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A NONCONFORMING FINITE ELEMENT METHOD OF STREAMLINE DIFFUSION TYPE FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS *

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Abstract

A nonconforming finite element method of streamline diffusion type for solving the stationary and incompressible Navier-Stokes equations is considered. Velocity field and pressure field are approximated by piecewise linear and piecewise constant functions, respectively. The existence of solutions of the discrete problem and the strong convergence of a subsequence of discrete solutions are established. Error estimates are presented for the uniqueness case.

§1. Introduction

Finite element schemes of streamline diffusion type are nowadays a common procedure for solving convection dominated problems in fluid mechanics such as transport problems[5],[9],[1], the Euler equations and Navier-Stokes equations with small viscosity for incompressible[5],[10] or compressible flow[6],[11].

In this note the incompressible stationary Navier-Stokes equations are addressed. Recently, Johnson and Saranen[10] considered streamline diffusion methods for the time-dependent case where discrete velocity fields are employed which are assumed to be exactly divergence free. In the stationary case, we consider a nonconforming FEM based on piecewise linear and piecewise constant approximations for the velocity and the pressure fields, respectively, satisfying the discrete LBB-condition and thus circumventing exact divergence free discrete velocity fields. For other approaches of upwind type concerning the incompressible Navier-Stokes equation, see e.g. [3] or [13].

The plan of the paper is the following. In Section 2 we give some notations. The FEM is presented in Section 3. Existence and uniqueness results for the discrete problems are given in Section 4. Convergence properties of the method are studied in Sections 5 and 6.

§2. Notations

Let $\Omega \subset R^N (N = 2,3)$ be a convex polygon or polyhedron with boundary $\Gamma = \partial\Omega$ and let $\nu = 1/\mathrm{Re} > 0$. We consider the incompressible Navier-Stokes equation:

Find (u,p) such that

$$\begin{cases} -\nu\Delta u + u\nabla u + \nabla p &= f \quad \text{in} \quad \Omega, \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

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The spaces for velocity and pressure are defined by

$$V = H_0^1(\Omega)^N, Q = \left\{q \in L^2(\Omega); \int_\Omega q dx = 0\right\},$$

$$W = \{v \in V; \nabla \cdot v = 0\}.$$

For vector-valued functions $u = (u_1, \cdots, u_N) \in W^{k,p}(\Omega)^N$, $v = (v_1, \cdots, v_N)$ belonging to $L^\infty(\Omega)^N$ we use the usual norms and seminorms, respectively,

$$\|u\|_{k,p}^p = \sum_{i=1}^N \|u_i\|_{k,p}^p, \quad |u|_{k,p}^p = \sum_{i=1}^N |u_i|_{k,p}^p,$$

$$\|v\|_{0,\infty} = \max_i \|v_i\|_{0,\infty}.$$

Furthermore, we introduce

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad \forall u, v \in V,$$

$$b(u,v,w) = \int_\Omega (u \cdot \nabla) v w dx, \qquad \forall u, v, w \in V,$$

$$\bar{b}(u,v,w) = \frac{1}{2}\{b(u,v,w) - b(u,w,v)\}, \quad \forall u, v, w \in V$$

satisfying

$$b(u,v,w) = \bar{b}(u,v,w), \quad \forall u \in W, v, w \in V.$$

The weak velocity-pressure formulation of (2.1) reads now;

$$\begin{cases} \text{Find } (u,p) \in V \times Q \text{ such that} \\ \nu a(u,v) + \bar{b}(u,u,v) - (p, \nabla \cdot v) = (f,v), \quad \forall v \in V, \\ (q, \nabla \cdot u) = 0, \quad \forall q \in Q. \end{cases} \tag{2.2}$$

From (2.2) we obtain the weak velocity formulation

$$\begin{cases} \text{Find } u \in W \text{ such that} \\ \nu a(u,v) + \bar{b}(u,u,v) = (f,v), \quad \forall v \in W. \end{cases} \tag{2.3}$$

It is well-known that (2.2) admits at least one solution and that this solution is unique provided that $\nu^{-2}\|f\|_{0,2}$ is sufficiently small.

