Journal of Computational Mathematics, Vol.8, No.2, 1990, 159–177.

ON NUMERICAL SIMULATION OF STRATIFIED FLOWS OF INCOMPRESSIBLE FLUID*

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Introduction

In the present paper, numerical algorithms for calculation of stratified fluids naturally adapted for parallel computations and allowing one to estimate changes of the river temperature condition downstream of hydroelectric stations according to temperature stratification of the reservoir and water intake conditions have been described. Practice has shown that building hydroelectric stations with deep-water reservoirs leads to appreciable changes of the hydrothermal river condition both up stream and downstream of the waterworks facility. In deepwater reservoirs, temperature stratification is established; water temperature changes appreciably with depth. Theoretical and experimental studies have shown that the flow pattern of a non-homogeneous fluid in the near dam part depends on the stratification character, water discharge and position of intake apertures.

Before describing numerical results, we shall briefly review numerical methods of simulation of flows of stratified fluids.

§1. A Review of Works on Numerical Simulation of Flows of Stratified Fluids

Reviws of the theory of stratified flows are given in [1,2]. Studies of flows of viscous incompressible density stratified fluids in a gravitational force field are based on the consideration of a complete set of the Navier-Stokes equations

$$\begin{aligned} &\frac{\partial \bar{V}}{\partial t} + (\bar{V}\cdot\nabla)\bar{V} = -\frac{1}{\rho}\nabla p + \nu\Delta\bar{V} + \bar{g}, \\ &\frac{\partial \rho}{\partial t} + \bar{V}\cdot\nabla\rho = 0, \\ &\mathrm{div}\bar{V} = 0. \end{aligned} \tag{1.1}$$

Here $\bar{V}$ is the velocity vector, p is the pressure, ρ is the density, ν is the coefficient of kinematic viscosity, and $\bar{g}$ is the gravity force. In describing dynamical processes, use is made of Oberbeque-Boussinesq model [3,4], according to which only the change of fluid density is taken account of in buoyancy forces

$$\begin{aligned} &\frac{\partial \bar{V}}{\partial t} + (\bar{V}\cdot\nabla)\bar{V} = -\frac{1}{\rho_0}\nabla p + \nu\Delta\bar{V} + \frac{\rho}{\rho_0}\bar{g}, \\ &\frac{\partial \rho}{\partial t} + \bar{V}\cdot\nabla\rho = 0, \\ &\mathrm{div}\bar{V} = 0, \end{aligned} \tag{1.2}$$

* Received September 14, 1987.

where $\rho_0 = \text{const.}$ is the characteristic density value. From the view-point of hydrodynamical theory, Oberbeque-Boussinesq equations differ little from Navier-Stokes equations for incompressible fluids. However, the small differences can result in rather appreciable effects and sometimes in initiation of motions impossible in the absence of stratification. Numerical schemes for a study of stratified flows retain all specific properties of schemes for equations of a homogeneous fluid and contain some specific properties associated with extra calculations of the density field.

At present, a great number of numerical methods of solving Navier-Stokes equations are known [5-9]. Among the methods of computational hydrodynamics, finite-difference methods are the most common which we shall confine ourselves to. Methods of solving Navier-Stokes equations can be divided into two main groups. The first is connected with the introduction of the function of the current ψ at the vorticity ω and transformation of the initial system of equations to the system of equations relative to (ψ, ω)

$$\begin{gathered}\frac{\partial \omega}{\partial t} + (\bar{V} \cdot \nabla)\omega = \nu \Delta \omega, \\ \Delta \psi = -\omega, \\ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.\end{gathered} \tag{1.3}$$

Here u, v are the velocity vector projections. The advantage of such an approach is that there is need to take care of solenoidality of the velocity field (the condition is fulfilled automatically). However, there arise difficulties associated with setting boundary conditions on the stream function and vorticity. Such methods are restricted to the case of two-dimensional flows.

The other group is a solution of Navier-Stokes equations in primitive variables "velocity-pressure". The main difficulty with such an approach consists in defining a boundary condition for the pressure. Historically the major share of numerical methods has been developed applicable to a system of equations in Helmholtz (1.3) form. Initially, the methods were based on using explicit schemes such as a scheme with differences upstream (with donor cells) [10]

$$\begin{aligned}\varphi_{i,j}^{n+1} &= \varphi_{i,j}^n - (u\varphi)_{i+1/2,j}^n + (u\varphi)_{i-1/2,j}^n - (v\varphi)_{i,j+1/2}^n + (v\varphi)_{i,j-1/2}^n \\ &+ a^2 \cdot \Delta t \left(\frac{\varphi_{i+1,j}^n - 2\varphi_{i,j}^n + \varphi_{i-1,j}^n}{\Delta x^2} + \frac{\varphi_{i,j+1}^n - 2\varphi_{i,j}^n + \varphi_{i,j-1}^n}{\Delta y^2} \right),\end{aligned} \tag{1.4}$$

where

$$(u\varphi)_{i+1/2,j}^n = \begin{cases} u_{i+1/2,j}^n \cdot \varphi_{i,j}^n \dfrac{\Delta t}{\Delta x} & \text{at } u_{i+1/2,j}^n > 0, \\ u_{i+1/2,j}^n \cdot \varphi_{i+1,j}^n \dfrac{\Delta t}{\Delta x} & \text{at } u_{i+1/2,j}^n \le 0, \end{cases}$$

$$(v\varphi)_{i,j+1/2}^n = \begin{cases} v_{i,j+1/2}^n \cdot \varphi_{i,j}^n \dfrac{\Delta t}{\Delta y} & \text{at } v_{i,j+1/2}^n > 0, \\ v_{i,j+1/2}^n \cdot \varphi_{i,j+1}^n \dfrac{\Delta t}{\Delta y} & \text{at } v_{i,j+1/2}^n \le 0, \end{cases}$$

the scheme "forward in time – central in space" [5,10]

$$\varphi^{n+1}_{i+1/2,j} = \varphi^n_{i+1/2,j} - \Delta t\frac{(u\varphi)^n_{i+1,j} - (u\varphi)^n_{i,j}}{\Delta x} - \Delta t\frac{(v\varphi)^n_{i+1/2,j+1/2} - (v\varphi)^n_{i+1/2,j-1/2}}{\Delta y}$$
$$+a^2\Delta t\Big(\frac{\varphi^n_{i+3/2,j} - 2\varphi^n_{i+1/2,j} + \varphi^n_{i-1/2,j}}{\Delta x^2} + \frac{\varphi^n_{i+1/2,j+1} - 2\varphi^n_{i+1/2,j} + \varphi^n_{i+1/2,j-1}}{\Delta y^2}\Big), \tag{1.5}$$

for the equation

$$\frac{\partial\varphi}{\partial t} + \frac{\partial(u\varphi)}{\partial x} + \frac{\partial(v\varphi)}{\partial y} = a^2\Big(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\Big), \tag{1.6}$$

and Lait's scheme [5,12] (for a transport equation)

$$\begin{aligned}\varphi^{n+1/2}_{i,j} &= \varphi^n_{i,j} - \frac{1}{2}C_x(\varphi^n_{i+1,j} - \varphi^n_{i-1,j}) + \frac{1}{2}C_x^2(\varphi^n_{i+1,j} - 2\varphi^n_{i,j} + \varphi^n_{i-1,j}),\\ \varphi^{n+1/2}_{i,j} &= \varphi^{n+1/2}_{i,j} - \frac{1}{2}C_y(\varphi^{n+1/2}_{i,j-1} - \varphi^{n+1/2}_{i,j-1}) + \frac{1}{2}C_y^2(\varphi^{n+1/2}_{i,j+1} - 2\varphi^n_{i,j} + \varphi^{n+1/2}_{i,j-1}),\end{aligned} \tag{1.7}$$

where

$$C_x = u^n_{i,j}\frac{\Delta t}{\Delta x}, \quad C_y = v^n_{i,j}\frac{\Delta t}{\Delta y}.$$

