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l_2–STABILITY OF DIFFERENCE MODELS FOR HYPERBOLIC INITIAL BOUNDARY VALUE PROBLEMS *

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Abstract

It is showed that, for many commonly used difference models on hyperbolic initial boundary value problems, the necessary and sufficient condition for GKS-stability (in the sense of Definition 3.3 of [1]) is a necessary condition for l_2-stability.

§1. Introduction

Consider the mixed initial boundary value problem

$$\begin{cases} \partial U(x,t)/\partial t = A\partial U(x,t)/\partial x, & x \geq 0,\ t > 0, \\ U(x,0) = f(x), & x \geq 0, \\ U^{\rm I}(0,t) = SU^{\rm II}(0,t) + g(t), & t \geq 0. \end{cases} \tag{1.1}$$

Here A is a constant square matrix, and

$$U(x,t) = (u^{(1)}(x,t), \cdots, U^{(N)}(x,t))^T$$

is a vector function. Furthermore, A is diagonalizable and of the form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \text{with } A_1 < 0,\ \ A_2 > 0. \tag{1.2}$$

$U^{\rm I}(0,t) = (u^{(1)}, \cdots, u^{(l)})^T$ and $U^{\rm II}(0,t) = (u^{(l+1)}, \cdots, u^{(N)})^T$ correspond to the partition of A , S is a rectangular matrix, and $f(x)$ and $g(t)$ are given functions.

We want to solve the above problem by the general consistent multistep difference model Q:

$$\begin{cases} Q_{-1}U_j^{n+1} = \sum\limits_{\sigma=0}^{s} Q_\sigma U_j^{n-\sigma}, & n \geq s,\ j = 1, 2, 3, \cdots, \\ U_j^\sigma = f_j^\sigma, \quad 0 \leq \sigma \leq s, & j = -r+1,\ -r+2, \cdots, \\ U_\mu^{n+1} = \sum\limits_{\sigma=-1}^{s} S_\sigma^{(\mu)} U_1^{n-\sigma} + g_\mu^n, & -r+1 \leq \mu \leq 0,\ \ n \geq s. \end{cases} \tag{1.3}$$

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Here $U_j^n \simeq U(jh, n\tau)$ denotes the difference solution, h and τ are mesh width of space step and time step, respectively. The ratio $\tau/h = \lambda$ is a constant.

$$Q_\sigma = \sum_{i=-r}^{p} A_{i\sigma} K^i, \quad KU_j^n = U_{j+1}^n, \quad -1 \le \sigma \le s$$

are difference operators with matrix coefficients, and Q_{-1} is invertible.

$$S_\sigma^{(\mu)} = \sum_{j=0}^{q} C_{j\sigma}^{(\mu)} K^j, \quad -r+1 \le \mu \le 0, \quad -1 \le \sigma \le s$$

are one-sided difference operators. The initial value f_j^σ and boundary value g_μ^n are given $(0 \le \sigma \le s, j \ge -r+1, n \ge s, -r+1 \le \mu \le 0)$. s, p, r, q are nonnegative integers. As usual, we need the following assumptions.

Assumption 1.1. The difference model (1.3) can be solved boundedly for U^{n+1}, i.e., there is a constant $M > 0$ sucy that, for every $G \in l_2(x)$, there is a unique solution $W \in l_2(x)$ of

$$\begin{cases} Q_{-1} W_j = G_j, & j = 1, 2, 3, \cdots, \\ W_\mu - S_{-1}^{(\mu)} W_1 = g_\mu, & -r+1 \le \mu \le 0, \end{cases}$$

With $\|W\|_x^2 \le M(\|G\|_x^2 + h \sum_{\mu=-r+1}^{0} |g_\mu|^2)$. Here $\|W\|_x^2 = \sum_{j=-r+1}^{\infty} |W_j|^2 h$ and $|W|^2 = \sum_{i=1}^{N} (W^{(i)})^2$.

Assumption 1.2. The matrices $\{A_{m\sigma}\}_{m=-r,\sigma=-1}^{p,\ s}$ are simultaneously diagonalizable.