§3. A Nonconforming Streamline Diffusion Finite Element Method

We consider a nonconforming finite element approximation due to Crouzeix and Raviart[2], and Temam[14]. Let $(\mathcal{J}_h)_h$ be a regular family of triangulations of Ω into N-simplices K_j

with $\bar{\Omega} = \bigcup_j \bar{K}_j$. The faces of K_j are denoted by S_{ij}, and the barycentres of S_{ij} by B_{ij}. Moreover, we shall need the inverse assumption on the family of triangulations

$$\frac{h}{h_j} \leq C, \quad \forall K_j \in \bigcup_h \mathcal{J}_h$$

where $h_j = \text{diam} K_j$ and $h = \max_j h_j$.

The finite element spaces for velocity and pressure are defined by

$$\begin{aligned} V_h &= \{v \in L^2(\Omega)^N; v|K_j \in P_1(K_j)^N, \quad \forall K_j \in \mathcal{J}_h, \\ &\qquad v \text{ continuous in } B_{ij}, v(B_{ij}) = 0 \text{ if } B_{ij} \in \Gamma\}, \\ Q &= \{q \in L_0^2(\Omega) \quad ; q|K_j \in P_0(K_j), \quad \forall K_j \in \mathcal{J}_h\}. \end{aligned}$$

Because of $V_h \not\subset V$ we have to extend the divergence operator and the forms $a(\cdot,\cdot), b(\cdot,\cdot), \bar{b}(\cdot,\cdot)$. This can be done by an elementwise calculation of the integrals over.

$$\begin{aligned} (q, \nabla_h \cdot v) &= \sum_j \int_{K_j} q(\nabla \cdot v) dx, & \forall v \in V_h, q \in Q_h, \\ a_h(u,v) &= \sum_j \int_{K_j} \nabla u \cdot \nabla v dx, & \forall u, v \in V + V_h, \\ b_h(u,v,w) &= \sum_j \int_{K_j} (u \cdot \nabla) v w dx, & \forall u, v, w \in V + V_h, \\ \bar{b}_h(u,v,w) &= \frac{1}{2}\{b_h(u,v,w) - b_h(u,w,v)\}, & \forall u, v, w \in V + V_h. \end{aligned}$$

Further, let

$$W_h = \{v \in V_h; \nabla_h \cdot v_{|K_j} = 0 \quad \forall K_j \in \mathcal{J}_h\}$$

be the space of discrete divergence free functions and

$$\|v\|_h = [a_h(v,v)]^{1/2}$$

the norm on V_h and W_h, respectively.

Following[10] we use test functions of the form $v + \delta_j(u_h \cdot \nabla v + \nabla q)$ giving extra control over gradients in the streamline direction. Note that in our method $\nabla q_{|K_j} = 0$ and $\Delta v_{|K_j} = 0$, $\forall K_j \in \mathcal{J}_h$. The streamline diffusion method then reads;

$$\begin{cases} \text{Find } (u_h, p_h) \in V_h \times Q_h \quad \text{such that} \\ B_\delta(u_h, u_h; u_h, v) - (p_h, \nabla_h \cdot v) = L_\delta(u_h; v), & \forall v \in V_h, \\ (q, \nabla_h \cdot u_h) = 0, & \forall q \in Q_h. \end{cases} \tag{3.1}$$

In (3.1) we used following notations

$$B_\delta(u, u_h; v, w) = \nu a_h(v,w) + \bar{b}_h(u,v,w) + \sum_j \delta_j(u \cdot \nabla v, u_h \cdot \nabla w)_{K_j}, \tag{3.2}$$

$$L_\delta(u_h; w) = (f, w) + \sum_j \delta_j(f, u_h \cdot \nabla w)_{K_j} \tag{3.3}$$

with the L^2-scalar product $(\cdot,\cdot)_{K_j}$ restricted on K_j. Setting $\delta_j = 0$ in (3.1) we obtain the classical method[2],[14]. In the following we denote by c a general constant independent of h and ν the value of which may be different from row to row, and $\delta = \max_j \delta_j$.

§4. Existence and Uniqueness of Discrete Solutions

Now let us consider the solvability of the discrete problem (3.1). The discrete velocity u_h satisfies

$$\begin{cases} \text{Find } u_h \in W_h \text{ such that} \\ B_\delta(u_h, u_h; u_h, v) = L_\delta(u_h; v), \quad \forall v \in W_h. \end{cases} \tag{4.1}$$

To prove the existence of a solution of (4.1) we shall apply a variant of Brouwer's fixed point theorem[14]. Taking into consideration that our finite element spaces fulfil the discrete LBB-condition:

$$\begin{cases} \text{There is a constant } \alpha > 0 \text{ independent of } h \text{ such that} \\ \sup_{v \in V_h} \dfrac{(q, \nabla_h \cdot v)}{\|v\|_h} \geq \alpha \|q\|_{0,2}, \quad \forall q \in Q_h \end{cases} \tag{4.2}$$

we obtain a unique pressure $p_h \in Q_h$ to each solution $u_h \in W_h$ of (4.1) such that (u_h, p_h) is a solution of (3.1).