These schemes possess the advantage that on passing from one temporal layer to the next one, a simple recount is required. However, the stability condition is rather limited and, for a number of problems, one has to use too small a time step. The implicit schemes [11]

$$\frac{\omega^{n+1} - \omega^n}{\tau} + u^n\Big(\frac{\partial\omega}{\partial x}\Big)^{n+1} + v^n\Big(\frac{\partial\omega}{\partial y}\Big)^{n+1} = \nu\Delta\omega^{n+1}, \quad \Delta\psi^{n+1} = -\omega^{n+1} \tag{1.8}$$

possess better stability criteria but are complicated since they require handling of tri-diagonal matrices. With an implicit approximation transport equations with a diffusion vorticity perturbation at one point instantly influence the whole area. Lately, the most popular are implicit schemes of the method of variable directions [7, 12, 13] and splitting schemes [14]. The scheme of the method of variable directions for a parabolic equation

$$\begin{aligned}&\frac{\partial\varphi}{\partial t} + A\varphi = 0,\\ &A = A_1 + A_2, \quad A_2 = -a^2\frac{\partial^2}{\partial x_2^2}\end{aligned} \tag{1.9}$$

has the form

$$\begin{aligned}&\frac{\varphi^{j+1/2} - \varphi^j}{\tau} + \frac{1}{2}(A_1\varphi^{j+1/2} + A_2\varphi^j) = 0,\\ &\frac{\varphi^{j+1} - \varphi^{j+1/2}}{\tau} + \frac{1}{2}(A_1\varphi^{j+1/2} + A_2\varphi^{j+1}) = 0\end{aligned} \tag{1.10}$$

(τ is the time step). The splitting algorithm based on the usage of implicit schemes of the first order of approximation in time, has the form

$$\begin{aligned}&\frac{\varphi^{j+1/n} - \varphi^j}{\tau} + A_1\varphi^{j+1/n} = 0,\\ &\frac{\varphi^{j+1} - \varphi^{j+(n-1)/n}}{\tau} + A_n\varphi^{j+1} = 0,\end{aligned} \tag{1.11}$$

where

$$A = \sum_{\alpha=1}^{n} A_\alpha, \quad A_\alpha \geq 0, \quad n \geq 2.$$

High order of approximation combined with unconditional stability permits one to obtain good results at large Reynolds numbers with comparatively small expenditures of machine time.

In the numerical simulation of a system in stream function-vorticity variables, as a rule, an overlapped grid is used, when all the unknown quantities are determined at the same points. In [15] use is made of hybrid grids with spaced velocities, which possess somewhat better dispersion properties [16]. A considerable problem in numerical simulation in variables of stream-function vorticity is the setting of a condition for the vorticity at the no-slip boundary which is absent in the physical statement of the problem. The condition of the first order of approximation, first suggested by Thom [17]

$$\omega_j = \frac{2(\psi_{j+1} - \psi_j)}{\Delta^2} \tag{1.12}$$

where Δ is the distance along a normal to the wall from the nearest to the boundary nodal point $j+1$ to its projection j on the wall, has successfully been used by many researchers. Conditions of a higher order of approximation are considered in [18, 19,7]

$$\begin{aligned} \omega_j &= \frac{3(\psi_{j+1} - \psi_j)}{\Delta^2} - \frac{1}{2}\omega_{j+1}, \\ \omega_j &= \frac{-7\psi_j + 8\psi_{j+1} - \psi_{j+2}}{2\Delta^2}. \end{aligned} \tag{1.13}$$

However, their application is solving some problems resulting in the occurence of instability. In [7] consideration is given to numerical boundary conditions ensuring immediate fulfilment of a no-slip condition at the solid boundary. The idea of this approach is as follows. Let the solution of system (1.3) be sought in the region Ω_0; Ω_1 is the auxiliary region ($\Omega_1 \subset \Omega_0$), whose boundaries are one grid interval from the boundary of the main region Ω_0. Equations for the vortex are solved in the region Ω_1, and equations for the stream function are solved in the region Ω_0. Boundary conditions for the vorticity at the boundary of the region Ω_1 are defined proceeding from the equation for the stream function. The magnitude of the stream function at the boundary Ω_1 is corrected with a difference analog of the condition $\left(\frac{\partial \psi}{\partial n}\right)_\Gamma = 0$. It should be noted that employment of the above boundary conditions in applying implicit schemes leads to a decrease of stability of the main scheme. For stationary problems an increase of stability is achieved by using relaxational formulae for boundary conditions [7], according to which the magnitude of the vorticity at the boundary is obtained in the form

$$\omega_j^{n+1} = \alpha f(\psi^{n+1}) + (1-\alpha)\omega_j^n, \tag{1.14}$$

where α is the relaxation parameter, varying within the limits $0 \leq \alpha \leq 1$, and $f(\psi^{n+1})$ is the dependence between the vorticity at the boundary and the stream function of the form (1.12) or (1.13).

For an unsteady problem, it is necessary to introduce an additional iteration process according to the boundary condition [7], which considerably increases the amount of computations. Recently a number of boundary procedures [8, 20] have been suggested, retaining the advantages of implicit schemes. In [8] a technique based on the employment of Green's formula for equation (1.2) is considered. In [20] an operator-difference approach of a numerical solution of unsteady problems is presented, based on the method of total apporximation. In realizing thus obtained schemes the restriction on a time step can be stipulated by the nonlinearity of the problem. The principal difficulty in a numerical solution of the Navier-Stokes equations in natural variables is associated with the calculation of pressure. A significant stage in the development of equation solving methods of an incompressible fluid is the idea of introducing an artificial compressibility into the continuity equation [14, 21]. To solve a steady problem, use is made of the following unsteady system

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} + \nabla p = \nu \Delta \vec{V}, \tag{1.15}$$

$$\frac{\partial q}{\partial t} + \operatorname{div} \vec{V} = 0, \tag{1.16}$$

where $q = p + \frac{V^2}{2}$ or $q = p$. The main technique of obtaining an equation for the pressure consists in applying the div operator to the conservation of momentum. As a result, one obtains an elliptic type for pressure

$$\Delta p = - \operatorname{div} ((\vec{V} \cdot \nabla)\vec{V}), \tag{1.17}$$

whose solution ensures a solenoidality of the velocity vector. Such an approach was considered in [22] when developing the MAC method and in [6, 23]. In the works [14, 24] it is suggested using an evolutionary equation for pressure (1.16) ($q = p$), associating a velocity divergence and a pressure derivative with respect to time, to solve unsteady problems. In the numerical solution of the Navier-Stokes equations in primitive variables, as a rule, a grid with staggered velocities is used which allows one to ensure a fulfilment of conservation laws in a difference form and to use central-difference approximations of the second order. For equations in primitive variables, one need not define the vorticity at the no-slip boundary [6]; physical conditions $\vec{V} = \vec{V}|_\Gamma$ are set at all the boundaries. However, to define pressure from the equation of an elliptical type, it is required to assign boundary conditions which are absent from the initial statement. Numerical schemes for equations in velocity-pressure variables are generalized for a case of spatial flows. Splitting methods are used widely in solving multidimensional problems [9, 14]. The idea of splitting associated with handling the pressure forms the basis of the particle method in cells [22, 25]. According to this scheme, an intermediate velocity field is calculated first from the momentum equation without accounting for the pressure

$$\frac{\vec{V}^{n+1/2} - \vec{V}^n}{\tau} + (\vec{V}^n \cdot \nabla)\vec{V}^n = \nu \Delta \vec{V}^n. \tag{1.18}$$

Then, this field is corrected to take account of the pressure gradient [15]

$$\vec{V}^{n+1} = \vec{V}^{n+1/2} - \tau \nabla p, \tag{1.19}$$

where p is the steady-state solution of the equation

$$\frac{\partial p}{\partial t} + \operatorname{div} \vec{V}^{n+1/2} = \tau \Delta p. \tag{1.20}$$

A scheme of fractional steps is suggested in [26] and used in [24], based on the projection method, where the idea of the splitting particle method in cells is expounded in the projection formulation:

$$\begin{aligned}
&\frac{\bar{V}^* - \bar{V}^n}{\tau} + (\bar{V} \cdot \nabla)\bar{V}^* = \frac{1}{\text{Re}}\Delta\bar{V}^*,\\
&\Delta p^{n+1} = \frac{1}{\tau}\,\text{div}\bar{V}^*, \qquad (1.21)\\
&\frac{\bar{V}^{n+1} - \bar{V}^*}{\tau} + \nabla p^{n+1} = 0.
\end{aligned}$$

In contrast to the splitting schemes used in the noted methods, in the work [6] is used an explicit splitting scheme by physical factors, consisting of three stages

$$\begin{aligned}
&\frac{\bar{V}^* - \bar{V}^n}{\tau} = -(\bar{V}^n \cdot \nabla)\bar{V}^n + \nu\Delta\bar{V}^n + \bar{f},\\
&\Delta p = -\frac{1}{\tau}\,\text{div}\,\bar{V}^*, \qquad (1.22)\\
&\frac{\bar{V}^{n+1} - \bar{V}^*}{\tau} = -\nabla p.
\end{aligned}$$

In [6] the following physical interpretation of this scheme is suggested. In the first stage, it is assumed that the transfer of momentum is achieved by convection and diffusion only. The intermediate velocity field thus obtained does not satisfy the incompressibility condition; still it provides the correct description of the vorticity characteristics. In the second stage, the pressure is sought from the preliminary velocity field subject to the solenoidality of the final velocity field. In the third stage, it is suggested that the velocity field be advanced by the pressure gradient only. To increase the stability margin, a variant is considered in which, in the first stage, use is made of an implicit scheme. A difference scheme permitting computation of the flow of a viscous incompressible fluid without using a boundary condition for the vorticity at a solid surface possesses a larger efficiency, other conditions being equal.

