Journal of Computational Mathematics, Vol.8, No.2, 1990, 135-146.

BIVARIATE POLYNOMIAL NATURAL SPLINE INTERPOLATION TO SCATTERED DATA *

Li Yue-sheng Guan Lü-tai
(*Department of Computer Science, Zhongshan University, Guangzhou, China*)

Abstract

By means of the theory of spline interpolation in Hilbert spaces, the bivariate polynomial natural spline interpolation to scattered data is constructed. The method can easily be carried out on a computer, and parallelly generalized to high dimensional cases as well. The results can be used for numerical integration in higher dimensions and numerical solution of partial differential equations, and so on.

There are many papers about multivariate spline interpolation and a comprehensive and important survey is given almost every two years in the world [1]-[5]. Now, B-splines, B-nets, thin plate splines and radial functions[6] are useful tools for interpolation, but a method which can easily be carried out on a computer with better variational properties is not found.

The first author of this paper has pointed out that based on variational consideration with multiple restrictions via a generalized Euler equaiton, a simple, convenient and practical method for multivariate spline interpolation could be obtained, and in fact a suitable generalized blending spline function space for solving problems of multivariate optimal interpolation to scattered data throughout a rectangle with continuous boundary conditions and discrete boundary conditions has been constructed in [7], [8]. But it needs to be revised and has not yet been published in magazines.

By means of the theory and methods of spline interpolation in Hilbert spaces, we treat again the bivariate polynomial natural spline interpolation to scattered data proposed in [8]. The method is simple and convenient; the solution has better variational properties and suitable smoothing properties. It suits the solution of the problem of bivariate interpolation to scattered data without boundary conditions and can be generalized to similar multivariate interpolation problems. The results can be used for numerical integration in higher dimensions, the computer solution of partial differential equations and so on.

The notations in this paper can be found in [9], [10].

§1. Selection of Spaces and Definition of Operators

Let $X = H^{m,n}(R) = \left\{u(x,y)\Big|\frac{\partial^{m+n}u}{\partial x^m\partial y^n} \in L_2(R), \frac{\partial^{\alpha+\beta}u}{\partial x^\alpha\partial y^\beta}\right.$ is a absolutely continuous function, $\left.\alpha=\overline{0,m-1}, \beta=\overline{0,n-1}; (x,y)\in R=[a,b]\times[c,d]\right\}$.

Let

$$Y = L_2(R)\times\prod_{\nu=0}^{n-1}L_2[a,b]\times\prod_{\mu=0}^{m-1}L_2[c,d]$$

* Received September 4, 1987.

be a product space where $\prod_{\nu=0}^{n-1} L_2[a,b]$ is a product space constructed by n Hilbert space $L_2[a,b]$, and $Z = R^p$ be a p-dimensional Euclidean space. Suppose that they are all Hilbert spaces.

We define a continuous linear operator $T : X \to Y$ with

$$T = t_0 \times \prod_{\nu=0}^{n-1} t_1^{(\nu)} \times \prod_{\mu=0}^{m-1} t_2^{(\mu)}$$

being a product operator mapping X onto product space Y, where

$$t_0 : X \to L_2(R), t_0(u) = u^{(m,n)}(x,y) = \frac{\partial^{m+n} u(x,y)}{\partial x^m \partial y^n},$$

$$t_1^{(\nu)} : X \to L_2[a,b], t_1^{(\nu)}(u) = u^{(m,\nu)}(x,d) = \frac{\partial^{m+\nu} u(x,y)}{\partial x^m \partial y^\nu}\Big|_{y=d},$$

$$\nu = 0, \cdots, n-1,$$

$$t_2^{(\mu)} : X \to L_2[c,d], t_2^{(\mu)}(u) = u^{(\mu,n)}(b,y) = \frac{\partial^{\mu+n} u(x,y)}{\partial x^\mu \partial y^n}\Big|_{x=b}$$

$$\mu = 0, \cdots, m-1,$$

and define a linear continuous operator $A : X \to Z$ with

$$Au = [u^{(\alpha,\beta)}(x_i, y_i)], \quad \alpha \in I_i, \beta \in J_i, i = \overline{1,N}$$

with $u(x,y) \in X$, the scattered interpolation nodes $(x_i, y_i) \in R$ and $I_i \subset I = \{0, \cdots, m-1\}$, $J_i \subset J = \{0, \cdots, n-1\}$ being arbitrary index sets. The total number of indices is denoted by p.

