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A NEW CLASS OF VARIATIONAL FORMULATIONS FOR THE COUPLING OF FINITE AND BOUNDARY ELEMENT METHODS *[1)]

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§1. Introduction

The coupling of finite element (FEM) and boundary element (BEM) methods has been widely used in the engineering [1, 2]. The coupling method preserves the good features of both FEM, such as applicability to nonhomogeneous and nonlinear problems, and BEM, such as applicability to problems on unbounded domains. At present, there are two different concepts for this coupling. One consists of subdividing the original domain $\Omega \subset R^n$ into a finite number of subregions and using either FEM or BEM in each of them, where the latter lies on all boundaries of the subregions. The second concept, which we will not consider, uses BEM for the modelling of special finite element functions; see Schnack [3]. Error estimates for a coupling of FEM-BEM have been presented by Johnson and Nedelec [4] for a special exterior boundary value problem with the Laplacian. Wendland [5] extended their approach to more general equations and problems but needed a rather strong assumption, namely a compact perturbation of the identity, for the coupling operator between FEM and BEM. Unfortunately, this assumption is violated for many practical problems in e.g. elasticity. In this case, Wendland required a more careful coupling of the elements instead by imposing higher regularity of the finite elements at the coupling boundary and requiring a faster mesh refinement of the boundary elements than of the finite elements. In all the cases cited, the error estimates hold only for $0 < h_1,\ h_2 \leq h_0$, where h_1, h_2 denote the parameters of mesh width corresponding to FEM and BEM, and $h_0 > 0$ is an unknown constant. Hence in the practical computation, it is very hard to determine how small h_1 and h_2 must be to ensure that the error estimates hold.

In this paper, we present a new formulation for coupling of FEM and BEM, which preserves the coercive property of the original problem. Therefore, the strong assumption for the coupling operator required by Wendland will no longer be needed. Furthermore, the error estimates hold for all $h_1 > 0$ and $h_2 > 0$.

§2. The New Variational Formulations

Let Ω^c be the complement of a bounded regular domain in R^2 with boundary Γ. We consider the following two problems as examples.

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2.1. Dirichlet problem for Poisson's equation

Consider the boundary value problem:

$$\begin{cases} -\Delta u = f, \text{ in } \Omega^c, \\ u = 0, \text{ on } \Gamma, \\ u \text{ is bounded, when } |x| \to +\infty \end{cases} \tag{2.1}$$

where f has its support in a bounded subdomain Ω_1 of Ω^c. Let Γ_2 be the boundary of Ω_1 in Ω^c and Ω_2 be $\Omega^c \setminus \bar{\Omega}_1$ (see Fig. 1). We solve the problem (2.1) by using the coupling of FEM and BEM. Consider the equivalent system of equations:

$$\begin{cases} -\Delta u_1 = f, & \text{in } \Omega_1, \\ -\Delta u_2 = 0, & \text{in } \Omega_2, \\ u_1 = u_2, & \text{on } \Gamma_2, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \lambda, & \text{on } \Gamma_2 \\ u_1 = 0, & \text{on } \Gamma, \\ u_2 \text{ is bounded, when } |x| \to \infty \end{cases} \tag{2.2}$$