In variables of stream function-vorticity for a spatial case, the vector potential is introduced: instead of the stream function,

$$\begin{aligned}
&\frac{\partial\bar{\omega}}{\partial t} + (\bar{V} \cdot \nabla)\bar{\omega} = (\bar{\omega} \cdot \nabla)\bar{V} + \frac{1}{\text{Re}}\Delta\bar{\omega} + \frac{1}{F_r^2}\,\text{rot}\,\bar{F},\\
&\nabla(\,\text{div}\,\bar{\psi}) - \Delta\bar{\psi} = \bar{\omega}, \qquad (1.23)\\
&\bar{V} = \text{rot}\,\bar{\psi} + \nabla\varphi,\\
&\Delta\varphi = 0.
\end{aligned}$$

Here $\bar{\omega}$ is the vortex vector, $\bar{\psi}$ is the vector potential, and φ is the scalar potential. With this formulation there arises a problem of assigning boundary conditions for the vector potential, which is investigated in [27, 28]. The problem of relation between the vorticity and the vector potential is considered in [29]. In [30,31] in vector potential-vorticity variables, a problem of convection in a rectangular space is investigated. In studying stratified flows, as a rule, use is made of the same difference schemes as in the case of a homogeneous fluid. Splitting methods for problem-solving of stratified dynamics are given in [6, 32, 33].

In [6, 34, 35] a collapse process of a homogeneous spot in a stratified fluid is studied. In [6] the impact of stratification on the character of the flow about a sphere with a viscous fluid is considered. In [36] a numerical simulation of the flow in a stratified reservoir is made when warm water is discharging into it. In [37] an algorithm is developed and a study of a spatial flow about a mountain with a stratified fluid is carried out. A study of the impact of stratification on the character of flow in a flowing reservoir with the account of a wind effect is carried out in [38]. In [39] the influence of stratification on the quality of the water intaken from the reservoir is studied. In the work [40] the process of flow past an obstacle in a stratified fluid is studied, and the origination of the flow choking effect under some conditions is noted. In [8] an example of calculation of a laminar flow in a channel with a stratified fluid is given. It is noted that, beyond the boundary of the jet, the flow is non-stationary.

At present, for problem-solving of continuous media there have been developed a large number of numerical algorithms. An a priori guarantee of the reliability of the numerical solution is based on two theoretical provisions-approximation by a discrete algorithm of a differential problem and stability of its linearized model. All algorithms are substantiated asymptotically only, at sufficiently small values of the discretization parameters, i.e. the spatial-temporal grid sizes, which are seldom obtainable in practice. Thus, there arises a problem of constructing difference schemes ensuring an assigned accuracy on realistic grids.

A difference scheme should reflect the main properties of the continuous medium. One of the natural requirements is the consistency of difference analogs of the main conservation laws. The significance of conservation is pointed out in [41, 42]. In [41] the notion of a completely conservative scheme is introduced, for which not only difference analogs of the main conservation laws hold, as for conservative schemes, but also extra grid relations are established, which are necessitated by physical considerations. In the work [42] group properties of difference schemes for problems of gas dynamics are studied on the basis of the first differential approximation. Classes of invariant schemes of a different order of approximation are constructed. The relationship between invariance properties and the complete conservation of difference schemes is investigated.

In solving problems of incompressible fluid dynamics, the question of defining the correct vorticity characteristics is of great importance. In a numerical solution it is necessary to ensure that the vorticity conservation law holds at a difference level. Approximating levels of impulses by conservative difference schemes does not always appear to ensure fulfilment of the noted property. Therefore, to conserve vorticity characteristics at a convective transfer, it is recommend to consider the momentum equations in the Gromek-Lamb form:

$$\frac{\partial \bar{V}}{\partial t} + \operatorname{rot} \bar{V} \times \bar{V} = \frac{1}{\mathrm{Re}} \Delta \bar{V} + \frac{1}{Fr^2} \bar{F} + \nabla P. \qquad (1.24)$$

In the framework of the noted approach, computations of an incompressible homogeneous fluid in a cubic pit have been carried out, revealing even on a course grid new structures characteristic of three-dimensional flows-corner vortices and vortices of the Taylor-Goertler type.

§2. Simulation of a Selective Water Intake of a Stratified Fluid

Now we shall describe results of the numerical simulation of velocity and temperature fields in stratified reservoirs under different conditions of water intake. Studies on a selective water intake from a two-layered fluid have been conducted by a number of authors both theoretically and experimentally [38, 43, 46]. Flows of many-layered stratified fluids with a continuous change of density were studied less intensively than of two-layered ones.

The mathematical model of a heavy non-homogeneous fluid flow used below is based on the following assumptions [47, 48]:

– temperature regime of the reservoir is known and the influence of heat exchange processes may be neglected and density is a function of temperature only;

– flows are laminar and the equations of viscous fluid flow are based on the Boussinesq approximation;

– surface waves do not affect the flow pattern and a free surface is replaced by a rigid lid.

2.1. Equations of non-homogeneous heavy fluid flow

A numerical study of viscous incompressible density-stratified fluid flow in the field of a gravitational force is based on the Navier-Stokes equations with a Boussinesq approximation. The system of Navier-Stokes equations may be written in two equivalent forms [5,6]. In primitive variables, "velocity-pressure":

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + (\bar{V} \cdot \nabla)\rho = 0,\\
&\frac{\partial \bar{V}}{\partial t} + (\bar{V} \cdot \nabla)\bar{V} = \frac{1}{Fr^2}(-\nabla p + \bar{F}) + \frac{1}{\mathrm{Re}}\Delta \bar{V},\\
&\mathrm{div}\ \bar{V} = 0.
\end{aligned} \tag{2.1}$$

Here (x, y, z) is a rectangular Cartesian coordinate system (the y axis is directed downward); t is the time; the coordinates are related to the characteristic lengh H (H is the depth of a reservoir); time is related to H/u_0, where u_0 is a characteristic velocity value; ρ is the deviation of density from the characteristic value ρ_0, related to $\Delta\rho(\Delta\rho = \rho_{\max} - \rho_{\min})$; p is the deviation of pressure from the hydrostatic $p_0 (p_0 = \rho_0 g y)$, related to $\Delta\rho g H$; $\bar{V} = (u, v, w)$ is the velocity vector, with u, v, w related to the characteristic velocity u_0; $\bar{F} = (0, \rho, 0)$; Re $= \dfrac{u_o H}{\nu}$ is the Reynolds number; $Fr = \dfrac{u_0}{\sqrt{gH\Delta\rho/\rho_0}}$ is the Froude number; ν is the kinematic viscosity coefficient; g is the acceleration due to gravity; $\nabla = \dfrac{\partial}{\partial x}\bar{i} + \dfrac{\partial}{\partial y}\bar{j} + \dfrac{\partial}{\partial z}\bar{k}$, $\bar{i}, \bar{j}, \bar{k}$ are the unit vectors of the Cartesian coordinate system. In vector potential-vorticity variables, the system of equations (2.1) has the form:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + (\bar{V} \cdot \nabla)\rho = 0,\\
&\frac{\partial \omega}{\partial t} + (\bar{V} \cdot \nabla)\bar{\omega} = (\bar{\omega} \cdot \nabla)\bar{V} + \frac{1}{\mathrm{Re}}\Delta\bar{\omega} + \frac{1}{Fr^2}\mathrm{rot}\ \bar{F},\\
&\nabla(\mathrm{div}\bar{\psi}) - \Delta\bar{\psi} = \bar{\omega},\\
&\bar{V} = \mathrm{rot}\ \bar{\psi} + \nabla\varphi,\\
&\Delta\varphi = 0,
\end{aligned} \tag{2.2}$$