Definition 1.1 We say the finite difference model Q is initial and boundary value l_2-stable, if for any given time $T > 0$, there exist constants $M > 0$ and $\tau_0 > 0$ such that, for any initial value f^σ and boundary value $g_\mu (0 \le \sigma \le s, -r+1 \le \mu \le 0)$, the estimate

$$\|U^n\|_x^2 \le M\Big(\sum_{\mu=-r+1}^{0} \|g_\mu\|_{t \le T}^2 + \sum_{\sigma=0}^{s} \|f^\sigma\|_x^2 \Big)$$

holds for all n and τ with $n\tau \le T, \tau \le \tau_0$. Here $\|g_\mu\|_{t\le T}^2 = \sum_{n=s}^{T/\tau} |g_\mu|^2 k$.

The aim of this paper is to show that, for many commonly used difference models defined by (1.3), the necessary and sufficient condition for GKS-stability (in the sense of definition 3.3 of [1]) is a necessary condition for l_2-stability.

We shall use some results of [1] and [2], and assume that the reader is familiar with those papers.

§2. Left-going and Right-going Signals

Let Q denote the difference scheme

$$Q_{-1} U_j^{n+1} = \sum_{\sigma=0}^{s} Q_\sigma U_j^{n-\sigma},$$

and let Z be the time shift operator

$$ZU_j^n = U_j^{n+1}.$$

Then Q can be written as

$$P(K,Z)U_j^n = \Big[\sum_{m=-r}^{p}\Big(\sum_{\sigma=0}^{s} A_{m\sigma}K^{m+r}Z^{s-\sigma} - A_{m(-1)}K^{m+r}Z^{s+1}\Big)\Big]U_j^n = 0. \tag{2.1}$$

The dispersion relation for (2.1) is

$$\det P(k,z) = \det \sum_{m=-r}^{p}\Big(\sum_{\sigma=0}^{s} A_{m\sigma}k^{m+r}z^{s-\sigma} - A_{m(-1)}k^{m+r}z^{s+1}\Big) = 0, \tag{2.2}$$

where the wave number ξ and frequency ω are defined by

$$k = \exp(-i\xi h), \quad z = \exp(i\omega\tau).$$

Assumption 2.1. Q is Cauchy stable.

Assumption 2.2. For all z_0 with $|z_0| \geq 1$, the polynomial

$$P_z(k) = P(k, z_0)$$

has nonsingular 0th and $(r+p)$th coefficients.

Throughout this paper, we assume Q and $\bar{Q}$ satisfy assumptions 1.1, 1.2, 2.1, and 2.2.

Under assumption 1.2, there exists a nonsingular matrix T, such that

$$\bar{A}_{m\sigma} = TA_{m\sigma}T^{-1} = \operatorname{diag}(a_{m\sigma}^{(1)}, \cdots, a_{m\sigma}^{(N)}), -r \leq m \leq p, -1 \leq \sigma \leq s$$

with $a_{m\sigma}^{(\alpha)} \in I\!R$ for all $-r \leq m \leq p, 1 \leq \alpha \leq N, -1 \leq \sigma \leq s$.

We denote TU_j^n by $\bar{U}_j^n$, and TPT^{-1} by $\bar{P}$. Then we have

$$\bar{P}(K,Z)\bar{U}_j^n = \Big[\sum_{m=-r}^{p}\Big(\sum_{\sigma=0}^{s} \bar{A}_{m\sigma}K^{m+r}Z^{s-\sigma} - \bar{A}_{m(-1)}K^{m+r}Z^{s+1}\Big)\Big]\bar{U}_j^n = 0.$$

Now $\bar{P}$ is a bivariate polynomial with diagonal matrix coefficients. This system is equivalent to the N scalar systems

$$\bar{P}^{(\alpha)}(K,Z)(\bar{U}^{(\alpha)})_j^n = \sum_{m=-r}^{p}\Big(\sum_{\sigma=0}^{s} a_{m\sigma}^{(\alpha)}K^{m+r}Z^{s-\sigma} - a_{m(-1)}^{(\alpha)}K^{m+r}Z^{s+1}\Big)(\bar{U}^{(\alpha)})_j^n = 0, \quad 1 \leq \alpha \leq N. \tag{2.3}$$