Given p real numbers $z_i^{\alpha\beta} (\alpha \in I_i, \beta \in J_i, i = \overline{1,N})$, we consider the following spline interpolation problem in Hilbert spaces, which is called bivariate polynomial natural spline interpolation problem, and its solution as bivariate polynomial natural spline interpolant.

Problem P. Find a function $\sigma(x,y) \in X$ satisfying

$$\|T\sigma\|_Y^2 = \min_{x \in I_z} \|Tx\|_Y^2$$

where

$$I_z = \{u \in X | Au = z, z = [z_i^{\alpha\beta}]_{i=1,\alpha\in I_i,\beta\in J_i}^N\}$$

and

$$\|Tx\|_Y^2 = \langle Tx, Tx\rangle_Y = \langle t_0 x, t_0 x\rangle_{L_2(R)} + \sum_{\nu=0}^{n-1} \langle t_1^{(\nu)} x, t_1^{(\nu)} x\rangle_{L_2[a,b]} + \sum_{\mu=0}^{m-1} \langle t_2^{(\mu)} x, t_2^{(\mu)} x\rangle_{L_2[c,d]}$$

$$= \iint_R (u^{(m,n)}(x,y))^2 dx dy + \sum_{\nu=0}^{n-1} \int_a^b (u^{(m,\nu)}(x,d))^2 dx + \sum_{\mu=0}^{m-1} \int_c^d (u^{(\mu,n)}(b,y))^2 dy.$$

Let us denote $\langle Tu, Tu\rangle_Y$ as $J(u)$. The problem is to find a function $\sigma(x,y) \in H^{m,n}(R)$ satisfying

$$\sigma^{(\alpha,\beta)}(x_i,y_i) = z_i^{\alpha\beta}, \quad \alpha \in I_i, \beta \in J_i, i = \overline{1,N},$$

and

$$J(\sigma) = \min J(u), \quad u(x,y) \in H^{m,n}(R), \quad u^{\alpha,\beta}(x_i,y_i) = z_i^{\alpha\beta}.$$

Definition 1. *A polynomial of the form* $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{ij}x^iy^j$ *is called an* (m,n) *order polynomial; the set of all such polynomials is called* (m,n) *order polynomial space, denoted by* $P\langle m,n\rangle$.

Theorem 1. *The null subspace of the operator* $T, N(T) = P < m,n >$.

Proof. According to the definition of $N(T) = \{u \in X | Tu = 0\}$, if $u \in N(T)$, then $u(x,y) \in H^{m,n}(R)$ and

$$u^{(m,n)}(x,y) \equiv 0, \tag{1.1}$$

$$u^{(m,\nu)}(x,d) \equiv 0, \quad \nu = 0,\cdots,n-1, \tag{1.2}$$

$$u^{(\mu,n)}(b,y) \equiv 0, \quad \mu = 0,\cdots,m-1. \tag{1.3}$$

From (1.1) we get

$$u(x,y) = \sum_{i=0}^{m-1} g_i(y)\phi_i(x) + \sum_{i=0}^{n-1} f_i(x)\psi_i(y)$$

with

$$\phi_i(x) = x^i, \quad i = 0,\cdots,m-1; \quad \psi_i(y) = y^i, \quad i = 0,\cdots,n-1$$

and

$$g_i(y) \in H^n[c,d], \quad i = 0,\cdots,m-1; \quad f_i(x) \in H^m[a,b], \quad i = 0,\cdots,n-1$$

as arbitrary functions. Substituting (1.2) and (1.3) into it, we immediately get

$$\sum_{i=0}^{n-1} f_i^{(m)}(x)\psi_i^{(\nu)}(d) = \sum_{i=0}^{m-1} g_i^{(\nu)}(d)\phi_i^{(m)}(x) + \sum_{i=0}^{n-1} f_i^{(m)}(x)\psi_i^{(\nu)}(d) = u^{(m,\nu)}(x,d) = 0,$$

$$\nu = 0,\cdots,n-1;$$

$$\sum_{i=0}^{m-1} g_i^{(n)}(y)\phi_i^{(\mu)}(y) = \sum_{i=0}^{m-1} g_i^{(n)}(y)\phi_i^{(\mu)}(b) + \sum_{i=0}^{n-1} f_i^{(\mu)}(b)\psi_i^{(n)}(y) = u^{(\mu,n)}(b,y) = 0,$$