where φ is the scalar potential, ψ is the vector potential, $\bar{\omega}$ is the vorticity , $\bar{\omega} = \mathrm{rot}\ \bar{V}$,

$$\mathrm{rot}\ \bar{V} = \nabla x \bar{V} = \begin{vmatrix} i & k & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix},$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In the two-dimensional case, if the flow occurs in the plane $(x, y), \bar{V} = (u, v, 0)$, and $\varphi \equiv 0$, the vector potential and the vorticity have one component only: $\bar{\psi} = (0, 0, \psi), \bar{\omega} = (0, 0, \omega)$. Then the system of equations in variables of the stream function ψ and vorticity ω for two-dimensional flows has the form:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + (\bar{V} \cdot \nabla)\rho = 0, \\ &\frac{\partial \omega}{\partial t} + (\bar{V} \cdot \nabla)\omega = \frac{1}{Fr^2}\frac{\partial \rho}{\partial x} + \frac{1}{\mathrm{Re}}\Delta\omega, \\ &\Delta\psi = -\omega, \\ &u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \end{aligned} \tag{2.3}$$

Numerical methods for solving two-dimensional problems are available for equations written in variables of the stream function-vorticity [5,6,7], whose common drawback is the use of this or another form of a boundary condition for the vorticity on a solid surface, being absent from a physical statement of the problem. In solving three-dimensional problems, the Navier-Stokes equations are mainly used in the primitive variables velocity-pressure. However, there are works [30, 31, 48, 49], in which for a solution of three-dimensional problems use is made of equations in vector potential-vorticity variables. In [30, 31, 48] problems of convection are considered in closed volumes; in [48] three-dimensional leakage problems in rectangular regions are studied.

2.2. Numerical simulation of two-dimensional flows of stratified fluid

Consider the problem of the flow of a viscous incompressible fluid with non-homogeneous density in a two-dimensional region with the boundary Γ (we restrict ourseleves to studies of flows in a rectangular region, shown in Fig.1). In the general case of a non-rectangular region, there exists a non-singular sufficiently smooth transformation of the coordinates, mapping the studied area into a rectangle for solving equations (2.3), written in new coordinates. One can use a numerical algorithm constructed for a rectangular region. At the location Γ_1 the fluid flows in; at the location Γ_2 it flows out. Γ_3 is the rigid part of the boundary. With the assumption that surface waves are small and do not affect the outflow pattern, the free surface can be substituted by a rigid "lid". The boundary conditions are : at Γ_1 are assigned density values, stream and vorticity functions; at Γ_2 – values of the stream function, for the vorticity and density "soft" boundary conditions $\frac{\partial f}{\partial n} = 0$ (n is the normal to the boundary) are set; at Γ_3 no-slip conditions $\psi = \text{const}$, $\frac{\partial \psi}{\partial n} = 0$ are set. Boundary conditions for the vorticity on rigid walls are defined from the no-slip condition by the relationship $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. At the initial moment, a density distribution by the depth $\rho(0, x, y) = \rho^0(y)$ is known, and the fluid is considered at rest: $-u = 0, v = 0$. To approximate differential equations by the difference ones, a spatial-temporal grid (t_n, x_i, y_j) is introduced: $t_{n+1} = t_n + \Delta t, x_{i+1} = x_i + \Delta x_i, y_{j+1} = y_j + \Delta y_j$, where Δt is the time step, and $\Delta x_i, \Delta y_j$ are the sizes of grid steps on spatial variables. Let the fields of the vorticity

ω^n, stream function ψ^n and density ρ^n be known at some moment of the time $t_n = n \cdot \Delta t$. To define the functions sought for at the temporal layer t_{n+1}, apply an explicit scheme for the splitting method by physical processes [6], [47, 48]. In the first stage, the diffusion equation for ω is solved:

$$\frac{\omega^* - \omega^n}{\Delta t} = \frac{1}{\text{Re}} \Delta \omega^n; \tag{2.4}$$

in the second stage, transport equations for ρ and ω are solved:

$$\begin{aligned} &\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{\partial u\rho}{\partial x} + \frac{\partial v\rho}{\partial y} = 0, \\ &\frac{\omega^{**} - \omega^*}{\Delta t} + \frac{\partial u\rho}{\partial x} + \frac{\partial v\omega}{\partial y} = 0; \end{aligned} \tag{2.5}$$

in the third stage, final values of the vorticity and the stream function are defined:

$$\begin{aligned} &\frac{\omega^{n+1} - \omega^{**}}{\Delta t} = \frac{1}{Fr^2} \frac{\partial \rho^{n+1}}{\partial x}, \\ &\Delta \psi^{n+1} = -\omega^{n+1}. \end{aligned} \tag{2.6}$$

Consider a finite-difference scheme using a hybrid grid (Fig.2). Values of the stream function ψ are determined at the mesh points, values of the density ρ and the vorticity ω at the centres of cells, values of the horizontal component of the velocity vector in the middles of lateral faces, and values of a vertical velocity component are determined in the middles of upper and lower faces. With such an approximation of velocities, a numerical analog of an incompressibility equation is satisfied identically. Further, we restrict ourselves to the grid uniform along the y axis and non-uniform along the x axis. A numerical analog of equation (2.4) has the form:

$$\begin{aligned} \omega^*_{i,j} &= \omega^n_{i,j} + \frac{\Delta t}{\text{Re}} 2 \frac{\Delta x_{i-1}(\omega^n_{i+1,j} - \omega^n_{i,j}) - \Delta x_i(\omega^n_{i,j} - \omega^n_{i-1,j})}{\Delta x_i \cdot \Delta x_{i-1}(\Delta x_i + \Delta x_{i-1})} \\ &+ \frac{\omega^n_{i,j+1} - 2\omega^n_{i,j} + \omega^n_{i,j-1}}{\Delta y^2}, \end{aligned} \tag{2.7}$$

with the stability condition

$$\Delta t \text{ dif} \le \frac{\text{Re } \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} \quad \text{at } \Delta x_i = \Delta x. \tag{2.8}$$

In the near boundary cells in approximating the derivatives $\frac{\partial^2 \omega}{\partial x^2}$ and $\frac{\partial^2 \omega}{\partial y^2}$, use is made of the vorticity values at the corresponding faces coinciding with the solid boundary $(2, j)$ (Fig. 2). A difference scheme for a solution of equations (2.5) is constructed on the basis of the integral conservation law. Applying for integrals an approximation of first order accuracy, we obtain in the uniform x grid a scheme with the upwind differences

$$f_{i,j}^{n+1} = f_{i,j}^n - (uF)_{i+1/2,j}^n + (uF)_{i-1/2,j}^n - (vF)_{i,j+1/2}^n + (vF)_{i,j-1/2}^n,$$

$$(uF)_{i+1/2,j}^n = \begin{cases} u_{i+1/2,j} f_{i,j} \dfrac{\Delta t}{\Delta x}, & u_{i+1/2,j}^n > 0, \\ u_{i+1/2,j}^n f_{i+1,j}^n \dfrac{\Delta t}{\Delta x}, & u_{i+1/2,j}^n \leq 0, \end{cases}$$

$$(vF)_{i,j+1/2}^n = \begin{cases} v_{i,j+1/2}^n f_{i,j}^n \dfrac{\Delta t}{\Delta y}, & v_{i,j+1/2}^n > 0, \\ v_{i,j+1/2}^n f_{i,j+1}^n \dfrac{\Delta t}{\Delta y}, & v_{i,j+1/2}^n \leq 0, \end{cases} \tag{2.10}$$

$$f^n = \begin{pmatrix} \rho^n \\ \omega^* \end{pmatrix}, \quad f^{n+1} = \begin{pmatrix} \rho^{n+1} \\ \omega^{**} \end{pmatrix}.$$