By assumption 2.2, for each scalar ordinary difference equation

$$\bar{P}^{(\alpha)}(K,z_0)(\bar{U}^{(\alpha)})_j^n = \sum_{m=-r}^{p}\Big(\sum_{\sigma=0}^{s} a_{m\sigma}^{(\alpha)}K^{m+r}z_0^{s-\sigma} - a_{m(-1)}^{(\alpha)}K^{m+r}z_0^{s+1}\Big)(\bar{U}^{(\alpha)})_j^n = 0, \quad 1 \leq \alpha \leq N, \tag{2.4}$$

with $z_0 \in \mathbb{C}, |z_0| \geq 1$, according to ordinary difference equation theory, there exist sequences

$$\bar{\phi}_j^{(\alpha)} = [k_i^{(\alpha)}]^j j^\delta, \quad \begin{array}{l} 1 \leq i \leq \mu^{(\alpha)}, \\ 0 \leq \delta \leq \nu_i^{(\alpha)} - 1, \\ 1 \leq \alpha \leq N, \end{array} \tag{2.5}$$

which are linearly independent solutions of $\bar{P}_{z_0}^{(\alpha)}(k)\bar{\phi}_j^{(\alpha)} = \bar{P}^{(\alpha)}(K, z_0)\bar{\phi}_j^{(\alpha)} = 0$, and span the linear space of all such solutions. Here $\{k_i^{(\alpha)}\}_{1 \leq i \leq \mu^{(\alpha)}}$ denote the distinct nonzero roots of $\bar{P}_{z_0}^{(\alpha)}(k) = 0$ with $k_i^{(\alpha)}$ of multiplicity $\nu_i^{(\alpha)}$.

Lemma 2.1. *Suppose that $z_0 \in \mathbb{C}$ is the root of $\bar{P}_{k_0}^{(\alpha)}(z) = \bar{P}^{(\alpha)}(k_0, z) = 0$ with multiplicity β and $k_0 \in \mathbb{C}$ is the root of $\bar{P}_{z_0}^{(\alpha)}(k) = \bar{P}^{(\alpha)}(k, z_0) = 0$ with multiplicity $\nu^{(\alpha)}$. Then for z and k which satisfy $\bar{P}^{(\alpha)}(k, z) = 0$ and are in a sufficiently small neighborhood of (k_0, z_0), the formula*

$$(z - z_0)^\beta = b(k - k_0)^{\nu^{(\alpha)}} + O((k - k_0)^{\nu^{(\alpha)}+1}), \quad b \neq 0$$

is valid. Here $b \in \mathbb{C}$ is a constant only dependent on (z_0, k_0)[1]*.*

Proof. The lemma can be easily proven by the Taylor expansion.

From (2.2)–(2.5), we know the linearly independent solutions of $P(K, z_0)\phi_j = 0$ are

$$\phi_j = [k_i^{(\alpha)}]^j j^\delta \psi^{(\alpha)}, \quad 1 \leq i \leq \mu^{(\alpha)}, 0 \leq \delta \leq \nu_i^{(\alpha)} - 1, 1 \leq \alpha \leq N, \sum_{i=1}^{\mu^{(\alpha)}} \nu_i^{(\alpha)} = p + r,$$

and they span the linear space of all such solutions, where $\{k_i^{(\alpha)}, \nu_j^{(\alpha)}\}_{1 \leq i \leq \mu^{(\alpha)}}^{1 \leq \alpha \leq N}$ are defined as above, and $\psi^{(\alpha)}$ denote $T^{-1}(\overbrace{0, \cdots, 0, 1}^{\alpha}, 0 \cdots, 0)^T$.