$$\mu = 0,\cdots,m-1.$$

Noticing that

$$\phi_i^{(j)}(x) = \psi_i^{(j)}(y) = 0, \quad j \geq i; \quad \phi_i^{(i-1)}(x) = \psi_i^{(i-1)}(y) = 1$$

and that the coefficient matrix of the previous equations (about $f_i^m(x)$ and $g_i^{(m)}(y)$) is triangular, we have

$$f_i^{(m)}(x) = g_j^{(n)}(y) = 0, \quad i = 0,\cdots,n-1; \quad j = 0,\cdots,m-1,$$

$$u(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_j^{(1)}y^jx^i + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} c_i^{(2)}x^jy^i = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{ij}x^iy^j.$$

That is,

$$u \in P < m, n > .$$

On the other hand, if $u \in P < m, n >$, direct examination leads to

$$u^{(m,n)}(x,y) = u^{(m,\nu)}(x,d) = u^{(\mu,n)}(b,y) = 0,$$

the same as when

$$u \in N(T).$$

Hence

$$N(T) = P < m, n > .$$

§2 Existence, Uniqueness, Characteristics and Variational Properties of the Natural Spline

Theorem 2. *The bivariate polynomial natural interpolation spline solution always exists.*

Proof. According to Theorem 1, we know that $N(T)$ is a finite dimensional space, and $N(T) + N(A)$ must be a closed set in X. Using Theorem 4.4.1 in [9], we can prove this existence theorem.

Denote

$$I_0 = \{u \in X | Au = 0\}.$$

Definition 2. *We say that problem* P *is* (m,n)*-poised, provided that* $u \in I_0 \cap P < m, n >$ *implies* $u \equiv 0$.

It is easy to find that problem P is (m,n)-poised if and only if the rank of the matrix $A = (a_{i,\mu\nu}^{\alpha\beta})_{\mu \in I, \nu \in J, \alpha \in I_i, \beta \in J_i, i=\overline{1,N}}, r(A) \geq mn$ where

$$a_{i,\mu\nu}^{\alpha\beta} = (x^\mu y^\nu)^{(\alpha,\beta)}\Big|_{(x,y)=(x_i,y_i)} = \frac{\partial^{\alpha+\beta}(x^\mu y^\nu)}{\partial x^\alpha \partial y^\beta}\Big|_{(x,y)=(x_i,y_i)}.$$

Theorem 3. *A sufficient and necessary condition for the existence of a unique bivariate polynomial natural spline interpolation solution is that problem* P *is* (m,n)*-poised.*

Proof. From the proof of Theorem 2, we know that $N(T) + N(A)$ must be a closed set in X. Now,

$$I_0 = \{u \in X | Au = 0\} = N(A).$$

According to Theorem 1, $P < m, n >= N(T)$. Since problem P is (m,n)-poised, we get $N(T) \cap N(A) = \{0\}$. Thus, we can prove this theorem by Theorem 4.4.2 in [9].

Corollary 1. A necessary condition for the existence of a unique bivariate polynomial interpolation natural spline solution is that the sum of total indices $p \geq mn$.

Proof. If it is not true, the rank of matrix A is less than mn, and problem P is thus not (m,n)-poised. Then the solutions are not unique.

Corollary 2. If $m, n \geq 2$, and there exists a unique bivariate polynomial interpolation natural spline solution without derivative conditions, then all of the interpolation nodes will not be in a line.

Proof. Since there are $a_1 = (1, \cdots, 1)^T, a_2 = (x_1, \cdots, x_N)^T$, $a_3 = (y_1, \cdots, y_N)^T$ in the column vectors of matrix A, if $y_i = a + k(x_i - b), i = 1, \cdots, N$ then a_1, a_2, a_3 are linear dependent. Then, $r(A) < mn$, and the solutions are not unique.