The present explicit scheme is conservative and approximates equation (2.5) with first order for all variables. A study of stability, conducted for a linear equation by the Fourier harmonic method and by the differential approximation method [42], has shown that at

$$\Delta t_{tr} \leq \left(\frac{|u|_{\max}}{\Delta x} + \frac{|v|_{\max}}{\Delta y}\right)^{-1} \tag{2.11}$$

the scheme is stable and also monotonic. In the concluding stage from the solution of equations (2.6), ω^{n+1}, ψ^{n+1} are defined:

$$\omega_{i,j}^{n+1} = \omega_{i,j}^{**} + \frac{\Delta t}{Fr^2} \frac{\Delta x_{i-1}^2(\rho_{i+1,j}^{n+1} - \rho_{i,j}^{n+1}) + \Delta x_i^2(\rho_{i,j}^{n+1} - \rho_{i-1,j}^{n+1})}{\Delta x_i \Delta x_{i-1}(\Delta x_i + \Delta x_{i-1})} \tag{2.12}$$

and ψ^{n+1} is found by the successive over-relaxation method [9]:

$$\begin{aligned} \psi_{i,j}^{\nu+1} &= \psi_{i,j}^{\nu} + \frac{K}{2(1 + \Delta y^2/\Delta x_i/\Delta x_{i-1})} \frac{2\Delta y^2}{\Delta x_i(\Delta x_i + \Delta x_{i-1})} \psi_{i+1,j}^{\nu} \\ &+ \frac{2\Delta y^2}{\Delta x_{i-1}(\Delta x_i + \Delta x_{i-1})} \psi_{i-1,i}^{\nu+1} + \psi_{i,j+1}^{\nu} + \psi_{i,j-1}^{\nu+1} \\ &-2\left(1 + \frac{\Delta y^2}{\Delta x_i \cdot \Delta x_{i-1}}\right) \psi_{i,j}^{\nu} + \Delta y^2 \cdot \omega_{i,j}. \end{aligned} \tag{2.13}$$

Here K is the relaxation parameter, $1 \leq K \leq 2$. An optimal value of the parameter K for solving the Dirichlet problem in a rectangular domain with the dimension $(I \times J)\Delta y$ is determined by the relationship [5, 73]

$$K_{\text{opt}} = 2/(1 + \sqrt{1 - g^2}), \tag{2.14}$$

$$g = \frac{1}{2}\Big[\cos(\pi/I) + \cos(\pi/J)\Big]. \tag{2.15}$$

For large values of I anf J,

$$K_{\text{opt}} \approx \frac{2}{1 + \pi\sqrt{(1/I^2 + 1/J^2)/2}}. \tag{2.16}$$

The time step for the whole algorithm is estimated from the condition

$$\Delta t \leq \min(\Delta t_{\text{dif}}, \Delta t_{\text{tr}}), \tag{2.17}$$

where Δt_{dif} is the time step for the diffusion equation, and Δt_{tr} is the time step for the transport equation.

Boundary conditions for the vorticity ω are determined at the cell face adjoining the wall. A unmerical analog of the boundary condition for the vorticity is obtained from the relationship $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$. For instance, on the vertical wall ($i = 3/2$, Fig.2) the condition $\dfrac{\partial u}{\partial y} = 0$ is satisfied, since at the boundary $u = 0$. Therefore, $\omega = \dfrac{\partial v}{\partial x}$. Since the values of v are known at all horizontal cell faces adjoining the boundary, and $v = 0$ at the boundary, to define ω the following relationships are obtained

$$\omega_{3/2,j} = \frac{v_{2,j+1/2} + v_{2,j-1/2}}{x} + O(\Delta x), \tag{2.18}$$

or

$$\omega_{3/2,j} = \frac{9(v_{2,j+1/2} + v_{2,j-1/2}) - (v_{3,j+1/2} + v_{3,j-1/2})}{6 \cdot \Delta x}. \tag{2.19}$$

Analogous expressions are obtained for other boundaries. Approximating the rest of the boundary conditions does not cause major difficulties. Numerical simulation of an incompressible fluid flow in the variables of the atream function-vorticity does not require the overcoming of difficulties arising in approximating boundary conditions for the vorticity. There exist different mays of solving this problem [5,7], associated with a representation of the vorticity through the stream function and using an approximation of the no-slip condition $\dfrac{\partial \psi}{\partial n} = 0$. The above approach with the use of a grid with staggered velocities possesses the following properties:

a) since the boundary value of the vorticity ω is determined by the known velocity projections, there is no need to require the condition $\dfrac{\partial \psi}{\partial n} = 0$ to be fulfilled which, with the difference approximation, leads to a zero flow rate between the solid boundary and the nearest point of the grid, or imposes rigid conditions on the profile of a boundary layer in the near boundary cells, if known algorithms are used [7];

b) in the works [7, 23] it is noted that the use of approximate boundary conditions for the vorticity can lead to a decrease of the maximum time step as compared to the time step obtained from a stability analysis of the linear equations.

For stationary two-dimensional flows of a heavy stratified non-viscous fluid, one can construct an analytical solution [47, 51] which can be used for a qualitative estimation of the stratified fluid flow pattern, and also as a test for numerical solutions.

2.3. Numerical algorithm for calculation of three-dimensional stratified flows

Consider the Navier-Stokes equations written in vector-potential vorticity variables. Note some properties of the equation for the vector potential $\bar{\psi}$ from (2.2):

1) for the existence of the solution it is necessary for the condition div $\bar{\omega} = 0$ to be satisfied;

2) without violating generality, one can estimate that div $\bar{\psi} = 0$ [59]. Therefore, instead of the third equation in (2.2) one can consider the equation

$$\Delta \bar{\psi} = -\bar{\omega}, \tag{2.20}$$

provided div $\bar{\psi} = 0$. If one requires that at the boundary (the surface of Γ is supposed to be piecewise smooth) of the region R div $\bar{\psi}|_{\Gamma} = 0$, a sufficiently smooth solution of equation (2.20) with the right side satisfying the condition div $\bar{\omega} = 0$ will satisfy the equation div $\bar{\omega} = 0$ in R. Indeed, having applied to equation (2.20) the operation div, we obtain the Laplace equation for div $\bar{\psi}$ with a zero condition at the border. In the sequel, instead of the third equation in (2.2) we shall use equation (2.20) subject to the properties 1) and 2). Note that equation (2.20) has a solution with div $\bar{\omega} \neq 0$, yet this solution will not satisfy the third equaiton in (2.20).

Set boundary conditions for the flow region in the form of a rectangle $[0 \leq x \leq X, 0 \leq z \leq Z]$ with the boundary Γ (Fig. 3). In the flow problem the following types of boundaries are possible: input and output jet boundaries, and solid surface (the free surface is substituted by a "solid lid"). At every section of the boundary the normal component of the velocity vector $(\bar{V} \cdot \bar{n})|_{\Gamma} = \gamma(\xi)$, where $\xi \in \Gamma$, and $\bar{n}$ is the normal to the boundary. From this condition and the relationship div $\bar{\psi}|_{\Gamma} = 0$, one finds conditions at Γ for scalar and vector potentials:

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma} = \gamma(\xi), \quad \xi \in \Gamma. \tag{2.21}$$

$$\psi_{\tau_1}\Big|_{\Gamma} = \psi_{\tau_2}\Big|_{\Gamma} = 0, \quad \left.\frac{\partial\psi_n}{\partial n}\right|_{\Gamma} = 0, \tag{2.22}$$

where ψ_n is the normal component of the vector potential, and $\psi_{\tau_1}, \psi_{\tau_2}$ are the tangent components.

To approximate equations (2.2), use is made of a grid with staggered velocities (Fig.4). Scalar potential, density and pressure are determined at the cell centres, vorticity vector and vector potential projections are determined at the middle of corresponding edges, and velocity projections are determined at the centre of faces. Vector velocity components are approximated by relationships of the form:

$$\begin{aligned} u_{i+1/2,j,k} = {} & \frac{(\psi_z)_{i+1/2,j+1/2,k} - (\psi_z)_{i+1/2,j-1/2,k}}{\Delta y} \\ & - \frac{(\psi_y)_{i+1/2,j,k+1/2} - (\psi_y)_{i+1/2,j,k-1/2}}{\Delta z} + \frac{\varphi_{i+1,j,k} - \varphi_{i,j,k}}{\Delta x}. \end{aligned} \tag{2.23}$$

Analogous expressions are used for v and w. If the fourth equation in (2.2) is solved exactly, in using approximations of (2.23) to form a unmerical analog of the equation, div $\bar{V} = 0$ is satisfied identically. It has been noted above that if instead of the third equations in (2.2), equation (2.20) is solved, then it is necessary at each step on time to ensure satisfaction of the condition div $\bar{\omega}^n \equiv 0$. Yet, if the second equation in (2.2) is approximated directly, then this condition is not identically fulfilled. As a result, the solution of equation (2.20) does not satisfy the third equation in (2.2).