Theorem 1. *Assume that $f \in L^2(\Omega)^N$ and $0 \leq \delta_j \leq c_1 \nu^{-1}$. Then there exists at least one solution $(u_h, p_h) \in V_h \times Q_h$ of (3.1). Moreover, there is a constant $c_2 > 0$ such that each solution $u_h \in W_h$ of (4.1) satisfies*

$$\nu \|u_h\|_h^2 + \sum_j \delta_j \|u_h \cdot \nabla u_h\|_{0,2,K_j}^2 \leq c_2 \nu^{-1} \|f\|_{0,2}^2. \tag{4.3}$$

Proof. Let $[v, w] = a_h(v, w)$ be the scalar product on W_h and $p; W_h \to W_h$ be the operator defined by

$$[Pv, w] = B_\delta(v, v; v, w) - L_\delta(v; w).$$

Using standard inequalities we have

$$[Pv, v] = \nu a_h(v, v) + \tilde{b}_h(v, v, v) + \sum_j \delta_j (v \cdot \nabla v, v \cdot \nabla v)_{K_j} - (f, v) - \sum_j \delta_j (f, v \cdot \nabla v)_{K_j},$$

$$[Pv, v] \geq \nu \|v\|_h^2 + \sum_j \delta_j \|v \cdot \nabla v\|_{0,2,K_j}^2 - \|f\|_{0,2} \|v\|_{0,2} - \sum_j \delta_j \|f\|_{0,2,K_j} \|v \cdot \nabla v\|_{0,2,K_j}.$$

Because

$$\|f\|_{0,2} \|v\|_{0,2} \leq \frac{\nu}{2} \|v\|_h^2 + \frac{c}{\nu} \|f\|_{0,2}^2$$

and

$$\sum_j \delta_j \|f\|_{0,2,K_j} \|v \cdot \nabla v\|_{0,2,K_j} \leq \frac{1}{2} \sum_j \delta_j \|v \cdot \nabla v\|_{0,2,K_j}^2 + \frac{c_1}{2\nu} \|f\|_{0,2}^2$$

we obtain

$$[Pv, v] \geq \frac{1}{2} (\nu \|v\|_h^2 + \sum_j \delta_j \|v \cdot \nabla v\|_{0,2,K_j}^2 - \frac{c_2}{\nu} \|f\|_{0,2}^2)$$

such that each solution u_h of $Pv = 0$ satisfies (4.3) and $[Pv, v] > 0$ for $\|v\|_h \geq \frac{c_2}{\nu}\|f\|_{0,2}$.

It reamins to show the continuity of P. For $u_1, u_2 \in W_h$ there holds

$$[Pu_1 - Pu_2, v] = \nu a_h(u_1 - u_2, v) + \bar{b}_h(u_1, u_1, v) - \bar{b}_h(u_2, u_2, v)$$
$$+R_1(u_1, u_1, u_1, v) - R_1(u_2, u_2, u_2, v) + R_2(u_1 - u_2, v)$$

where we used the abbreviations

$$R_1(u, v, w, z) = \sum_j \delta_j (u \cdot \nabla v, w \cdot \nabla z)_{K_j}, \tag{4.4}$$

$$R_2(u, v) = \sum_j \delta_j (f, u \cdot \nabla v)_{K_j}. \tag{4.5}$$

By means of the continuity of a_h and $\bar{b}_h$ we can estimate

$$|\nu a_h(u_1 - u_2, v) + \bar{b}_h(u_1, u_1, v) - \bar{b}_h(u_2, u_2, v)| \leq (\nu + c(\|u_1\|_h + \|u_2\|_h))\|u_1 - u_2\|_h \|v\|_h. \tag{4.6}$$

Using an inverse inequality and a discrete version of Sobolev's embedding theorem[4] we conclude the estimate

$$\|u\|_{0,\infty} \leq ch^{-\varkappa}\|u\|_h \quad \forall u \in V_h \tag{4.7}$$

with $\varkappa = \varkappa(N)$, $\varkappa(2) > 0$ arbitrary and $\varkappa(3) = 1/2$.