Lemma 2.2[2]**.** *Suppose that Q admits a solution*

$$U_j^n = z_0^n k_0^j \psi^{(\alpha)} = \exp(i(\omega_0 t - \xi_0 x))\psi^{(\alpha)}, \quad x = jh, \ t = n\tau$$

with $|z_0| = |k_0| = 1, \psi^{(\alpha)} = T^{-1}(\overbrace{0, \cdots, 0, 1}^{\alpha}, 0, \cdots, 0)^T$. Then the group velocity of $\exp(i(\omega_0 t - \xi_0 x))\psi^{(\alpha)}$ defined by

$$c(k_0, z_0) = \frac{d\omega}{d\xi}\bigg|_{(\omega_0, \xi_0)} = -\frac{1}{\lambda}\frac{dz}{dk}\frac{k}{z}\bigg|_{(k_0, z_0)}$$

exists and is real. Furthermore, $c(k_0, z_0) = 0$ if and only if k_0 is a multiple root of the polynomial $\bar{P}_{z_0}^{(\alpha)}(k) = \bar{P}^{(\alpha)}(k, z_0) = 0$. Here k and z satisfy $\bar{P}^{(\alpha)}(k, z) = 0$.

Definition 2.1[2]**.** Suppose that Q admits a solution

$$U_j^n = z_0^n k_0^j j^\delta \psi^{(\alpha)}, \quad \begin{array}{ll} 0 \leq \delta \leq \nu - 1, & \text{if } |z_0| > 1, \\ 0 \leq \delta \leq (\nu + 1)/2, & \text{if } |z_0| = 1. \end{array}$$

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1) Note. By asumption, Q_{-1} is invertible in l_2, and the coefficient of the first term of polynomial $\bar{P}^{(\alpha)}(k_0, z)$ is not zero for k_0 with $|k_0| = 1$ (and hence, by continuity, for $|k_0|$ sufficiently close to 1).

Here ν is the multiplicity of k_0 as a root of $P^{(\alpha)}(k, z_0) = 0$. The strictly right-going (left-going) and right-going (left-going) signals are defined as follows:

Table D

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<tr><td>$|z_0| > 1$
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$c < 0$</td><td>$|z_0| = 1$
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<tr><td></td><td></td><td></td><td>$\delta = \nu_p$
$= \nu_r + 1$</td><td>$\delta \leq \min\{\nu_p, \nu_r\}$</td><td>$\delta = \nu_r$
$= \nu_p + 1$</td><td></td><td></td><td></td></tr>
<tr><td colspan="2">strictly left-going</td><td colspan="5"></td><td colspan="2">strictly right-going</td></tr>
<tr><td colspan="5">left-going</td><td colspan="4"></td></tr>
<tr><td colspan="4"></td><td colspan="5">right-going</td></tr>
</table>

Here $c = c(k_0, z_0)$ denotes group velocity of $z_0^n k_0^j j^\delta \psi^{(\alpha)}$, and ν_r and ν_p are defined as in Theorem 2.3.2 of [2], and satisfy

$$\nu_r = \nu_p = \nu/2, \qquad \text{if } \nu \text{ is even,}$$

$$\nu_r = \nu_p + 1 = (\nu+1)/2 \text{ or } \nu_r = \nu_p - 1 = (\nu-1)/2, \qquad \text{if } \nu \text{ is odd.}$$

Lemma 2.3[1]. *For all $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$, the number of linearly independent right-going signals of Q is equal to Nr.*

For given $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$, put the right-going signals in order. By Lemma 2.3, the general right-going solution of the form $U_j^n = z_0^n \phi_j$ admitted by Q can be written as

$$U_j^n = z_0^n \sum_{i=1}^{Nr} a_i k_i^j j^{\delta_i} \psi_i, \quad |z_0| \geq 1. \tag{2.6}$$

Inserting (2.6) into the homogeneous boundary formula

$$U_\mu^{n+1} = \sum_{\sigma=-1}^{s} S_\sigma^{(\mu)} U_1^{n-\sigma}, \quad -r+1 \leq \mu \leq 0,$$

we obtain a linear system of equations

$$\sum_{i=1}^{N_r} [\cdots] a_i = 0,$$

where each term in brackets is a given N_r-vector.

If we write

$$a^{[r]} = (a_1, \cdots, a_{Nr})^T,$$

then these equations take the form:

$$E(z_0) a^{[r]} = 0,$$

where $E(z_0)$ is an $N_r \times N_r$ matrix.