Consider an algorithm for solving system (2.2), (2.20), which provides the vorticity vector solenoidality at every step on time, does not require setting boundary conditions for the vorticity at solid surfaces and consists of three stages:

In the first stage, the intermediate field of the velocity vector

$$\frac{\bar{V}^* - \bar{V}^n}{\Delta t} = -(\bar{V}^n \cdot \nabla)\bar{V}^n + \frac{1}{\mathrm{Re}}\Delta\bar{V}^n + \frac{1}{Fr^2}F \tag{2.24}$$

and final density values

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = -(\bar{V} \cdot \nabla)\rho. \quad (2.25)$$

are defined. The intermediate velocity field $\bar{V}^*$ has a quite definite physical meaning [6]. If we apply the operator rot to the second equation in (2.2) and to equation (2.24) and take account of the fact that rot $(\nabla p) \equiv 0$, we obtain $\bar{\omega}^{n+1} =$ rot $\bar{V}^{n+1} =$ rot $\bar{V}^*$. Therefore, even in the first stage the intermediate velocity field determines the correct vorticity characteristics for the flow region under study.

In the second stage, from the equation $\bar{\omega}^{n+1} =$ rot $\bar{V}^*$ the vorticity is found by a relationship of the form

$$(\omega_x)_{i,j+1/2,k+1/2} = \frac{W_{i,j+1,k+1/2} - W_{i,j,k+1/2}}{\Delta y} - \frac{V_{i,j+1/2,k+1} - V_{i,j+1/2,k}}{\Delta z}. \quad (2.26)$$

For the vorticity, a numerical analog of the condition div $\bar{\omega} = 0$ is identically satisfied at all points within the region. Since the vorticity is defined through the velocities, no boundary conditions for it are required at no-slip boundaries.

In the third stage, vector potential components are found from the solution of a mixed boundary problem (2.20), (2.22). The scalar potential is defined from the solution of Neuman's problem (the fourth equation in (2.2) and condition (2.21)). Note that for stationary boundary conditions it is sufficient to solve this problem once. Finite values of the velocity vector are defined by formulae of the form (2.23).

The pressure field at any step on time can be found from the equation

$$\Delta p^n = Fr^2 \cdot \operatorname{div} \frac{1}{\mathrm{Re}} \Delta \bar{V}^n + \frac{1}{Fr^2} \bar{F}^n - (\bar{V}^n \cdot \nabla)\bar{V}. \quad (2.27)$$

Solutions of discrete analogs of elliptical equations for vector potential components, scalar potential and pressure with corresponding boundary conditions are obtained by the over-relaxation method. To solve equations (2.24), (2.25), use is made of explicit difference schemes of (2.7), (2.10) type. The present algorithm is naturally adapted for parallel calculations.

§3. Numerical Results

Stratified fluid flows in rectangular reservoirs are studied for different values of Reynolds and Froude numbers using the described numerical algorithm.

For reservoirs whose length is much more than the depth, a quasi-uniform grid was applied. Computations showed that the main parameter determining the flow pattern is the density Froude number that characterizes the relation of inertia forces to bouyancy forces. For laminar flows viscosity has a considerably less influence. Numerical experiments allowed one to determine critical values of the Froude number $Frcr$. With values $Fr \leq Frcr$ for the case of a surface position of the discharge aperture the flow splits into two characteristic regions: the upper one involves the flow into the discharge aperture; the lower one consists of low velocities circulating in the bottom region of the reservoir (Fig. 5). With further decrease of the Froude number, the thickness of the fluid layer associated with the intake aperture decreases. As a test, use was made of an analytical solution of a steady-state problem for a case of a linear density dependent on the stream function [47,51]. A non-homogeneous fluid flow pattern depends on the nature of the density (temperature) stratification. In the presence of a strongly pronounced thermo-clyne, separation of fluid arises in the region of the largest density gradient. If the temperature of water smoothly changes with depth, the thickness of the layer associated with the aperture is continuously decreasing with the decreasing Froude number. The same thickness of the fluid layer flowing into the aperture in these two cases was observed at different values of the Froude number. For instance, for temperature distributions with depth, depicted in Fig. 5, the thickness of the intaken layer is half the depth at $Fr = 0.2$ for the case "1" and at $Fr = 0.26$ for the curve "2". Computations for a problem on a homogeneous viscous fluid flow in the cavity with a moving upper boundary have been made. The pattern of stream lines with an increasing number of grid points changed little. The value $\psi_{\max}$ changed appreciably.

With the given algorithm, calculations of spatial flows of a nonviscous stratified fluid in the reservoir of a rectangular form (Fig. 3) have been carried out. A case with the position of the water intake near the surface has been considered. At the initial moment, a distribution of the density $\rho(0, x, y, z) = \rho_0(y)$ is assigned; the fluid is considered at rest. Influence of values of the Froude density number, form and position of the water intake aperture on the character of the flow in the reservoir have been studied. For finite values of the Froude number according to the character of fluid involvement in the aperture flows can be divided into two types:

a) at $Fr > 0.35$ into the discharge aperture, fluid is involved from all the layers with depth but at the bottom region velocity is less the closer the value of the Froude number is to 0.35;

b) at $Fr < 0.3$ fluid separates into two regions; the upper one is involved into a discharge aperture; the lower one circulates with low velocities. The flow pattern is also affected by the dimensions and position of the intake aperture. If an aperture of a rectangular form extends throughout the width of the reservoir, then (in a non-viscous case) a two-dimensional flow occurs. When the aperture is considerably narrower than the reservoir width and is positioned symmetrically to the plane $z = 1/2$, then at $Fr < 0.3$ the flow separation in the vicinity of this plane is more singular than at the periphery. At the lateral walls thickness of the boundary layer is bigger but flow velocities are less. At the bottom part of reservoir, a non-stationary circulation flow of a complex structure is formed.

The author expresses his gratitude to V. M. Belolipetsky and V. Yu. Kostyuk for their assistance in preparing the present paper.