Now we have

$$\begin{aligned} |R_1(u, v, w, z)| &\leq \|u\|_{0,\infty}\|w\|_{0,\infty} \sum_j \delta_j |v|_{1,2,K_j} |z|_{1,2,K_j} \\ &\leq ch^{-2\varkappa}\|u\|_h\|w\|_h \Big(\sum_j \delta_j |v|_{1,2,K_j}\Big)^{1/2} \Big(\sum_j \delta_j |z|_{1,2,K_j}\Big)^{1/2} \\ &\leq c\delta h^{-2\varkappa}\|u\|_h\|v\|_h\|w\|_h\|z\|_h \end{aligned} \tag{4.8}$$

and

$$|R_2(u, v)| \leq \|u\|_{0,\infty} \sum_j \delta_j \|f\|_{0,2,K_j} |v|_{1,2,K_j} \leq c\delta h^{-2\varkappa}\|f\|_{0,2}\|u\|_h\|v\|_h. \tag{4.9}$$

Consequently, it follows that

$$\begin{aligned} |R_1(u_1, u_1, u_1, v) - R_1(u_2, u_2, u_2, v)| &\leq |R_1(u_1 - u_2, u_1, u_1, v)| + |R_1(u_2, u_1 - u_2, u_1, v)| \\ +|R_1(u_2, u_2, u_1 - u_2, v)| &\leq c\delta h^{-2\varkappa}(\|u_1\|_h^2 + \|u_1\|_h\|u_2\|_h + \|u_2\|_h^2)\|u_1 - u_2\|_h\|v\|_h \end{aligned} \tag{4.10}$$

and

$$|R_2(u_1 - u_2, v)| \leq c\delta h^{-\varkappa}\|f\|_{0,2}\|u_1 - u_2\|_h\|v\|_h. \tag{4.11}$$

Combining (4.6), (4.10) and (4.11) we finally obtain

$$|[Pu_1 - Pu_2, v]| \leq c(\delta, \nu, h, \|u_1\|_h, \|u_2\|_h)\|u_1 - u_2\|_h\|v\|_h$$

i.e. the bounded Lipschitz continuity of P.

Brouwer's fixed point theorem yields the existence of at least one solution $u_h \in W_h$ of $Pu_h = 0$ satisfying (4.3). The existence of a unique $P_h \in Q_h$ corresponding to this solution u_h can be established by means of (4.2) in the usual way.

A uniqueness result is given by

Lemma 1. *Let f belong to $L^2(\Omega)^N$ and δ_j satisfy $0 \le \delta_j \le \nu^{-1}h^{2N}c_1(h)$ with $\lim_{h\to 0} c_1(h) = 0$. Then there are positive constants $\nu_0 = \nu_0(\|f\|_{0,2})$ and h_0 such that for $\nu \ge \nu_0$ and $h \le h_0$ the solution of (4.1) is unique.*

Proof. Let $u_i \in W_h, i = 1, 2$, be two solutions of (4.2) and $u = u_1 - u_2 \in W_h$. Then, it follows that

$$\begin{aligned}\nu\|u\|_h^2 &\le B_\delta(u_1, u_1, ; u, u) = B_\delta(u_1, u_1; u_1, u) - B_\delta(u_1, u_1; u_2, u)\\ &\le L_\delta(u_1; u) - L_\delta(u_2; u) + B_\delta(u_2, u_2; u_2, u) - B_\delta(u_1, u_1; u_2, u)\\ &\le R_2(u, u) + \bar{b}_h(u, u_2, u) + R_1(u_2, u_2, u_2, u) - R_1(u_1, u_1, u_2, u)\end{aligned}$$

with the abbreviations R_1, R_2 defined by (4.4), (4.5). Applying the estimates (4.8), (4.9) and taking into consideration the continuity of $\bar{b}_h$ we conclude

$$\nu\|u\|_h^2 \le c\delta h^{-N}\|f\|_{0,2}\|u\|_h^2 + c\|u_2\|_h\|u\|_h^2 + c\delta h^{-2N}(\|u_2\|_h^2 + \|u_1\|_h\|u_2\|_h)\|u\|_h^2.$$

By means of the a priori estimate stated in Theorem 1 we obtain

$$\nu\|u\|_h^2(1 - c\nu^{-2}\|f\|_{0,2} - c\delta\nu^{-1}h^{-N}\|f\|_{0,2} - c\delta\nu^{-3}h^{-2N}\|f\|_{0,2}^2) \le 0.$$

The term in brackets is positive provided that $\nu_0^2 > c\|f\|_{0,2}$ and h_0 is sufficiently small. Consequently, we have $u = u_1 - u_2 = 0$.