§3. Necessary l_2-stability Conditions

Theorem GKS[1]. *The necessary and sufficient condition for GKS-stability of $\bar{Q}$ (in the sense of Definition 3.3 of [1]) is that for all $z \in \mathbb{C}$ with $|z| \geq 1$, the matrix $E(z)$ is nonsingular, i.e.*

$$\det E(z) \neq 0, \quad \forall |z| \geq 1.$$

Proof. See [1].

Let Q_* denote the homogeneous boundary difference model

$$\begin{cases} Q_{-1} U_j^{n+1} = \sum_{\sigma=0}^{s} Q_\sigma U_j^{n-\sigma}, \\ U_\mu^{n+1} = \sum_{\sigma=-1}^{s} S_\sigma^{(\mu)} U_1^{n-\sigma}, \quad -r+1 \leq \mu \leq 0. \end{cases}$$

Theorem GRT (Godunov-Ryabenkii, Trefethen). *A necessary condition for initial and boundary value l_2-stability of $\bar{Q}$ is that the difference model Q_* admits no strictly right-going solution for all $z \in \mathbb{C}$ with $|z| \geq 1$, where the strictly right-going solution means that the solution is the linear combination of strictly right-going signals.*

Proof. See Theorems 4.2.1, 4.2.3 and 4.2.4 in [2].

Theorem 3.1. *A necessary condition for initial and boundary value l_2-stability of $\bar{Q}$ is that Q_* admits no sloution in the following form:*

$$U_j^n = z_0^n \sum_{i=1}^{l} a_i k_i^j \psi_i, \quad a_i \neq 0, \quad 1 \leq i \leq l, \quad |z_0| \geq 1. \tag{3.1}$$

Here for each i, $\psi_i = T^{-1}(\overbrace{0,\cdots,1}^{\alpha_i},0,\cdots,0)^T$ and $z_0^n k_i^j \psi_i$ is a right-going signal of Q. When $|z_0| = 1$, (a) the multiplicity of each k_i as a root of $P^{(\alpha_i)}(k, z_0) = 0$ is not larger than two, and (b) for the z which is sufficiently close to z_0, the following estimate is satisfied

$$\|U^0(z)\|_x^2 = \sum_{j=-r+1}^{+\infty} \Big|\sum_{i=1}^{l} a_i k_i^j(z)\psi_i\Big|^2 h \geq \bar{c}h/|z - z_0|^{\frac{1}{2}}, \tag{3.2}$$

where $\bar{c}$ is a positive constant, and $z^n k_i^j(z)\psi_i$ $(1 \leq i \leq l)$ are all right-going signals of Q.

Proof. Assume the difference scheme Q_* admits the solution (3.1).

If $|z_0| > 1$, then by Lemma 2.2.1 of [2] and Theorem GRT, $\bar{Q}$ is l_2-unstable. In the following, we assume $|z_0| = 1$.

Let $z = (1+\varepsilon)z_0$ and $k_i(z)$ satisfy $P^{(\alpha_i)}(k_i(z), z) = 0$ $(1 \leq i \leq l)$, where ε is a small enouth positive number.

By assumption (a), Lemma 2.1, and Lemma 2.2.1 and Theorem 2.3.2 of [2], we have

$$|k_i(z)| < 1, \ |k_i(z) - k_i(z_0)| \leq M\varepsilon^{\frac{1}{2}}, \ 1 \leq i \leq l, \tag{3.3}$$

where M is a constant.

Choosing the initial value

$$f_j^\sigma = z^\sigma \sum_{i=1}^{l} a_i k_i^j(z)\psi_i, \quad z = (1+\varepsilon)z_0, \quad 0 \le \sigma \le s,$$

and solving the Cauchy problem

$$\begin{cases} Q_{-1}U_j^{n+1} = \sum_{\sigma=0}^{s} Q_\sigma U_j^{n-\sigma}, \\ U_j^\sigma = f_j^\sigma, \quad j = 0, \pm 1, \pm 2, \cdots, 0 \le \sigma \le s, \end{cases}$$

we have the solution

$$U_j^n = z^n \sum_{i=1}^{l} a_i k_i^j(z)\psi_i.$$

Let

$$g_\mu^n = U_\mu^{n+1} - \sum_{\sigma=-1}^{s} S_\sigma^{(\mu)} U_1^{n-\sigma}, \quad -r+1 \le \mu \le 0, \quad n \ge s. \tag{3.4}$$