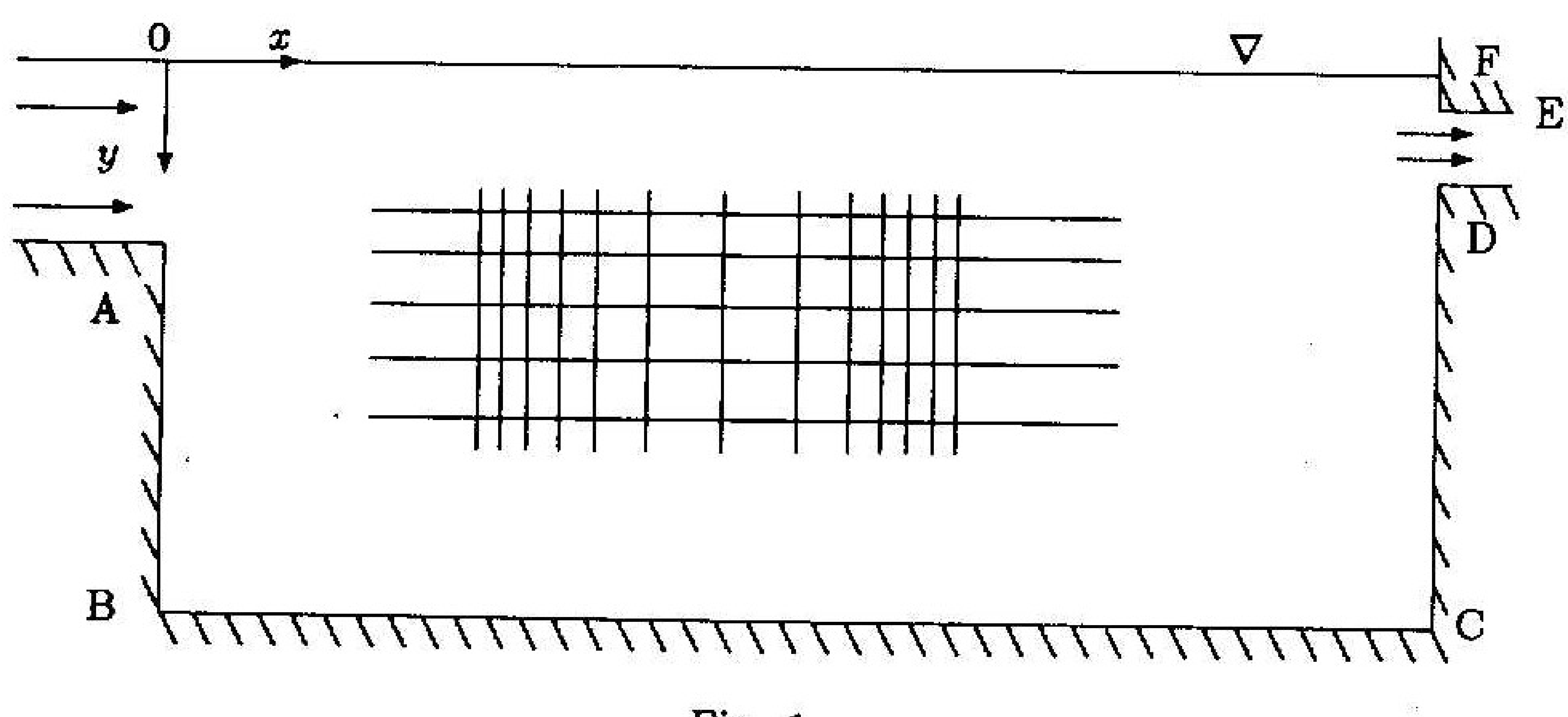

Fig. 1

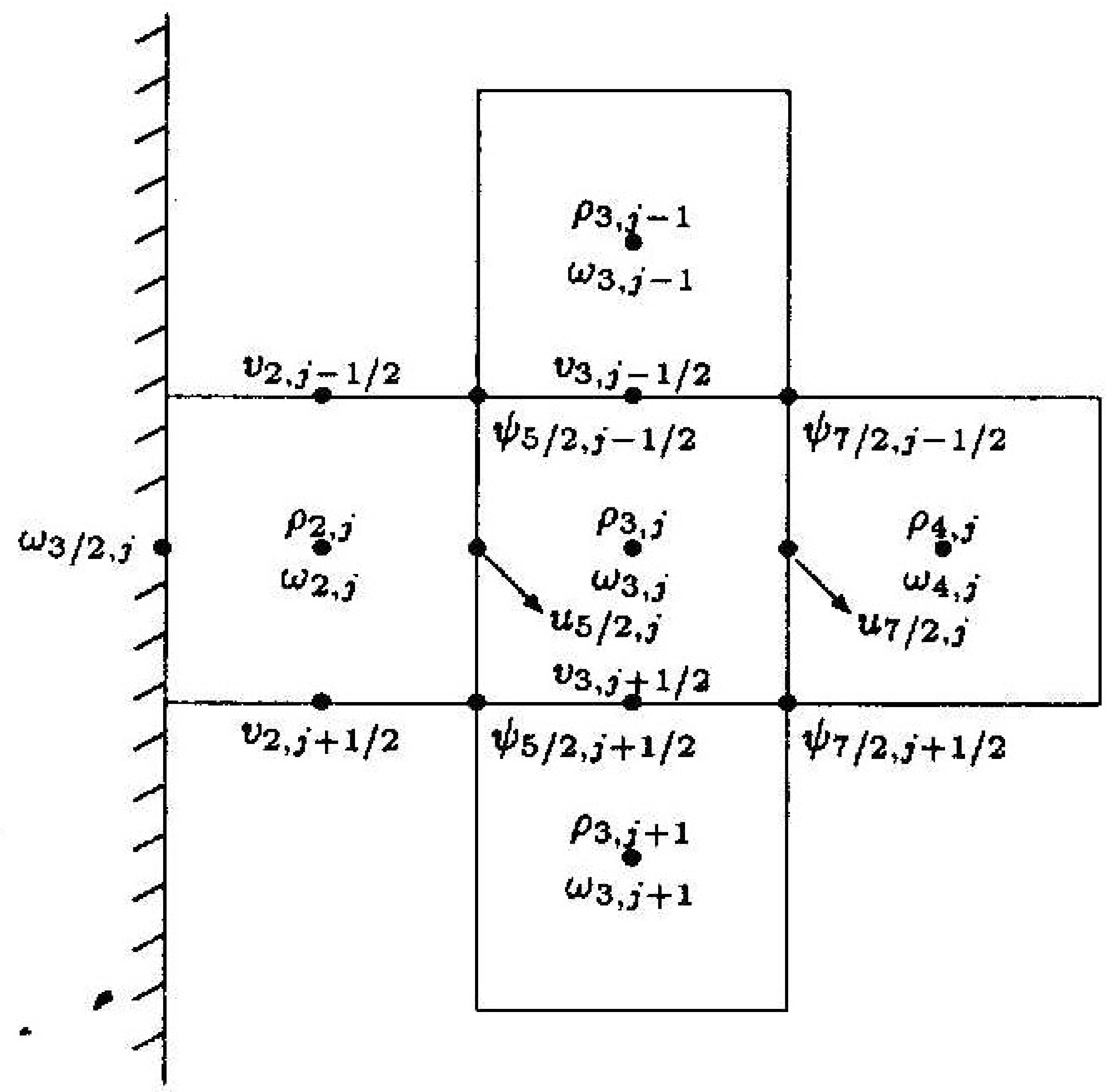

Fig. 2

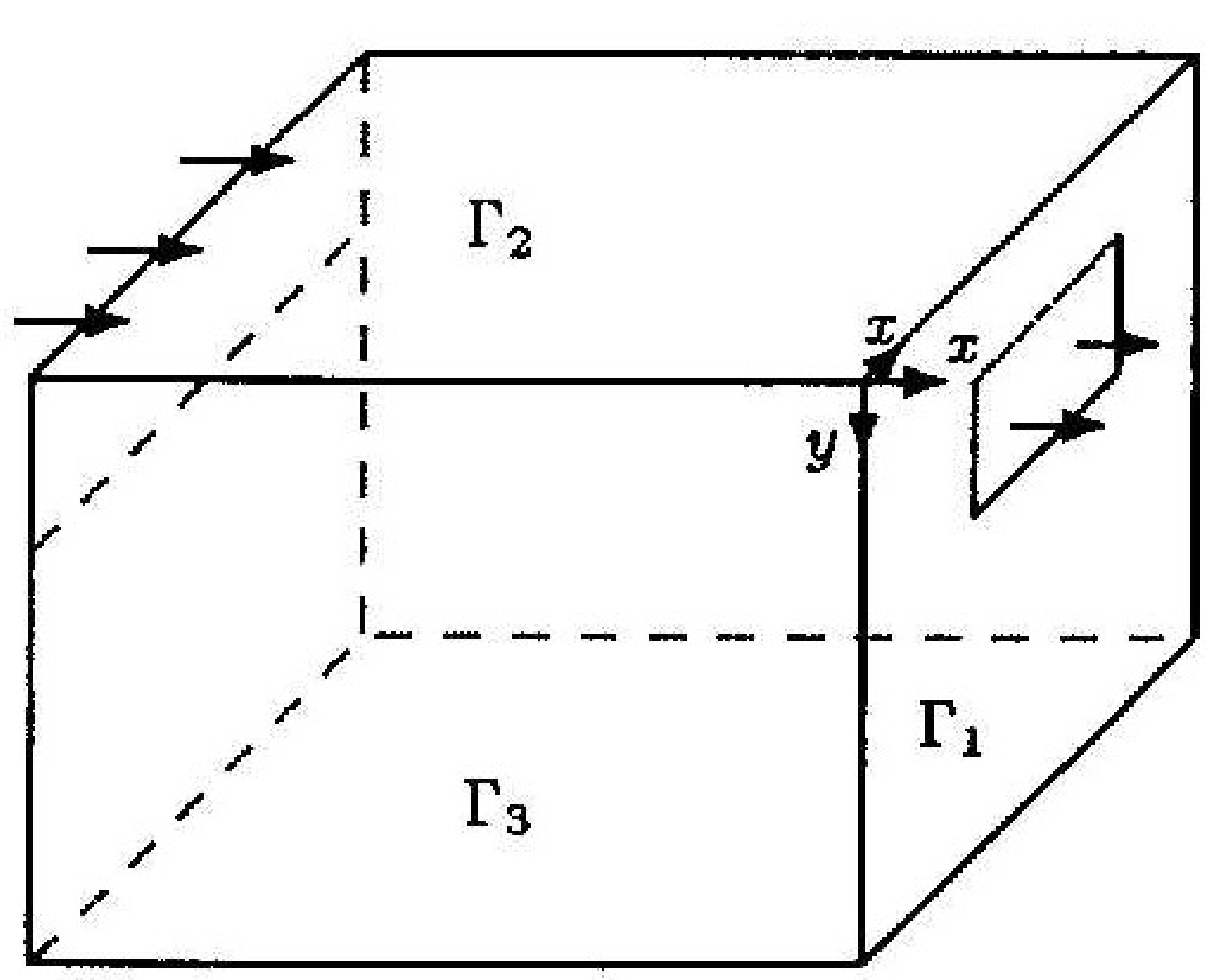

Fig. 3