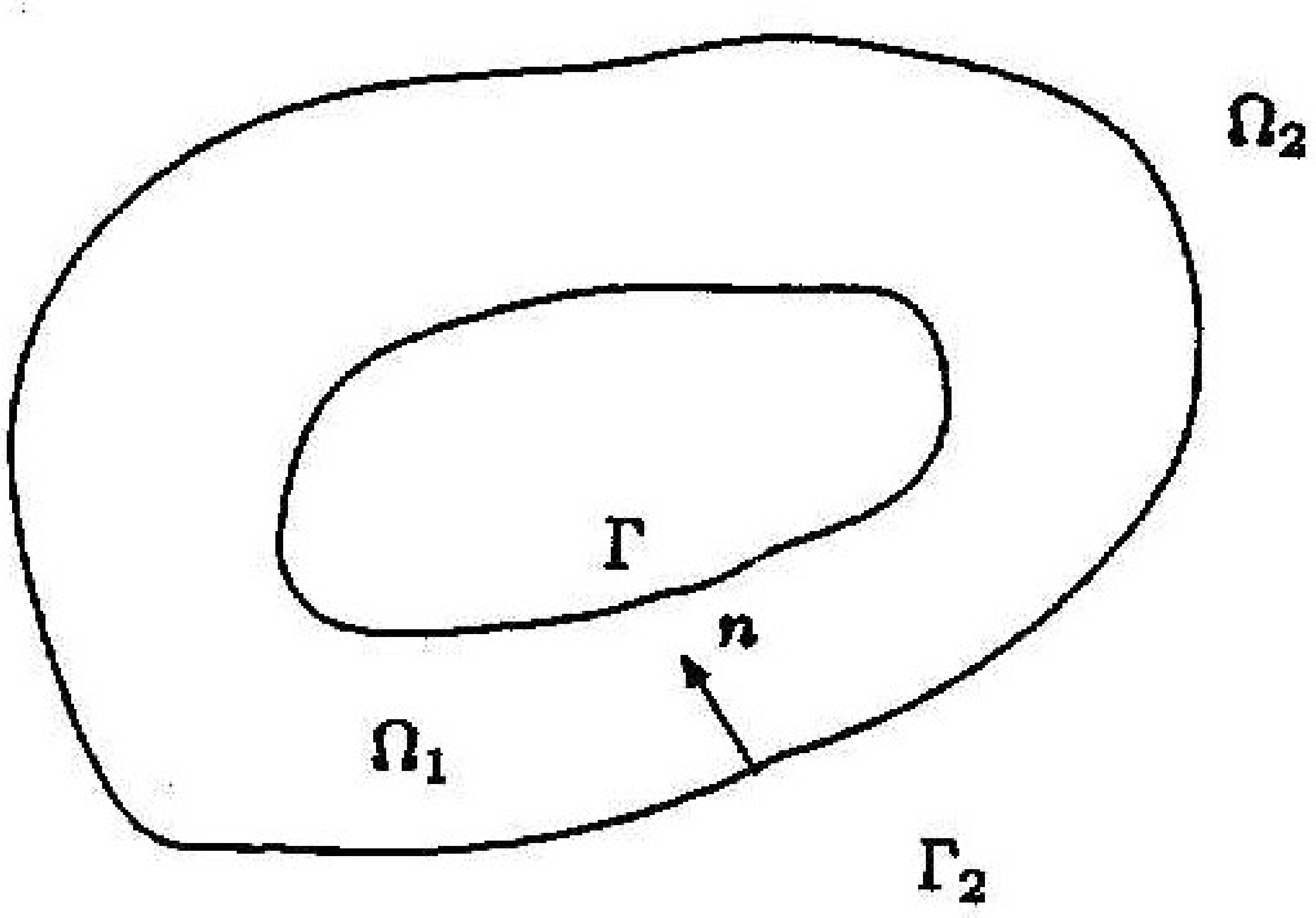

Fig. 1

where $u_i = u|_{\Omega_i}, i = 1, 2$, and $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative to $\Gamma_2 = \partial\Omega_2$ (see FIg. 1). Since $-\Delta u = 0$ in Ω_2, using Green's formula we obtain

$$u(x) = \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u(y) ds_y - \int_{\Gamma_2} G(x,y)\lambda(y) ds_y + \alpha, \quad \forall x \in \Omega_2, \tag{2.3}$$

where $G(x,y) = \dfrac{1}{2\pi} \log|x-y|, x \neq y; n_y$ denotes the outward unit normal to $\Gamma_2 = \partial\Omega_2$ at $y \in \Gamma_2$, and α is a constant. By the properties of the single-layer and double-layer potentials, we obtain the following relationship between λ and $u|_{\Gamma_2}$:

$$\frac{1}{2}u(x) = \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u(y) ds_y - \int_{\Gamma_2} G(x,y)\lambda(y) ds_y + \alpha \quad \forall x \in \Gamma_2. \tag{2.4}$$

Furthermore, using properties of the derivatives of the single-layer and double-layer potentials [6, 7], we get

$$\frac{1}{2}\lambda = \int_{\Gamma_2} \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y) ds_y - \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_x} \lambda(y) ds_y, \quad \forall x \in \Gamma_2, \tag{2.5}$$

where

$$\int_{\Gamma_2} \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y) ds_y = \frac{d}{ds_x} \int_{\Gamma_2} G(x,y) \frac{du(y)}{ds_y} ds_y, \quad \forall x \in \Gamma_2.$$