§5. Convergence of the Discrete Solutions

In this section, we study convergence properties of the solutions (u_h, p_h) of (3.1). Let $I_h : V + V_h \to L^2(\Omega)^{N+N^2}$ be the embedding operator defined on each element K_j by

$$(I_h w)(x) = (w(x), (\nabla w)(x)), \quad \forall x \in \overset{\circ}{K}_j$$

satisfying $\|I_h w\|_{0,2} \le c\|w\|_h, \quad \forall w \in V + V_h$.

Theorem 2. *Assume that $0 \le \delta_j \le \nu^{-1}h^{2N}c_1(h)$ with $\min_{h\to 0} c_1(h) = 0$. Let $\{(u_h, p_h)\}$ be a sequence of discrete solutions of (3.1) with $h \to 0$. Then there is a subsequence $\{(u_{h'}, p_{h'})\}$ and an element (u, p) belonging to $V \times Q$ such that $I_{h'}p_{h'}$ converges to $(u, \nabla u)$ in $L^2(\Omega)^{N\times N^2}$, $p_{h'}$ converges to p weakly in $L^2(\Omega)$ and the pair (u, p) is a solution of the continuous problem (2.2). Moreover, if (u, p) belongs to $W^{2,2}(\Omega)^N \times W^{1,2}(\Omega)$, the pressure p'_h converges to p also strongly in $L^2(\Omega)$.*

Proof. (i) Because of Theorem 1, $\|I_h u_h\|_{0,2}$ is uniformly bounded with respect to h. By means of the discrete LBB–condition (4.2) we conclude from (3.1) for all h

$$\|p_h\|_{0,2} \le C.$$

Consequently, there is a weakly converging subsequence $\{(I_{h'}u_{h'}, p_{h'})\}$ benceforth for simplicity again denoted by $\{(I_h u_h, p_h)\}$. It was proven in [14] that there exists an element $u \in W$ such that

$$I_h(u_h - u) \to 0 \quad \text{in} \quad L^2(\Omega)^{N+N^2}, \quad h \to 0.$$

In the following step, we show that (u,p) is indeed a solution of (2.2). Let $r_h; W \to W_h$ and $r_h; V \to V_h$, respectively, be the restriction operator defined by

$$(r_h v)(B_{ij}) = \frac{1}{\text{means}(S_{ij})} \int_{S_{ij}} v\,ds.$$

For $N \le 3$ and $v \in C_0^\infty(\Omega)^N$ there hold

$$\lim_{h\to 0}(p_h, \nabla_h \cdot r_h v) = (p, \nabla \cdot v), \tag{5.1}$$

$$\lim_{h\to 0} \bar{b}_h(u_h, u_h, r_h v) = \bar{b}(u,u,v), \quad \lim_{h\to 0} a_h(u_h, r_h v) = a(u,v) \tag{5.2}$$

and

$$\lim_{h\to 0}(f, r_h v) = (f, v). \tag{5.3}$$

It remains to consider the terms

$$R_1(u_h, u_h, u_h, r_h v) = \sum_j \delta_j (u_h \cdot \nabla u_h, u_h \cdot \nabla r_h v)_{K_j},$$

$$R_2(u_h, r_h v) = \sum_j \delta_j (f, u_h \cdot \nabla r_h v)_{K_j}.$$

Because of the estimates (4.8) , (4.9), and the boundedness of $\|u_h\|_h$ and $\|r_h v\|_h$ we get

$$|R_1(u_h, u_h, u_h, r_h v)| \le C\delta h^{-2\varkappa}, \tag{5.4}$$

$$|R_2(u_h, r_h v)| \le C\delta h^{-\varkappa}. \tag{5.5}$$

Combining (5.1)– (5.5), we have in the limit of (3.1)

$$\nu a(u,v) + \bar{b}(u,u,v) - (p, \nabla\cdot v) = (f,v) \qquad \text{for all } v \in C_0^\infty(\Omega)^N,$$

$$(q, \nabla \cdot u) = 0 \qquad \text{for all } q \in Q.$$

Since $C_0^\infty(\Omega)$ is a dense subset of $W_0^{1,2}(\Omega), (u,p)$ is a solution of the continuous problem (2.2).