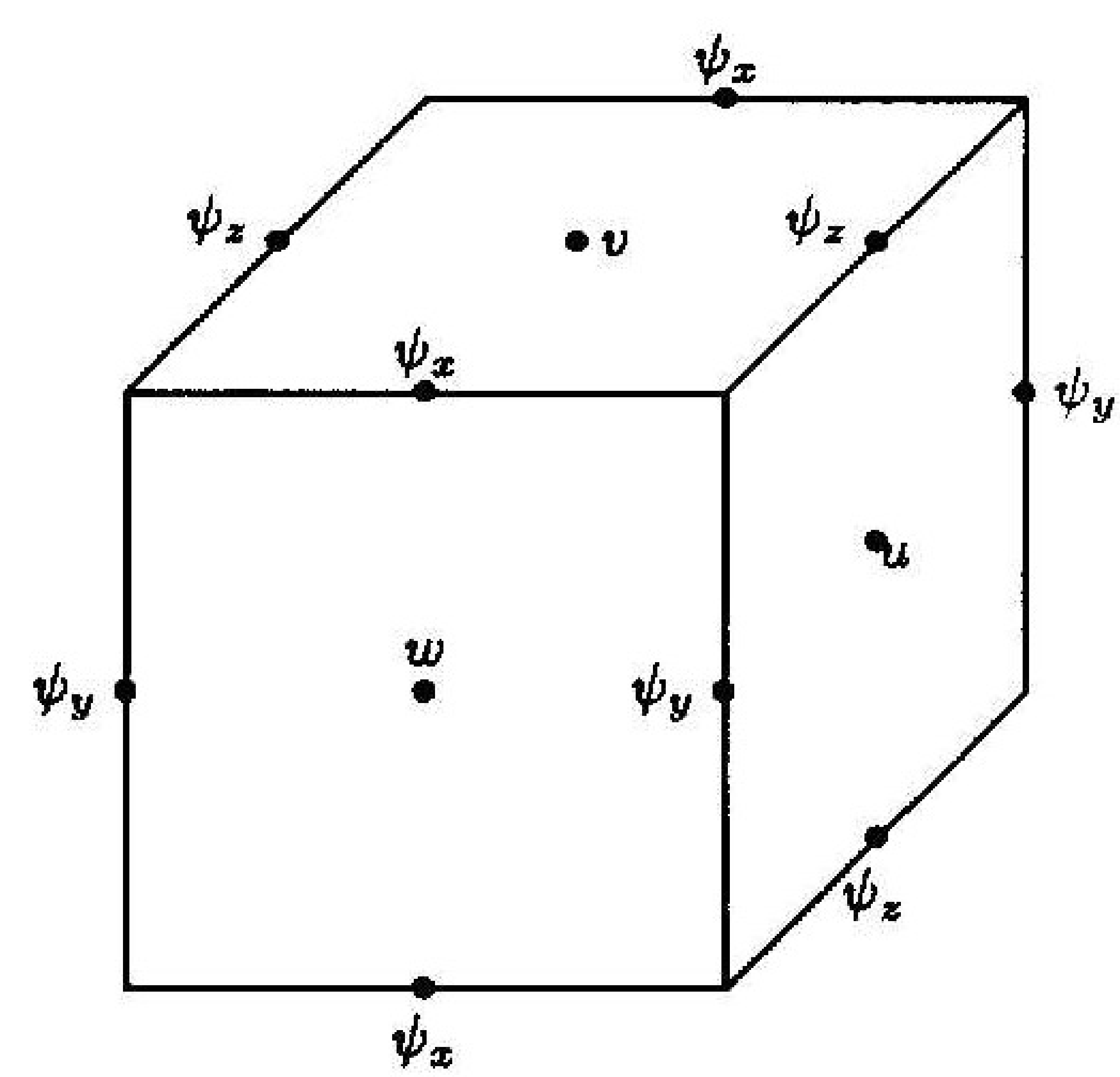

Fig. 4

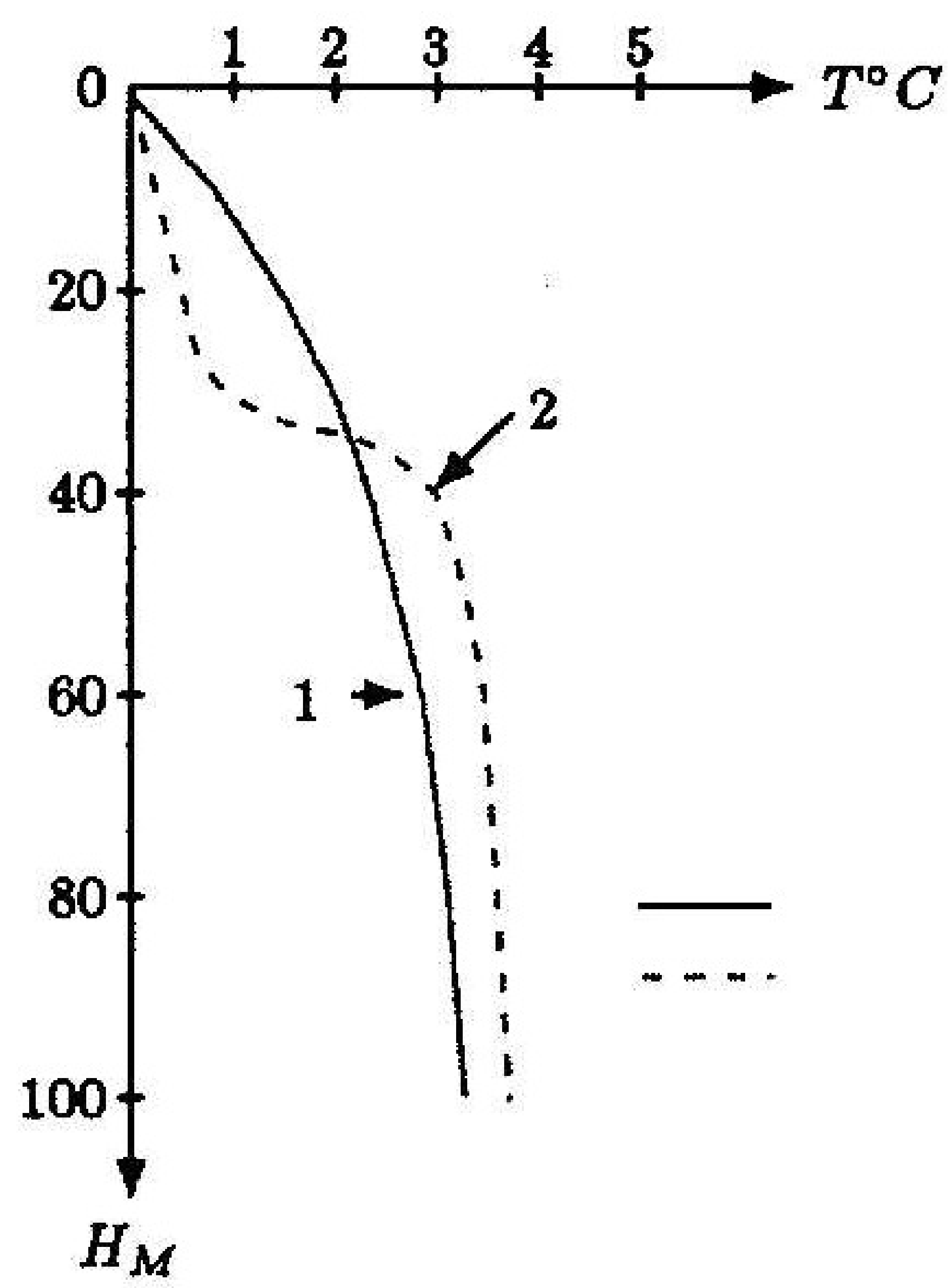

Fig. 5

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