Multiplying (2.4) by a function $\mu(x)$ satisfying $\int_{\Gamma_2} \mu ds = 0$, and integrating over Γ_2, we have

$$-\frac{1}{2} < u, \mu > + \int_{\Gamma_2} \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u(y)\mu(x) ds_y ds_x + b(\lambda, \mu) = 0, \tag{2.6}$$

where

$$b(\lambda, \mu) = - \int_{\Gamma_2} \int_{\Gamma_2} G(x,y)\lambda(y)\mu(x) ds_y ds_x, \tag{2.7}$$

$$< u, \mu > = \int_{\Gamma_2} u(x)\mu(x) ds_x. \tag{2.8}$$

Multiplying (2.5) by a function $v(x) \in H^{1/2}(\Gamma_2)$, we obtain

$$\frac{1}{2} < v, \lambda > = b\Big(\frac{du}{ds}, \frac{dv}{ds}\Big) - \int_{\Gamma_2} \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_x} \lambda(y) v(x) ds_y ds_x. \tag{2.9}$$

Let

$$\overset{*}{H}{}^{1}(\Omega) = \{v \in H^1(\Omega_1), \quad v = 0 \text{ on } \Gamma\},$$

$$\overset{*}{H}{}^{-1/2}(\Gamma_2) = \{\mu \in H^{-1/2}(\Gamma_2), \quad \int_{\Gamma_2} \mu ds = 0\},$$

$$V^* = \overset{*}{H}{}^{1}(\Omega) \times \overset{*}{H}{}^{-1/2}(\Gamma_2), \text{ with norm } \|(v, \mu\|_V^2 = \|v\|_{1,\Omega_1}^2 + \|\mu\|_{-1/2,\Gamma_2}^2.$$

Multiplying $-\Delta u = f$ by the function $v \in \overset{*}{H}{}^{1}(\Omega_1)$ and integrating over Ω_1, we get

$$a(u,v) + \int_{\Gamma_2} \lambda v ds_x = \iint_{\Omega_1} f v dx, \qquad \forall v \in \overset{*}{H}{}^{1}(\Omega_1), \tag{2.10}$$

where

$$a(u,v) = \iint_{\Omega_1} \nabla u \cdot \nabla v dx.$$

Inserting (2.9) into (2.10), the formulation (2.10) can be rewritten as follows:

$$a(u,v) + \frac{1}{2} < v, \lambda > + b\Big(\frac{du}{ds}, \frac{dv}{ds}\Big) - \int_{\Gamma_2} \int_{\Gamma_2} \frac{\partial G}{\partial n_x}(x,y)\lambda(y)v(x) ds_y ds_x = \iint_{\Omega_*} f v dx,$$

$$\forall v \in \overset{*}{H}{}^{1}(\Omega_1). \tag{2.11}$$

Combining (2.11) and (2.6), we get the following variational problem:

$$\begin{cases} \text{Find } (u, \lambda) \in V, \text{ such that} \\ a(u,v) + \frac{1}{2} < v, \lambda > + b\Big(\frac{du}{ds}, \frac{dv}{ds}\Big) - \int_{\Gamma_2} \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_x} \lambda(y) v(x) ds_y ds_x \\ \qquad = \iint_{\Omega_1} f v dx, \qquad \forall v \in \overset{*}{H}{}^{1}(\Omega_1), \\ -\frac{1}{2} < u, \mu > + b(\lambda, \mu) + \int_{\Gamma_2} \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u(y)\mu(x) ds_y ds_x = 0, \quad \forall \mu(x) \in \overset{*}{H}{}^{-1/2}(\Gamma_2). \end{cases} \tag{2.12}$$

The variational problem (2.12) can be rewritten as follows:

$$\begin{cases} \text{Find } (u,\lambda) \in V, \text{ such that} \\ A(u,\lambda;v,\mu) = \iint_{\Omega_1} f(x)v(x)dx, \qquad \forall (v,\mu) \in V, \end{cases} \tag{2.12*}$$

where

$$A(u,\lambda;v,\mu) = a(u,v) + b(\frac{du}{ds},\frac{dv}{ds}) + b(\lambda,\mu) + \frac{1}{2} < v,\lambda > - \frac{1}{2} < u,\mu >$$
$$- \int_{\Gamma_2}\int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_x}\lambda(y)v(x)ds_y ds_x + \int_{\Gamma_2}\int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y}u(y)\mu(x)ds_y ds_x.$$

For the variational problem (2.12)*, we obtain

Theorem 2.1. *Suppose $f \in H^{-1}(\Omega_1)$. Then variational problem (2.12)* has a unique solution $(u,\lambda) \in V$; moreover*

$$\|(u,\lambda)\|_V \le C\|f\|_{-1,\Omega_1}, \tag{2.13}$$

where $C > 0$ is a constant.