(ii) Now we prove the strong convergence of $I_h(u_h - u)$ in $L^2(\Omega)^{N+N^2}$.

$$\begin{aligned}\|u_h - r_h u\|_h^2 &= a_h(u_h,u_h) - 2a_h(u_h, r_h u) + a_h(r_h u, r_h u)\\ &= \nu^{-1}((f,u_h) + \sum_j \delta_j (f - u_h \cdot \nabla u_h, u_h \cdot \nabla u_h)_{K_j})\\ &- 2a_h(u_h, r_h u) + a_h(r_h u, r_h u)\end{aligned}$$

and therefore

$$\|u_h - r_h u\|_h^2 \le \nu^{-1}((f,u_h) + \sum_j \delta_j (f, u_h \cdot \nabla u_h)_{K_j}) - 2a_h(u, r_h u) + a_h(r_h u, r_h u).$$

As in the proof of (5.2) and (5.4), we find

$$\lim_{h\to 0} a_h(u, r_h u) = \lim_{h\to 0} a_h(r_h u, r_h u) = a(u,u)$$

and

$$\lim_{h\to 0}\sum_j \delta_j(f, u_h\cdot\nabla u_h)_K = 0$$

such that

$$\lim_{h\to 0}\|u_h - r_h u\|_h = 0.$$

Together with the triangle inequality, we deduce the strong convergence of $I_h(u_h - u)$ to zero in $L^2(\Omega)^{N+N^2}$.

(iii) The strong convergence in $L^2(\Omega)$ of the pressure p_h in the case $(u,p)\in W^{2,2}(\Omega)^N\times W^{1,2}(\Omega)$ follows in the way below. Multiplying the equation

$$-\nu\Delta u + u\cdot\nabla u + \nabla p = f,$$

which now holds in $L^2(\Omega)^N$, with $v\in V_h$, integrating over K_j, applying Green's formula and summing up over all finite elements K_j we get

$$\nu a_h(u,v) + \bar b_h(u,u,v) - (p,\nabla_h\cdot v) = (f,v) + I_h(v)$$

for all $v\in V_h$ where $I_h(v)$ is defined by

$$I_h(v) = \sum_j\int_{\partial K_j}\left(\nu\frac{\partial u}{\partial n}v - \frac{1}{2}(u\cdot n)(u\cdot v) - p(v\cdot n)\right)ds.$$

Together with (3.1) we have for each $v\in V_h$

$$\begin{aligned}(p-p_h,\nabla_h\cdot v) &= (p,\nabla_h\cdot v) - (p_h,\nabla_h\cdot v)\\ &= \nu a_h(u-u_h,v) + \bar b_h(u-u_h,u,v) + \bar b_h(u_h,u-u_h,v)\\ &\quad+\sum_j\delta_j(f-u_h\cdot\nabla u_h, u_h\cdot\nabla v)_{K_j} - I_h(v).\end{aligned}$$

In [14] it was already shown that

$$|I_h(v)|\le ch\|v\|_h \quad\text{for all}\quad v\in V_h.$$

The continuity of $a_h, \bar b_h$ and the estimates (4.8), (4.9) imply

$$\begin{aligned}|(p-p_h,\nabla_h\cdot v)| &\le (\nu + c(\|u\|_h + \|u_h\|_h))\|u-u_h\|_h\|v\|_h\\ &\quad+c\delta h^{-N}\|f\|_{0,2}\|u_h\|_h\|v\|_h + c\delta h^{-2N}\|u_h\|_h^3\|v\|_h + ch\|v\|_h.\end{aligned}$$

Using a priori estimates for u and u_h, respectively, we get

$$|(p-p_h,\nabla_h\cdot v)|\le (c\|u-u_h\|_h + c_1(h) + ch)\|v\|_h$$

and with the L^2-projection $\tilde p_h$ of p on Q_h

$$|(p_h-\tilde p_h,\nabla_h\cdot v)|\le (C\|u-u_h\|_h + c_1(h) + ch)\|v\|_h.$$

The discrete LBB–condition applied on $p_h-\tilde p_h\in Q_h$ and the triangle inequality yield

$$\|p-p_h\|_{0,2}\le\|p-\tilde p_h\|_{0,2} + \alpha^{-1}(C\|u-u_h\|_h + c_1(h) + ch) \tag{5.7}$$

such that the discrete pressure p_h converges to p in the L^2-norm.

§6. Error Estimates in the Uniqueness Case

For sufficiently large ν both problems (2.2) and (3.1) admit a unique solution. We shall study the order of convergence now.