Since $\sum_{i=1}^{l} a_i z_0^n k_i^j(z_0)\psi_i$ satisfies the boundary formula:

$$U_\mu^{n+1} = \sum_{\sigma=-1}^{s} S_\sigma^{(\mu)} U_1^{n-\sigma}, \quad -r+1 \le \mu \le 0,$$

it follows that

$$\sum_{i=1}^{l} a_i z_0 k_i^\mu(z_0)\psi_i - \sum_{\sigma=-1}^{s}\sum_{i=1}^{l} a_i z_0^{-\sigma} S_\sigma^{(\mu)} k_i^1(z_0)\psi_i = 0, \quad -r+1 \le \mu \le 0.$$

From this formula and (3.4) and (3.3), we obtain

$$\begin{aligned} |g_\mu^n|^2 \le (1+\varepsilon)^{2n} \Big| \sum_{i=1}^{l} a_i \Big[z k_i^\mu(z) - z_0 k_i^\mu(z_0)\Big]\psi_i \\ - \sum_{\sigma=-1}^{s}\sum_{i=1}^{l} a_i S_\sigma^{(\mu)} \Big[z^{-\sigma}k_i^1(z) - z_0^{-\sigma}k_i^1(z_0)\Big]\psi_i\Big|^2 \le M_0(1+\varepsilon)^{2n}\varepsilon, \\ -r+1 \le \mu \le 0, \end{aligned}$$

where M_0 is a constant independent of ε.

For any given h, set $\bar{N}+1 = 1/h, \varepsilon = h^{\frac{1}{2}}$. Then, on the one hand, we have

$$\|g_\mu\|_{t\le\lambda}^2 \le M_0 k \sum_{n=s}^{\bar{N}+1} (1+\varepsilon)^{2n}\varepsilon \le M_0 k(1+\varepsilon)^{2(\bar{N}+1)}/2, \quad \varepsilon < 1, \ -r+1 \le \mu \le 0; \tag{3.5}$$

on the other hand, by assumption (b), we have

$$\|U^0\|_x^2 = \|f^0\|_x^2 \ge \bar{c}h/\varepsilon^{\frac{1}{2}}, \tag{3.6}$$

where $\bar{c}$ is a positive constant.

From (3.5) and (3.6), we get

$$\frac{\|U^{\bar{N}+1}\|_x^2}{\sum\limits_{\mu=-r+1}^{0}\|g_\mu\|_{t\le\lambda}^2+\sum\limits_{\sigma=0}^{s}\|f^\sigma\|_x^2}\ge\frac{(1+\varepsilon)^{2(\bar{N}+1)}\|U^0\|_x^2}{rM_0k(1+\varepsilon)^{2(\bar{N}+1)}/2+\|U^0\|_x^2\sum\limits_{\sigma=0}^{s}(1+\varepsilon)^{2\sigma}}$$

$$\ge\min\left\{\frac{\|U^0\|_x^2}{rM_0k},\frac{(1+\varepsilon)^{2(\bar{N}+1)}}{2\sum\limits_{\sigma=0}^{s}(1+\varepsilon)^2}\right\}\ge\min\left\{\frac{\bar{c}}{rM_0\lambda h^{\frac{1}{4}}},\frac{(1+h^{\frac{1}{2}})^{2/h}}{2^{2s+1}(s+1)}\right\}\longrightarrow+\infty,$$

when $h\to+0$.

It follows that $\bar{Q}$ is initial and boundary value l_2-unstable.

Theorem 3.2. *A necessary condition for initial and boundary value l_2-stability of $\bar{Q}$ is that Q_* admits no solution in the following form:*

$$U_j^n=\sum_{i=1}^{l}a_iz_0^nk_i^j\psi_i,\quad a_i\ne0,\quad 1\le i\le l,\quad |z_0|\ge1,\tag{3.7}$$

where for each i, $\psi_i=T^{-1}(\overbrace{0,\cdots,1}^{\alpha_i},0,\cdots,0)$ and $z_0^nk_i^j\psi_i$ is a right-going signal of Q; k_i is the simple root of $\bar{P}^{(\alpha_i)}(k,z_0)=0$.