Proof. Firstly we recall that (see [8]) $b(\lambda,\mu)$ is a bounded bilinear form on $H^{-1/2}(\Gamma_2) \times H^{-1/2}(\Gamma_2)$. Furthermore, $b(\lambda,\mu)$ is $\overset{*}{H}{}^{-1/2}(\Gamma_2)$-elliptic, i.e., there exists a positive constant β such that

$$b(\mu,\mu) \ge \beta\|\mu\|^2_{-1/2,\Gamma_2}, \qquad \forall \mu \in \overset{*}{H}{}^{-1/2}(\Gamma_2). \tag{2.14}$$

Then it is straightforward to check that $A(u,\lambda;v,\mu)$ is a bounded bilinear form on $V \times V$, i.e. there is a constant $M > 0$, such that

$$|A(u,\lambda;v,\mu)| \le M\|(u,\lambda)\|_V\|(v,\mu)\|_V, \quad \forall (u,\lambda),(v,\mu) \in V. \tag{2.15}$$

Furthermore, we have

$$\begin{aligned} A(v,\mu;v,\mu) &= a(v,v) + b(\frac{dv}{ds},\frac{dv}{ds}) + b(\mu,\mu) \ge |v|^2_{1,\Omega_1} + \beta\|\mu\|^2_{-1/2,\Gamma_2} \\ &\ge \beta^*\{\|v\|^2_{1,\Omega_1} + \|\mu\|^2_{-1/2,\Gamma_2}\}, \qquad \forall (v,\mu) \in V \end{aligned}$$

with the constant $\beta^* > 0$, i.e.

$$A(v,\mu,v,\mu) \ge \beta^*\|(v,\mu)\|^2_V, \quad \forall (v,\mu) \in V. \tag{2.16}$$

Hence the conclusion follows immediately by the Lax-Milgram Theorem [9].

Suppose $u(x)$ is the solution of the problem (2.1). Then we know that $u_1(x) = u(x)|_{\Omega_1} \in \overset{*}{H}{}^1(\Omega_1)$, $\lambda_1 == \frac{\partial u}{\partial n}\Big|_{\Gamma_2} \in \overset{*}{H}{}^{-1/2}(\Gamma_2)$. Moreover, $(u_1(x),\lambda_1)$ is a solution of the variational problem (2.12)*. By the uniqueness of the variational problem (2.12)*, we know that the boundary value problem (2.1) is equivalent to the variational problem (2.12)*.

2.2. Displacement problem in linear elasticity

Let $\boldsymbol{u} = (u_1, u_2)^T$ denote the displacement field which is given by the Navier system in the two-dimensional case:

$$\begin{cases} -\mu^*\Delta \boldsymbol{u} - (\lambda^* + \mu^*) \text{ grad div } \boldsymbol{u} = \boldsymbol{f}, \text{ in } \Omega^c, \\ \boldsymbol{u} = 0, \text{ on } \Gamma, \\ \boldsymbol{u} \text{ is bounded, when } |x| \to +\infty \end{cases} \tag{2.17}$$

where $\lambda^* > -\mu^* < 0$ are the Lamé constants, $\boldsymbol{f} = (f_1, f_2)^T$ has its support in a bounded subdomain Ω_1 of Ω^c, and $\Omega_2 = \Omega^c \setminus \Omega_1$. Now we solve the following equivalent problem

$$\begin{cases} -\mu^*\Delta \boldsymbol{u}^1 - (\lambda^* + \mu^*) \text{ grad div} \boldsymbol{u}^1 = \boldsymbol{f}, \text{ in } \Omega_1, \\ -\mu^*\Delta \boldsymbol{u}^2 - (\lambda^* + \mu^*) \text{ grad div } \boldsymbol{u}^2 = 0, \text{ in } \Omega_2, \\ \boldsymbol{u}^1 = 0, \text{ on } \Gamma, \\ \boldsymbol{u}^1 = \boldsymbol{u}^2, \text{ on } \Gamma_2, \\ \boldsymbol{T}(\boldsymbol{u}^1) = \boldsymbol{T}(\boldsymbol{u}^2) = \boldsymbol{\lambda}, \text{ on } \Gamma_2, \\ \boldsymbol{u}^2 \text{ is bounded, when } |x| \to +\infty \end{cases} \tag{2.18}$$

where $\boldsymbol{u}^i = \boldsymbol{u}|_{\Omega_i}, i = 1, 2,$

$$\boldsymbol{T}(\boldsymbol{u}) = \lambda^*(\text{ div } \boldsymbol{u})\boldsymbol{n} + 2\mu^*\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + \mu^* \boldsymbol{n} \wedge \text{Curl}\,\boldsymbol{u}, \tag{2.19}$$

and $\boldsymbol{n}$ denotes the outward normal on $\Gamma_2 = \partial\Omega_2$. Let $G(x, y)$ denote the fundamental solution of the Navier system in the plane. Then we have

$$G(x,y) = \frac{\lambda^* + 3\mu^*}{4\pi\mu^*(\lambda^* + 2\mu^*)}\left\{\log\frac{1}{|x-y|}I + \frac{\lambda^* + \mu^*}{\lambda^* + 3\mu^*}\frac{(x-y)(x-y)^T}{|x-y|^2}\right\}. \tag{2.20}$$