Theorem 4 (Characterization Theorem). *$\sigma \in I_z$ is a bivariate polynomial interpolation natural spline iff*

$$\iint_R \sigma^{(m,n)}(x,y)u^{(m,n)}(x,y)dxdy + \sum_{\nu=0}^{n-1}\int_a^b \sigma^{(m,\nu)}(x,d)u^{(m,\nu)}(x,d)dx$$
$$+\sum_{\mu=0}^{m-1}\int_c^d \sigma^{(\mu,n)}(b,y)u^{(\mu,n)}(b,y)dy = 0$$

for any $u(x,y) \in I_0 = N(A)$, i.e. for an arbitrary function $u(x,y)$ satisfying zero interpolation conditions.

Proof. By our definition , the condition in theorem is $< T\sigma, Tu >_Y = 0$. From Theorem 4.5.1 in [9], we get this conclusion.

Definition 3. *We call*

$$S = \Big\{ s(x,y) \in H^{m,n}(R) \Big| \iint_R s^{(m,n)}(x,y)u^{(m,n)}(x,y)dxdy$$
$$+\sum_{\nu=0}^{n-1}\int_a^b s^{(m,\nu)}(x,d)u^{(m,\nu)}(x,d)dx + \sum_{\mu=0}^{m-1}\int_c^d s^{(\mu,n)}(b,y)u^{(\mu,n)}(b,y)dy = 0, \forall u \in N(A)\Big\}.$$

a bivariate polynomial natural spline function space.

Theorem 5 (Existence and uniqueness). *If the interpolation problem P is (m,n)-poised, then for any given real vector $z = [z_i^{\alpha\beta}] \in R^p$, there exists a unique element $\sigma(x,y)$ in the bivariate polynomial natural spline function space satisfying $\sigma^{(\alpha,\beta)}(x_i,y_i) = z_i^{\alpha\beta}, \alpha \in I_i, \beta \in J_i, i = \overline{1,N}$.*

Proof. From Theorem 2, we get the existence of a bivariate polynomial interpolation natural spline solution, and in Theorem 4 we pointed out that this solution is in spline space S. Thus, the argument about the existence of this theorem is proved.

Suppose that there are $s_1, s_2 \in S$ such that $s_1^{(\alpha,\beta)}(x_i,y_i) = s_2^{(\alpha,\beta)}(x_i,y_i) = z_i^{\alpha\beta}$. Let $s = s_1 - s_2$. Then

$$s^{(\alpha,\beta)}(x_i,y_i) = 0, \quad s \in I_0 = N(A),$$
$$J(s_2) = J(s_1) + J(s) + 2a(s_1,s)$$

where

$$a(s_1,s) = \iint_R s_1^{(m,n)}(x,y)s^{(m,n)}(x,y)dxdy + \sum_{\nu=0}^{n-1}\int_a^b s_1^{(m,\nu)}(x,d)s^{(m,\nu)}(x,d)dx$$
$$+\sum_{\mu=0}^{m-1}\int_c^d s_1^{(\mu,n)}(b,y)s^{(\mu,n)}(b,y)dy.$$

According to Theorem 4, $a(s_1,s) = 0$ and $J(s_2) = J(s_1)$. Hence

$$J(s) = J(s_2) - J(s_1) = 0, \quad s \in N(T) = P < m,n > .$$

Since problem P is (m,n)-poised, we get $s = 0$.

Theorem 6 (First integral Relation). *Suppose that $\sigma(x,y)$ is a bivariate polynomial interpolation natural spline solution, $s(x,y) \in S$, $u(x,y) \in I_z$. Then*

$$J(u-s) = J(u-\sigma) + J(\sigma - s).$$

Proof. Since

$$J(u-s) = J(u-\sigma+\sigma-s) = J(u-\sigma) + J(\sigma-s) + a(u-\sigma, \sigma-s),$$

$u(x,y) \in I_z, \sigma(x,y) \in I_z$ and $s(x,y) \in S, \sigma(x,y) \in S$, it is easy to see $u(x,y) - \sigma(x,y) \in I_0, \sigma(x,y) - s(x,y) \in S, a(u-\sigma, \sigma-s) = 0$. Thus, $J(u-s) = J(u-\sigma) + J(\sigma-s)$ and the proof is complete. Especially, set $s = 0$. Then

$$\iint_R (u^{(m,n)}(x,y))^2 dxdy + \sum_{\nu=0}^{n-1} \int_a^b (u^{(m,\nu)}(x,d))^2 dx + \sum_{\mu=0}^{m-1} \int_c^d (u^{(\mu,n)}(b,y))^2 dy$$