Theorem 3. *Assume that* $0 \le \delta_j \le \nu^{-1}h^{2\varkappa}c_1(h)$ *with* $\lim_{h\to 0} c_1(h) = 0$ *and let the exact solution* (u,p) *belong to*

$$(W^{1,\infty}(\Omega)\cap W^{2,2}(\Omega))^N \times W^{1,2}(\Omega).$$

Then there are positive constants $\nu_0 = \nu_0(\|f\|_{0,2})$ *and* h_0 *such that for* $\nu \ge \nu_0$ *and* $h \le h_0$ *the problems* (2.2) *and* (3.1) *admit unique solutions which satisfy*

$$\nu\|u-u_h\|_h^2 + \sum_j \delta_j\|u_h\cdot\nabla(u-u_h)\|_{0,2,K_j}^2 \le K(h^2+\delta^2 h^{-2\varkappa}),$$

$$\|p-p_h\|_{0,2} \le K(h+\delta h^{-2\varkappa}).$$

Proof. Let $r_h u$ be the restriction of the exact solution $u\in W$ introduced in the proof of Theorem 2 and let u_h be the unique solution of (4.1). Then for $\varsigma = r_h u - u_h$ it holds that

$$\begin{aligned} S_0 &= \nu\|\varsigma\|_h^2 + \sum_j \delta_j\|u_h\cdot\nabla\varsigma\|_{0,2,K_j}^2 = B_\delta(u_h,u_h;r_h u,\varsigma) - B_\delta(u_h,u_h;u_h,\varsigma) \\ &= B_\delta(u_h,u_h;r_h u,\varsigma) - B_\delta(u,u_h;u,\varsigma) + B_\delta(u,u_h;u,\varsigma) - L_\delta(u_h;\varsigma). \end{aligned} \tag{6.1}$$

The first difference on the right hand side can be bounded as follows

$$\begin{aligned} S_1 &= B_\delta(u_h,u_h;r_h u,\varsigma) - B_\delta(u,u_h;u,\varsigma) \\ &= \nu a_h(r_h u - u,\varsigma) + \bar b_h(u_h, r_h u - u,\varsigma) + \bar b_h(u_h - u, u,\varsigma) \\ &\quad + \sum_j \delta_j(u_h\cdot\nabla(r_h u - u), u_h\cdot\nabla\varsigma)_{K_j} + \sum_j \delta_j((u_h-u)\cdot\nabla u, u_h\cdot\nabla\varsigma)_{K_j}. \end{aligned}$$

Applying a priori estimates for u, u_h and the triangle inequality

$$\|u-u_h\|_h \le \|u - r_h u\|_h + \|\varsigma\|_h$$

we obtain

$$\begin{aligned} |S_1| \le\ & \nu\|r_h u - u\|_h\|\varsigma\|_h + c\nu^{-1}\|f\|_{0,2}\|r_h u - u\|_h\|\varsigma\|_h \\ & + c\nu^{-1}\|f\|_{0,2}\|\varsigma\|_h^2 + \frac{1}{2}\sum_j \delta_j\|u_h\cdot\nabla\varsigma\|_{0,2,K_j}^2 \\ & + c\sum_j \delta_j\|u_h\cdot\nabla(r_h u - u)\|_{0,2,K_j}^2 + c\sum_j \delta_j\|(u_h-u)\cdot\nabla u\|_{0,2,K_j}^2. \end{aligned}$$

Now let K be a constant which may depend on ν, $\|f\|_{0,2}$ and $\|\nabla u\|_{0,\infty}$. Taking into account the approximation properties of V_h we have

$$\begin{aligned} |S_1| \le\ & \frac{\nu}{4}\|\varsigma\|_h^2 + c\nu^{-1}\|f\|_{0,2}\|\varsigma\|_h^2 + Kh^2 + c\delta\|u_h\|_{0,\infty}^2 h^2 \\ & + \frac{1}{2}\sum_j \delta_j\|u_h\cdot\nabla\varsigma\|_{0,2,K_j}^2 + c\delta\|\nabla u\|_{0,\infty}^2\|u_h - u\|_{0,2}^2 \end{aligned}$$

and by means of (4.7)

$$|S_1| \le \frac{\nu}{4}\|\varsigma\|_h^2 + c\nu^{-1}\|f\|_{0,2}\|\varsigma\|_h^2 + c\delta\|\nabla u\|_{0,\infty}^2\|\varsigma\|_h^2 + \frac{1}{2}\sum_j \delta_j\|u_h\cdot\nabla\varsigma\|_{0,2,K_j}^2 + Kh^2. \quad (6.2)$$