Since $-\mu^*\Delta \boldsymbol{u} - (\lambda^* + \mu^*)$ grad div $\boldsymbol{u} = 0$ in Ω_2, using the Betti formulation we obtain

$$\boldsymbol{u}(x) = \int_{\Gamma_2} G(x,y)\,\boldsymbol{\lambda}(y)ds_y - \int_{\Gamma_2} (\boldsymbol{T}_{(y)}(G(x,y)))^T \boldsymbol{u}(y)ds_y + \boldsymbol{\alpha}, \quad \forall x \in \Omega_2 \tag{2.21}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$ is a constant vector. On the boundary Γ_2, we have[10]

$$\frac{1}{2}\boldsymbol{u}(x) = \int_{\Gamma_2} G(x,y)\,\boldsymbol{\lambda}(y)ds_y - \int_{\Gamma_2} (\boldsymbol{T}_{(y)}(G(x,y)))^T \boldsymbol{u}ds_y + \boldsymbol{\alpha}. \tag{2.22}$$

Furthermore, by the behavior of the $\boldsymbol{T}$-operator acting on the single-layer and double-layer potentials [7, 10, 11], we have

$$\frac{1}{2}\boldsymbol{\lambda}(x) = \int_{\Gamma_2} (\boldsymbol{T}_{(x)}(G(x,y)))\,\boldsymbol{\lambda}(y)ds_y - \int_{\Gamma_2} \boldsymbol{T}_{(x)}(\boldsymbol{T}_{(y)}(G(x,y)))^T \boldsymbol{u}(y)ds_y, \quad \forall x \in \Gamma_2. \tag{2.23}$$

The kernel in the last integral of (2.23) has a singularity which is of order $\frac{1}{|x-y|^2}$ when x and y are close. Hence the last integral in (2.23) must be interpreted as a finite part [7].

Let

$\overset{*}{\boldsymbol{H}}^1(\Omega_1) = \overset{*}{H}^1(\Omega_1) \times \overset{*}{H}^1(\Omega_1)$,

$\boldsymbol{H}^\gamma(\Gamma_2) = H^\gamma(\Gamma_2) \times H^\gamma(\Gamma_2)$, where γ is a real number,

$\boldsymbol{R} = \left\{ \boldsymbol{\mu} = \binom{\alpha_1}{\alpha_2} + \beta\binom{-x_2}{x_1}; \alpha_1, \alpha_2, \beta \text{ are constants} \right\}$,

$\overset{*}{\boldsymbol{H}}^{-1/2}(\Gamma_2) = \left\{ \boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\Gamma_2), \int_{\Gamma_2} \mu_i ds = 0, \ i = 1,2, \ \int_{\Gamma_2} (-\mu_1 x_2 + \mu_2 x_1) ds = 0 \right\}$,

$\boldsymbol{V} = \overset{*}{\boldsymbol{H}}^1(\Omega_1) \times \overset{*}{\boldsymbol{H}}^{-1/2}(\Gamma_1)$ with norm $\|(\boldsymbol{v}, \boldsymbol{\mu})\|_{\boldsymbol{V}}^2 = \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega_1)} + \|\boldsymbol{\mu}\|^2_{\boldsymbol{H}^{-1/2}(\Gamma_2)}$.