$$= \iint_R (u^{(m,n)}(x,y) - \sigma^{(m,n)}(x,y))^2 dxdy + \sum_{\nu=0}^{n-1} \int_a^b (u^{(m,\nu)}(x,d) - \sigma^{(m,\nu)}(x,d))^2 dx$$

$$+ \sum_{\mu=0}^{m-1} \int_c^d (u^{(\mu,n)}(b,y) - \sigma^{(\mu,n)}(b,y))^2 dy + \iint_R (\sigma^{(m,n)}(x,y))^2 dxdy$$

$$+ \sum_{\nu=0}^{n-1} \int_a^b (\sigma^{(m,\nu)}(x,d))^2 dx + \sum_{\nu=0}^{m-1} \int_c^d (\sigma^{(\mu,n)}(b,y))^2 dy.$$

It is the generalization of the pp"first integral relation" in single variate functions.

Theorem 7 (Optimal Approximation Property). *Suppose that $z \in R^p$ is an arbitrarily given element, σ is an element in S, satisfying interpolation conditions:*

$$\sigma^{(\alpha,\beta)}(x_i, y_i) = z_i^{\alpha\beta}, \quad \alpha \in I_i, \beta \in J_i, i = \overline{1,N}.$$

Then (1)$J(\sigma - u) = \min\limits_{s \in S} J(s-u)$, $\forall u \in I_z$. *If $\tilde{\sigma}$ satisfies this property in S, then $\sigma - \tilde{\sigma}$ is an (m,n)-order polynomial.* (2)$J(\sigma - s) = \min\limits_{u \in I_z} J(u-s)$, $\forall s \in S$, *and σ is the unique element satisfying this property in I_z.*

Proof. By the expression in Theorem 6, noticing

$$\min_{s \in S} J(\sigma - s) = \min_{u \in I_z} J(u - \sigma) = 0$$

and $J(\sigma - \tilde{\sigma}) = 0$ gives $\sigma - \tilde{\sigma} \in N(T) = P < m,n >$.

If there is another element $\tilde{\sigma} \in I_z$ such that $J(\tilde{\sigma}) = J(\sigma)$, then $J(\sigma - \tilde{\sigma}) = 0, \sigma - \tilde{\sigma} \in I_0 = N(A)$ and $\sigma - \tilde{\sigma} \in P < m,n >= N(T)$. Since problem P is (m,n)-poised, we deduce that $\sigma = \tilde{\sigma}$.

This theorem can be obtained by Theorem 4.5.7 in [9]. The second expression in this theorem is also called minimum-norm property. From it we have

Corollary 3.

$$J(\sigma) = \min_{u \in I_z} J(u).$$

Since this is a result of setting $s = 0$ in the previous expression, it says that the element σ satisfying interpolation conditions in S has the minimum norm in all of the elements satisfying the interpolation conditions in $H^{m,n}(R)$ and the solution of problem P can be obtained from finding an element satisfying the interpolation conditions in spline space S.

Theorems 6,7 and Corollary 3 are called variational properties of natural splines.

§3. Construction of Natural Splines

We know $Z = R^p$ is a finite dimensional space. Suppose that there exist $k_i^{\alpha\beta} \in X$ satisfying $\langle k_i^{\alpha\beta}, u\rangle_X = u^{(\alpha,\beta)}(x_i, y_i), \alpha \in I_i, \beta \in J_i, i = \overline{1,N}$ for $\forall u \in X$. Thus, $Au = [\langle k_i^{\alpha\beta}, u\rangle_X]_{\alpha\in I_i,\beta\in J_i,i=\overline{1,N}}$, we have

Lemma. *If $\sigma(x,y)$ is a bivariate polynomial interpolation natural spline solution, then there must exist coefficients $\lambda_i^{\alpha\beta}, \alpha \in I_i, \beta \in J_i, i = \overline{1,N}$ such that*

$$T^*T\sigma = \sum_{i=1}^{N}\sum_{\alpha\in I_i}\sum_{\beta\in J_i}\lambda_i^{\alpha\beta}k_i^{\alpha\beta}$$

where T^* is the adjoint operator T.