In order to estimate the second difference in (6.1) we start with

$$\sum_j(-\nu\Delta u + u\cdot\nabla u + \nabla p - f, \varsigma + \delta_j u_h\cdot\nabla\varsigma)_{K_j} = 0$$

provided the exact solution (u,p) belongs to $W^{2,2}(\Omega)^N \times W^{1,2}(\Omega)$ and $f \in L^2(\Omega)^N$. Green's formula yields

$$\begin{aligned}\nu a_h\ (u,\varsigma) + \bar{b}_h(u,u,\varsigma) - (p,\nabla_h\varsigma) + \sum_j \delta_j(u\cdot\nabla u, u_h\cdot\nabla\varsigma)_{K_j}\\ = (f,\varsigma) + \sum_j \delta_j(f, u_h\cdot\nabla\varsigma)_{K_j} + \sum_j \delta_j(-\nu\Delta u + \nabla p, u_h\cdot\nabla\varsigma)_{K_j} + I_h(\varsigma)\end{aligned}$$

i.e.

$$S_2 := B_\delta(u,u_h;u,\varsigma) - L_\delta(u_h,\varsigma) = (p,\nabla_h\cdot\varsigma) + I_h(\varsigma) + \sum_j \delta_j(-\nu\Delta u + \nabla p, u_h\cdot\nabla\varsigma)_{K_j}. \quad (6.3)$$

In [14] it was already shown that

$$|I_h(\varsigma)| \le ch\|\varsigma\|_h$$

and with the L^2-projection $\tilde{p}_h$ of p on Q_h we have

$$|(p,\nabla_h\cdot\varsigma)| = |(p-\tilde{p}_h,\nabla_h\cdot\varsigma)| \le c\|p-\tilde{p}_h\|_{0,2}\|\varsigma\|_h \le ch\|\varsigma\|_h.$$

It remains to estimate the third term in (6.3). Because

$$\begin{aligned}&\Big|\sum_j \delta_j(-\nu\Delta u + \nabla p, u_h\cdot\nabla\varsigma)_{K_j}\Big|\\ &\qquad \le \|u_h\|_{0,\infty}\sum_j \delta_j\|-\nu\Delta u + \nabla p\|_{0,2,K_j}|\varsigma|_{1,2,K_j} \le \frac{\nu}{4}\|\varsigma\|_h^2 + K\delta^2h^{-2N}.\end{aligned}$$

we have the estimate

$$|S_2| \le \frac{\nu}{2}\|\varsigma\|_h^2 + K(h^2 + \delta^2h^{-2N}). \quad (6.4)$$

Combining (6.1), (6.2), (6.4) we deduce

$$\left(\frac{\nu}{4} - \frac{c}{\nu}\|f\|_{0,2} - c\delta\|\nabla u\|_{0,\infty}^2\right)\|\varsigma\|_h^2 + \frac{1}{2}\sum_j \delta_j\|u_h\cdot\nabla\varsigma_h\|_{0,2,K_j}^2 \le K(h^2 + \delta^2h^{-2N})$$

and get by means of the triangle inequality for sufficiently small $h \le h_0$ the error estimate stated in Theorem 3.

The error estimate for the pressure follows in the same way as (5.7) was proven.

Remark 6.1. For optimal order of convergence we choose $\delta = \max_j \delta_j = 0(\nu^{-1}h^{1+2\varkappa})$ and obtain the first order convergence.

Remark 6.2. Theorem 3 also gives additional control over gradients in the streamline direction; more precisely, with the choice $\delta_j = c\nu^{-1}h^{1+2\varkappa}$ we have

$$\sum_j \|u_h \cdot \nabla(u - u_h)\|^2_{0,2,K_j} \leq Kh^{1-2\varkappa}.$$

§7. Concluding Remarks

The proposed streamline diffusion method is characterized by additional control over gradients in the streamline direction and therefore by improved stability compared with Galerkin's method or special upwind techniques for the convection term [13]. Furthermore, in contrast to [10] exact divergence free discrete velocity approximations have been avoided.

An extension of the presented method to higher-order approximations for the velocity and pressure satisfying the discrete LBB-condition is straightforward. For a streamline diffusion approach with circumvention of the LBB-condition following [7] for the Stokes equation, see [12].

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