An application of the Betti formulation on domain Ω_1 yields

$$W(\boldsymbol{u}, \boldsymbol{v}) + \int_{\Gamma_2} \boldsymbol{\lambda} \cdot \boldsymbol{v} ds = \iint_{\Omega_1} \boldsymbol{f} \cdot \boldsymbol{v} dx, \quad \forall \boldsymbol{v} \in \overset{*}{\boldsymbol{H}}^1(\Omega), \tag{2.24}$$

where

$$W(\boldsymbol{u}, \boldsymbol{v}) = \iint_{\Omega_1} \left\{ \lambda^* \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} + 2\mu^* \Big(\sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} \frac{\partial v_i}{\partial x_i} \Big) + \mu^* \Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \Big) \Big(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \Big) \right\} dx. \tag{2.25}$$

Inserting (2.23) into (2.24), and combining with (2.22), we obtain the following variational problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{V}, \text{ such that} \\ W(\boldsymbol{u}, \boldsymbol{v}) + \frac{1}{2} \int_{\Gamma_2} \boldsymbol{\lambda} \cdot \boldsymbol{v} ds - \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{v}(x)^T \boldsymbol{T}_{(x)} (\boldsymbol{T}_{(y)}) \boldsymbol{G}(x,y))^T \boldsymbol{u}(y) ds_y ds_x \\ \quad + \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{v}(x)^T \boldsymbol{T}_{(x)} (\boldsymbol{G}(x,y)) \boldsymbol{\lambda}(y) ds_y ds_x = \iint_{\Omega_1} \boldsymbol{f} \cdot \boldsymbol{v} dx, \quad \forall \boldsymbol{v} \in \overset{*}{\boldsymbol{H}}^1(\Omega_1), \\ -\frac{1}{2} \int_{\Gamma_2} \boldsymbol{\mu} \cdot \boldsymbol{u} ds + \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{\mu}(x)^T \boldsymbol{G}(x,y) \boldsymbol{\lambda}(y) ds_y ds_x \\ \quad - \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{\mu}(x)^T (\boldsymbol{T}_{(y)} (\boldsymbol{G}(x,y))^T \boldsymbol{u}(y) ds_y ds_x = 0, \quad \forall \boldsymbol{\mu} \in \overset{*}{\boldsymbol{H}}^{-1/2}(\Gamma_2). \end{cases} \tag{2.26}$$

Let

$$D_1(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{\mu}(x)^T \boldsymbol{G}(x,y) \boldsymbol{\lambda}(y) ds_y ds_x,$$

$$D_2(\boldsymbol{v}, \boldsymbol{u}) = - \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{v}(x)^T \boldsymbol{T}_{(x)} (\boldsymbol{T}_{(y)} (\boldsymbol{G}(x,y))^T \boldsymbol{u}(y) ds_y ds_x,$$

$$< \boldsymbol{\mu}, \boldsymbol{v} >= \int_{\Gamma_2} \boldsymbol{\mu} \cdot \boldsymbol{v} dx,$$

$$I(\boldsymbol{u}, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{\mu}) = W(\boldsymbol{u}, \boldsymbol{v}) + D_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) + D_2(\boldsymbol{u}, \boldsymbol{v}) + \frac{1}{2} < \boldsymbol{\lambda}, \boldsymbol{v} > -\frac{1}{2}(\boldsymbol{\mu}, \boldsymbol{u})$$
$$+ \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{v}(x)^T \boldsymbol{T}_{(x)} (\boldsymbol{G}(x,y)) \boldsymbol{\lambda}(y) ds_y ds_x - \int_{\Gamma_2} \int_{\Gamma_2} \boldsymbol{\mu}(x)^T (\boldsymbol{T}_{(y)} (\boldsymbol{G}(x,y))^T \boldsymbol{u}(y) ds_y ds_x. \tag{2.27}$$

Then we can prove the following lemma.

Lemma 2.1. *$I(u, \lambda; v, \mu)$ is a bounded bilinear form on $V \times V$, i.e. there exists a constant $M > 0$, such that*

$$|I(u, \lambda; v, v)| \leq M\|(u, \lambda)\|_V \|(v, \mu)\|_V, \qquad (u, \lambda), (v, \mu) \in V. \tag{2.28}$$

Furthermore, there is a constant $\alpha > 0$, such that

$$I(v, \mu, v, \mu) \geq \alpha\|(v, \mu)\|_V^2, \quad \forall (v, \mu) \in V. \tag{2.29}$$

Proof. We recall that $W(u, v)$ is a bounded bilinear form on $H^1(\Omega_1) \times H^1(\Omega_1)$ and is $\overset{*}{H}{}^1(\Omega_1)$-elliptic, namely there exist two constants $M_1 > 0$ and $\alpha_1 > 0$ such that

$$|W(u, v)| \leq M_1 \|u\|_{1,\Omega_1} \|v\|_{1,\Omega_1}, \qquad \forall u, v \in \overset{*}{H}{}^1(\Omega_1)$$

$$W(v, v) \geq \alpha_1 \|v\|_{1,\Omega_1}^2, \qquad v \in \overset{*}{H}{}^1(\Omega_1).$$