Proof. From the characterization theorem, we have

$$\langle T\sigma, Tu\rangle_Y = 0, \quad \forall u(x,y) \in N(A).$$

Hence

$$\langle T^*T\sigma, u\rangle_X = 0, \quad T^*T\sigma \in N(A)^{\perp}.$$

Since

$$Au = [\langle k_i^{\alpha\beta}, u\rangle_X]_{\alpha\in I_i,\beta\in J_I,i=\overline{1,N}}$$

and $N(A)^{\perp}$ is a finite dimensional space with basis elements $k_i^{\alpha\beta}, \alpha \in I_i, \beta \in J_i, i = \overline{1,N}$, there must exist coefficients $\lambda_i^{\alpha\beta}$ such that

$$T^*T\sigma = \sum_{i=1}^{N}\sum_{\alpha\in I_i}\sum_{\beta\in J_i}\lambda_i^{\alpha\beta}k_i^{\alpha\beta}.$$

For the sake of easy application, we consider simple interpolation without derivative conditions first, in which case

$$Au = [\langle k_i, u\rangle_X]_{i=\overline{1,N}}, \quad T^*T\sigma = \sum_{i=1}^{N}\lambda_i k_i.$$

Theorem 8 (Construction Theorem). *If $\sigma(x,y)$ is a bivariate polynomial simple interpolation natural spline solution, then it can be expressed as*

$$\sigma(x,y) = \sum_{i=1}^{N}\lambda_i g_i(x,y) + \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}c_{\mu\nu}x^{\mu}y^{\nu}$$

where

$$g_i(x,y) = \tilde{G}(x,y;x_i,y_i),$$

$$\tilde{G}(x,y;t,\tau) = (-1)^{m+n}\frac{(x-t)_+^{2m-1}}{(2m-1)!}\cdot\frac{(y-\tau)_+^{2n-1}}{(2n-1)!}$$
$$+\sum_{\nu=0}^{n-1}(-1)^m\frac{(x-t)_+^{2m-1}}{(2m-1)!}\cdot\frac{(y-d)^\nu}{\nu!}\left[\frac{(\tau-d)^\nu}{\nu!}-(-1)^n\frac{(d-\tau)^{2n-\nu-1}}{(2n-\nu-1)!}\right]$$
$$+\sum_{\mu=0}^{m-1}(-1)^n\frac{(y-\tau)_+^{2n-1}}{(2n-1)!}\cdot\frac{(x-b)^\mu}{\mu!}\left[\frac{(t-b)^\mu}{\mu!}-(-1)^m\frac{(b-t)^{2m-\mu-1}}{(2m-\mu-1)!}\right].$$

The coefficients $[\lambda_i]^T_{i=\overline{1,N}} = \lambda, [c_{\mu\nu}]^T_{\mu=\overline{0,m-1},\nu=\overline{0,n-1}} = c$ *are determined by the following linear system:*

$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}\begin{bmatrix} \lambda \\ c \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}$$

where z *is a given real number set* $[z_i]^T_{i=\overline{1,N}}$*; matrix* $A = (a_{ij})_{i=\overline{1,N},j=\overline{1,N}}$ *is an* $(N\times N)$ *order matrix; matrix* $B = (b_{i,\mu\nu})_{i=\overline{1,N},\mu=\overline{0,m-1},\nu=\overline{0,n-1}}$ *is a matrix with* N *rows and* mn *columns; matrix* B^* *is the transpose of* B*;* $a_{ij} = g_j(x_i,y_i), b_{i,\mu\nu} = x_i^\mu y_i^\nu$ *and* 0 *is a zero element matrix.*

Proof. From the Lemma, we have

$$T^*T\sigma = \sum_{i=1}^N \lambda_i k_i.$$

For any fixed $v\in N(T)$,

$$\langle T^*T\sigma, v\rangle_X = \langle T\sigma, Tv\rangle_Y = \langle T\sigma,\theta\rangle_Y = 0.$$

Hence

$$\langle T^*T\sigma, v\rangle_X = \langle\sum_{i=1}^N\lambda_i k_i, v\rangle = \sum_{i=1}^N\lambda_i\langle k_i, v\rangle = \sum_{i=1}^N\lambda_i v(x_i,y_i) = 0.$$

Any element in $N(T)$ can be expressed as $\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} d_{\mu\nu}(x-b)^\mu(y-d)^\nu$. Then using the previous formula we deduce

$$\sum_{i=1}^N\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}\lambda_i d_{\mu\nu}(x_i-b)^\mu(y_i-d)^\nu = 0. \tag{3.1}$$