Proof. Assume the difference scheme Q_* has the solution (3.7) with $|z_0|=1$. Let $z=(1+\varepsilon)z_0$ with sufficiently small ε.

Similarly to (3.3), now we have

$$|k_i(z)|<1,\quad |k_i(z)-k_i(z_0)|<M\varepsilon,\quad 1\le i\le l,$$

where $M>0$ is a constant independent of ε, and z and $k_i(z)$ satisty $P^{(\alpha_i)}(k_i(z),z)=0$.

Choosing the initial value

$$f_j^\sigma=z^\sigma\sum_{i=1}^{l}a_ik_i^j(z)\psi_i,\quad z=(1+\varepsilon)z_0,\quad 0\le\sigma\le s,$$

and solving the Cauchy problem:

$$\begin{cases}Q_{-1}U_j^{n+1}=\sum\limits_{\sigma=0}^{s}Q_\sigma U_j^{n-\sigma},\\ U_j^\sigma=f_j^\sigma,\quad j=0,\pm1,\pm2,\pm3,\cdots,\quad 0\le\sigma\le s,\end{cases}$$

we have the solution

$$U_j^n=z^n\sum_{i=1}^{l}a_ik_i^j(z)\psi_i.$$

Let

$$g_\mu^n = U_\mu^{n+1} - \sum_{\sigma=-1}^{s} S_\sigma^{(\mu)} U_1^{n-\sigma}, \quad -r+1 \le \mu \le 0, \quad n \ge s.$$

Then, as earlier, we get

$$|g_\mu^n|^2 \le M_0(1+\varepsilon)^{2n}\varepsilon^2,$$

where $M_0 > 0$ is also a constant independent of ε.

For any given $h > 0$, set $\bar{N}+1 = 1/h$, $\varepsilon = h^{\frac{1}{3}}$. Then, on the one hand, we have

$$\|g_\mu\|_{t\le\lambda}^2 \le M_0\varepsilon^2 k \sum_{n=s}^{\bar{N}+1}(1+\varepsilon)^{2n} \le M_0\varepsilon k(1+\varepsilon)^{2(\bar{N}+2)}/3, \quad \varepsilon < 1, -r+1 \le \mu \le 0; \quad (3.8)$$

on the other hand, we have

$$\|U^0\|_x^2 = \|f^0\|_x^2 \ge \bar{c}h, \quad (3.9)$$

where $\bar{c}$ is a positive constant independent of h.

By (3.8) and (3.9), we get

$$\frac{\|U^{\bar{N}+1}\|_x^2}{\sum_{\mu=-r+1}^{0}\|g_\mu\|_{t\le\lambda}^2 + \sum_{\sigma=0}^{s}\|f^\sigma\|_x^2} \ge \frac{(1+\varepsilon)^{2(\bar{N}+1)}\|U^0\|_x^2}{rM_0\varepsilon k(1+\varepsilon)^{2(\bar{N}+1)}/3 + \|U^0\|_x^2\sum_{\sigma=0}^{s}(1+\varepsilon)^{2\sigma}}$$

$$\ge \min\left\{\frac{\|U^0\|_x^2}{2rM_0\varepsilon k/3}, \frac{(1+\varepsilon)^{2(\bar{N}+1)}}{2\sum_{\sigma=0}^{s}(1+\varepsilon)^{2\sigma}}\right\} \ge \min\left\{\frac{3\bar{c}}{2rM_0\lambda h^{\frac{1}{3}}}, \frac{(1+h^{\frac{1}{3}})^{2/h}}{2^{2s+1}(s+1)}\right\} \longrightarrow +\infty,$$

when $h \to +0$.

This has proved the theorem.