Moreover, the bilinear form $D_2(v, u)$ [7] can be rewritten as follows:

$$D_2(v, u) = \sum_{i,j=1}^{2} \int_{\Gamma_2} \int_{\Gamma_2} U_{ij}(x, y) \frac{du_i(y)}{ds} \cdot \frac{dv_j(x)}{ds} ds_y ds_x,$$

where

$$U_{ij}(x, y) = \frac{2\mu^*(\lambda^* + \mu^*)}{\pi(\lambda^* + 2\mu^*)} \left(\log \frac{1}{|x - y|} \delta_{ij} + \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} \right)$$

and $D_2(v, u)$ is bounded on $H^{1/2}(\Gamma_2) \times H^{1/2}(\Gamma_2)$. Furthermore, we have

$$D_2(v, v) \geq \alpha_2 \|v\|^2_{H^{1/2}(\Gamma_2)/R}, \quad \forall v \in H^{1/2}(\Gamma_2), \ \alpha_2 > 0.$$

For the bilinear form $D_1(\mu, \lambda)$, it is straightforward to check that $D_1(\mu, \lambda)$ is a bounded bilinear form on $\overset{*}{H}{}^{-1/2}(\Gamma_2) \times \overset{*}{H}{}^{-1/2}(\Gamma_2)$ and is $\overset{*}{H}{}^{-1/2}(\Gamma_2)$-elliptic, i.e. there exist two constants $M_3 > 0$ and $\alpha_3 > 0$ such that

$$|D_1(\mu, \lambda)| \leq M_3 \|\mu\|_{-1/2,\Gamma_2} \|\lambda\|_{-1/2,\Gamma_2}, \quad \forall \mu, \lambda \in \overset{*}{H}{}^{-1/2}(\Gamma_2),$$

$$|D_1(\mu, \lambda) \geq \alpha_3 \|\mu\|^2_{-1/2,\Gamma_2}, \quad \forall \mu \in \overset{*}{H}{}^{-1/2}(\Gamma_2).$$

Then the conclusion follows immediately with $\alpha = \min(\alpha_1, \alpha_3)$.

The problem (2.25) can be rewritten as follows:

$$\begin{cases} \text{Find } (u, \lambda) \in V, \text{ such that} \\ I(u, \lambda; v, \mu) = \iint_{\Omega_1} f \cdot v dx, \quad \forall (v, \mu) \in V. \end{cases} \tag{2.25*}$$

An application of the Lax-Milgram Theorem yields:

Theorem 2.2. *Suppose $f \in H^{-1}(\Omega_1)$. The variational problem (2.25)* has a unique solution $(u, \lambda) \in V$; furthermore,*

$$\|(u, \lambda)\|_V \leq \frac{M}{\alpha}\|f\|_{-1,\Omega_1}.$$

Finally, we point out that the variational problem (2.25)* is equivalent to the boundary value problem (2.17).

§3. The Discrete Approximation of Problem (2.12)

As an example, we consider the discrete approximation of problem (2.12). Let H_{h_1} denote a finite dimensional subspace of $\overset{*}{H}{}^1(\Omega_1)$, satisfying

$$\inf_{v_h \in H_{h_1}} \|u - v_h\|_{1,\Omega_1} \leq C_1 h_1^{\gamma_1}|u|_{\gamma_1+1,\Omega_1}, \quad \forall u \in \overset{*}{H}{}^1(\Omega_1) \cap H^{\gamma_1+1}(\Omega_1), \tag{3.1}$$

and let B_{h_2} denote a finite dimensional subspace of $\overset{*}{H}{}^{-1/2}(\Gamma_2)$ satisfying

$$\inf_{\mu_h \in B_{h_2}} \|\lambda - \mu_h\|_{-1/2,\Gamma_2} \leq C_2 h_2^{\gamma_2}|\lambda|_{\gamma_2-1/2,\Gamma_2}, \quad \forall \lambda \in \overset{*}{H}{}^{-1/2}(\Gamma_2) \cap H^{\gamma_2-1/2}(\Gamma_2), \tag{3.2}$$

where $\gamma_1 > 0, \gamma_2 > 0$ are two integers, and C_1 and C_2 are two positive constants independent of h_1 and h_2. Let $V_h = H_{h_1} \times B_{h_2}$. We consider the discrete problem

$$\begin{cases} \text{Find } (u_h, \lambda_h) \in V_h, \text{ such that} \\ A(u_h, \lambda_h; v_h, \mu_h) = \iint_{\Omega_1} f v_h dx, \quad \forall (v_h, \mu_h) \in V_h \end{cases} \tag{3.3}$$

