t_{i+1} \leq t \leq t_i,$$

where c_1, c_2, c_3 are constants.

4.4. Collocation using splines of class (IV)

Theorem 4.5. *For the collocation splines of class (IV) associated with differential operator $L(D) = (D - \lambda_3)(D - \lambda)^2$, the collocation scheme (4.2) is symplectic if and only if*

$$\alpha_+ \alpha_- I = \beta_+ \beta_- (K^{-1}B)^2,$$

where

$$\alpha_\pm = \lambda_3 (e^{(\lambda_3 - \lambda)h} - 1)(1 \pm e^{\lambda h}) - \lambda(\lambda_3 - \lambda)(1 \pm e^{\lambda_3 h})h,$$
$$\beta_\pm = (e^{(\lambda_3 - \lambda)h} - 1)(1 \pm e^{\lambda h}) - (\lambda_3 - \lambda)(1 \pm e^{\lambda_3 h})h.$$

Corollary 4.3. There does not exist a collocation spline of class (IV) such that the corresponding collocation scheme (4.2) is symplectic for any canonical equations (4.1).

The above result suggests that the collocation splines of class (IV) are not suitable for the Hamiltonian system in view of the symplectic method.

The collocation spline which always gives a symplectic scheme for any linear canonical equations will be called a symplectic spline.

Table 1. lists the symplectic splines of order three

class	Differential operator $L(D)$	Symplectic spline
(I)	$\prod_{i=1}^{3}(D - \lambda_i), \quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$	$c_1 + c_2 e^{\lambda t} + c_3 e^{-\lambda t}, \quad \lambda \neq 0$
(II)	$(D - \lambda)^3, \quad \lambda$ real	$c_1 + c_2 t + c_3 t^2$
(III)	$(D - \lambda_3)(D - \lambda)(D - \bar{\lambda}), \quad \lambda_3$ real, λ complex	$c_1 + c_2 \sin \lambda t + c_3 \cos \lambda t, \quad \lambda \neq 0$
(IV)	$(D - \lambda_3)(D - \lambda)^2, \quad \lambda_3 \neq \lambda$ real	No exist

§5. Collocation Schemes Using Operator Splines of Order Three: for nonlinear Equations

Consider nonlinear canonical equations

$$\frac{dz}{dt} = K^{-1} H_z(z) \tag{5.1}$$

with nonsingular anti-symmetric matrix K of order $2n$ and Gateaux differentiable functional $H(z), z \in R^{2n}$.

The results of the previous section show that only the splines listed in Table 1 (i.e., the symplectic splines) among operator splines of order three are suitable for the Hamiltonian system. For this reason, in this section we only consider symplectic splines.

Let $S(t)$ be a symplectic spline of order three, and construct the collocation scheme for (5.1) as follows:

$$S_0' = K^{-1}H_z(S_0), \tag{5.2}$$

$$\frac{1}{2}(S_i' + S_{i+1}') = K^{-1}H_z(\frac{1}{2}(S_i + S_{i+1})). \tag{5.3}$$

Obviously, the above scheme reduces to scheme (4.2) in the case that $H_z(z)$ is linear.

Theorem 5.1. *The collocation scheme* (5.2)–(5.3) *using symplectic splines of class* (I) *is always symplectic, and*

$$H(S_{i+1}) = H(S_i) + O(h^2).$$

Proof. For the symplectic spline in class (I), the following relation is valid:

$$(\text{ch}\lambda h - 1)(S_{i+1}' + S_i') = (\lambda\text{sh}\lambda h)(S_{i+1} - S_i).$$

In view of (5.2)–(5.3), we have

$$\frac{\partial S_{i+1}}{\partial S_i} = \left[\lambda\text{sh}\lambda hI - (\text{ch}\lambda h - 1)K^{-1}H_{zz}\left(\frac{S_i + s_{i+1}}{2}\right)\right]^{-1}$$
$$\times\left[\lambda\text{sh}\lambda hI + (\text{ch}\lambda h - 1)K^{-1}H_{zz}\left(\frac{S_i + s_{i+1}}{2}\right)\right]$$

which is a K-canonical transition. Therefore, the corresponding scheme (5.2)–(5.3) is symplectic.

By the mean value theorem, we have

$$H(S_{i+1}) - H(S_i) = 2\left(\frac{S_{i+1} - S_i}{2}\right)^T H_z\left(\frac{S_i + S_{i+1}}{2}\right)^- + \frac{1}{2}\left(\frac{S_{i+1} - S_i}{2}\right)^T (H_{zz}(\bar{S})$$
$$- H_{zz}(\tilde{\tilde{S}}))\left(\frac{S_{i+1} - S_i}{2}\right) = \frac{2\lambda\text{sh}\lambda h}{\text{ch}\lambda h - 1}\left(\frac{S_{i+1} - S_i}{2}\right)^T K\left(\frac{S_{i+1} - S_i}{2}\right)$$
$$+\frac{1}{8}(S_{i+1} - S_i)^T(H_{zz}(\bar{S}) - H_{zz}(\tilde{\tilde{S}}))(S_{i+1} - S_i)$$
$$= \frac{1}{8}(S_{i+1} - S_i)^T(H_{zz}(\tilde{S}) - h_{zz}(\tilde{\tilde{S}}))(S_{i+1} - S_i).$$

By noting that the scheme (5.2)–(5.3) has the first order accuracy, we get

$$H(S_{i+1}) - H(S_i) = O(h^2).$$

This completes the proof.

A similar argument gives the following results:

Theorem 5.2. *The scheme* (5.2)–(5.3) *using symplectic splines in class* (II) *or class* (III) *is always symplectic, and*

$$H(S_{i+1}) = H(s_i) + O(h^2).$$

Acknowledgment. The author wishes to express his sincere appreciation to Professor Li Yue–sheng and Professor Huang You–qian for their constant help and encouragement. The author also wishes to thank Dr. Ge Zhong for his helpfur discussions.

References

[1] Feng Kang, On difference scheme and symplectic geometry, in Computation of Partial Differential Equations, Proc. 5th Int. Sympo. On D.G. and D.E., Ed. Feng Kang, Science Press, Beijing, 1985, 42–58.

[2] D.Greenspan, Conservative numerical methods for $\ddot{x} = f(x)$, *J. Comput. phys.*, **56** (1984), 28–41.

[3] Xie Zhi-yun, Hamiltonian system and spline function, Thesis for Master's Degree, Zhongshan University, 1986.

[4] Feng Kang, Difference schemes for Hamiltonian formalism and symplectic geometry, Proc. of 10th Int. Meeting on Comput, 1986.

[5] Ge Zhong and Feng Kang, On the approximation of linear Hamiltonian system, *J. Comput. Math.*, **6** : 1 (1988), 88–97.