Theorem 3.3. *For the difference models coupling the following difference schemes to arbitrary boundary formulas, a necessary condition for l_2-stability is the same as the necessary and sufficient condition for GKS-stability.*

Difference schemes (We need only consider scalar difference schemes):

(a) LF-Leap Frog:

$$U_j^{n+1} - U_j^{n-1} = \lambda a(U_{j+1}^n - U_{j-1}^n),$$
$$z - z^{-1} = \lambda a(k - k^{-1});$$

(b) CN-Crank Nicolson:

$$U_j^{n+1} - U_j^n = \lambda a(U_{j+1}^{n+1} - U_{j-1}^{n+1} + U_{j+1}^n - U_{j-1}^n)/4,$$
$$z - 1 = \lambda a(zk - zk^{-1} + k - k^{-1})/4;$$

(c) BE-Backwards Euler:

$$U_j^{n+1} - U_j^n = \lambda a(U_{j+1}^{n+1} - U_{j-1}^{n+1})/2,$$
$$z - 1 = \lambda a(zk - zk^{-1})/2;$$

(d) UW-Upwind:

$$U_j^{n+1} - U_j^n = \lambda a\big(U_{j+1}^n - U_j^n\big), \text{ if } a > 0,$$
$$z - 1 = \lambda a(k-1);$$

(e) Bx-BOx:

$$U_j^{n+1} + U_{j+1}^{n+1} - U_j^n - U_{j+1}^n = \lambda a(U_{j+1}^n + U_{j+1}^{n+1} - U_j^n - U_j^{n+1}),$$
$$z + zk - 1 - k = \lambda a(k + zk - 1 - z).$$

Proof. For any fixed difference scheme listed above, and for any fixed z_0 with $|z_0| = 1$, the difference scheme has only one right-going signal $z_0^n k^j$. It always satisfies the following condition:

(i) k is a simple root of $P(k,z)|_{z=z_0} = 0$, where $P(k,z) = 0$ denotes the corresponding dispersion relation of that difference scheme; or

(ii) $k = i$ (resp. $-i$), and the multiplicity of root k is two; furhtermore, the estimate (3.2) is satisfied.

So, by Theorems 3.2 and 3.1 and the definition of $E(z)$, a necessary l_2-stability condition for the considered difference models is $\det E(z) \neq 0, \forall |z| \geq 1$.

Theorem 3.4. *For the difference models coupling the Lax-Friedrichs difference scheme to arbitrary-boundary formulas, a necessary l_2-stability condition is*

$$\det E(z) \neq 0, \quad \forall \quad |z| \geq 1.$$

Proof. The L-F scheme is

$$U_j^{n+1} = \frac{1}{2}(U_{j+1}^n + U_{j-1}^n) + \frac{1}{2}\lambda a(U_{j+1}^n - U_{j-1}^n).$$

Its dispersion relation is

$$z = \frac{1}{2}(k + k^{-1}) + \frac{1}{2}\lambda a(k - k^{-1}). \tag{3.10}$$

Now we prove for all z with $|z| = 1$ that the roots k_i of equation (3.10) are simple.

(3.10) is equivalent to

$$2zk = (k^2 + 1) + (\lambda a)(k^2 - 1).$$

If k_0 is a multiple root, then it satisfies

$$z = k_0 + \lambda a k_0,$$

i.e.

$$k_0 = z/(1 + \lambda a). \tag{3.11}$$

Inserting (3.11) into (3.10), we get

$$\begin{aligned} 2z^2(1+\lambda a) &= z^2 + (1+\lambda a)^2 + (\lambda a)z^2 - (\lambda a)(1+\lambda a)^2 \\ &= z^2(1+\lambda a) + (1-\lambda a)(1+\lambda a)^2, \end{aligned}$$

that is

$$z^2 = (1+\lambda a)(1-\lambda a) = 1 - (a\lambda)^2.$$

Since λa is real and nonzero, we must have $|z| < 1$. This proves that, for all z_0 with $|z_0| = 1$, the roots k of $P(k, z_0) = 0$ are simple. By Theorem 3.2, this theorem follows.

Theorem 3.5. *For difference models coupling the Lax-Wendroff scheme to arbitrary boundary formulas, a necessary l_2-stability condition is also*

$$\det E(z) \neq 0, \quad \forall\ |z| \geq 1.$$

Proof. The proof is similar to that of Theorem 3.4.

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