or

$$\begin{cases} \text{Find } (u_h, \lambda_h) \in V_h, \text{ such that} \\ a(u_h, v_h) + \frac{1}{2} < \lambda_h, v_h > + b\left(\frac{du_h}{ds}, \frac{dv_h}{ds}\right) \\ - \int_{\Gamma_2}\int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_x}\lambda_h(y)v_h(x)ds_y ds_x = \iint_{\Omega_1} f v_h ds, \quad \forall v_h \in H_{h_1}, \\ -\frac{1}{2} < \mu_h, u_h > + b(\lambda_h, \mu_h) \\ + \int_{\Gamma_2}\int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u_h(y)\mu_h(x)ds_y ds_x = 0, \quad \forall \mu_h \in B_{h_2}. \end{cases} \tag{3.3*}$$

Then we have

Theorem 3.1. *Suppose that (μ, λ), the solution of problem (2.12), satisfies $u \in H^{\gamma_1+1}(\Omega_1), \lambda \in H^{\gamma_2-1/2}(\Gamma_2)$. Then problem (3.3) has a unique solution $(u_h, \lambda_h) \in V_h$, and the following error estimate holds*

$$\|(u - u_h, \lambda - \lambda_h)\|_V \leq \frac{CM}{\beta^*}\left\{h_1^{\gamma_1}|u|_{\gamma_1+1,\Omega_1} + h_2^{\gamma_2}|\lambda|_{\gamma_2-1/2,\Gamma_2}\right\}, \tag{3.4}$$

where $C = \max\{C_1, C_2\}$.

Proof. By the coercive property of $A(u,\lambda;v,\mu)$, the existence and uniqueness of problem (3.3) follows immediately. From (2.10) and (3.3) we obtain

$$A(u-u_h,\lambda-\lambda_h;v_h,\mu_h)=0, \quad \forall(v_h,\mu_h)\in V_h.$$

Moreover, we have

$$A(u-u_h,\lambda-\lambda_h;u-u_h,\lambda-\lambda_h)=A(u-u_h,\lambda-\lambda_h;u-v_h,\lambda-\mu_h), \quad \forall(v_h,\mu_h)\in V_h. \tag{3.5}$$

On the other hand, by (2.13) we get

$$\beta^*\|(u-u_h,\lambda-\lambda_h)\|_V^2 \le A(u-u_h,\lambda-\lambda_h;u-u_h,\lambda-\lambda_h). \tag{3.6}$$

Combining (3.6), (3.5) and (2.12), we have

$$\begin{aligned}
\|(u-u_h,\lambda-\lambda_h)\|_V &\le \frac{M}{\beta^*}\inf_{(v_h,\mu_h)\in V_h}\|(v-v_h,\lambda-\mu_h)\|_V \\
&\le \frac{M}{\beta^*}\inf_{(v_h,\mu_h)\in V_h}\{\|u-v_h\|_{1,\Omega_1}+\|\lambda-\mu_h\|_{-1/2,\Gamma_2}\} \\
&= \frac{M}{\beta^*}\Big\{\inf_{v_h\in H_{h_1}}\|u-v_h\|_{1,\Omega_1}+\inf_{\mu_h\in B_{h_2}}\|\lambda-\mu_h\|_{-1/2,\Gamma_2}\Big\} \\
&\le \frac{CM}{\beta^*}\Big\{h_1^{\gamma_1}|u|_{\gamma_1+1,\Omega_1}+h_2^{\gamma_2}|\lambda|_{\gamma_2-1/2,\Gamma_2}\Big\}.
\end{aligned}$$

The last inequality follows from (3.1) and (3.2). The proof is complete.

These new variational formulations for the coupling of FEM and BEM have the following obvious advantages. We can choose the subspaces H_{h_1} and B_{h_2} independently. For any family of subspaces the discrete solution will converge to the true solution provided only that $H_{h_1}\to \overset{*}{H}{}^1(\Omega_1)$ when $h_1\to 0$ and $B_{h_2}\to \overset{*}{H}{}^{-1/2}(\Gamma)$ when $h_2\to 0$. For optimal error estimate, we should take $h_1^{\gamma_1}=h_2^{\gamma_2}$.

This approach can be extended to more general equations and problems. This will be discussed in a separate paper.

Finally, we should mention that recently M. Costable and E.P. Stephan [12–13] presented another new formulation for coupling of FEM and BEM, which is symmetric, but is not coercive. Our formulation is coercive but it is not symmetric.

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