Especially, if the basis elements of $N(T)$ are $x^\mu y^\nu$, then

$$\sum_{i=1}^N \lambda_i x_i^\mu y_i^\nu = 0,\quad \mu=0,\cdots,m-1;\nu=0,\cdots,n-1.$$

This linear system expressed in matrix form is

$$B^*\lambda = 0. \tag{3.2}$$

Now let us expand $u(x,y)$ at $x = b$ according to the Taylor expansion with Peano integral remainder first:

$$\begin{aligned} u(x,y) &= u(b,y) + (x-b)u^{(1,0)}(b,y) + \cdots + \frac{(x-b)^{m-1}u^{(m-1,0)}(b,y)}{(m-1)!} \\ &+ \int_a^b (-1)^m \frac{(\tau - x)_+^{m-1}}{(m-1)!} u^{(m,0)}(\tau,y)d\tau. \end{aligned}$$

Secondly, $u(b,y), \cdots, u^{(m,0)}(\tau,y)$ are expanded at $y = d$ according to the Taylor formula:

$$\begin{aligned} u(x,y) &= \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} \frac{(x-b)^\mu}{\mu!}\frac{(y-d)^\nu}{\nu!}u^{(\mu,\nu)}(b,d) \\ &+ \sum_{\nu=0}^{n-1}\int_a^b (-1)^m \frac{(\tau-x)_+^{m-1}}{(m-1)!}\frac{(y-d)^\nu}{\nu!}u^{(m,\nu)}(\tau,d)d\tau \\ &+ \sum_{\mu=0}^{m-1}\int_c^d (-1)^n \frac{(t-y)_+^{n-1}}{(n-1)!}\frac{(x-b)^\mu}{\mu!}u^{(\mu,n)}(b,t)dt \\ &+ \int_a^b\int_c^d (-1)^{m+n}\frac{(t-x)_+^{m-1}}{(m-1)!}\cdot\frac{(t-y)_+^{n-1}}{(n-1)!}\cdot u^{(m,n)}(\tau,t)d\tau dt. \end{aligned} \tag{3.3}$$

Then, for any function $u(x,y)$ in $H^{m,n}(R)$ we have

$$\langle T\sigma, Tu\rangle_Y = \langle T^*T\sigma, u\rangle_X = \langle \sum_{i=1}^N \lambda_i k_i, u\rangle_X = \sum_{i=1}^N \lambda_i\langle k_i, u\rangle_X = \sum_{i=1}^N \lambda_i u(x_i, y_i). \tag{3.4}$$

From (3.1),

$$\sum_{i=1}^N\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} \lambda_i u^{(\mu,\nu)}(b,d)(x_i-b)^\mu(y_i-d)^\nu = 0. \tag{3.5}$$

Substituting x_i, y_i into (3.3) and using (3.4) and (3.5) we get

$$\begin{aligned} \langle T\sigma, Tu\rangle_Y &= \iint_R (-1)^{m+n}\sum_{i=1}^N \lambda_i \frac{(\tau-x_i)_+^{m-1}}{(m-1)!}\frac{(t-y_i)_+^{n-1}}{(n-1)!}u^{(m,n)}(\tau,t)d\tau dt \\ &+ \sum_{\nu=0}^{n-1}\int_a^b (-1)^m \sum_{i=1}^N \lambda_i \frac{(\tau-x_i)_+^{m-1}}{(m-1)!}\frac{(y_i-d)_+^\nu}{\nu!}u^{(m,\nu)}(\tau,d)d\tau \\ &+ \sum_{\mu=0}^{m-1}\int_c^d (-1)^n \sum_{i=1}^N \lambda_i \frac{(t-y_i)_+^{n-1}}{(n-1)!}\frac{(x_i-b)^\mu}{\mu!}u^{(\mu,n)}(b,t)dt \\ &= \langle TG, Tu\rangle_Y. \end{aligned}$$

Changing the integral variables τ, t to x, y, from the definition we know that

$$G(x,y) = \sum_{i=1}^{N} \lambda_i g_i(x,y)$$

where g_i satisfy

$$g_i^{(m,n)}(x,y) = (-1)^{m+n} \frac{(x-x_i)_+^{m-1}}{(m-1)!} \frac{(y-y_i)_+^{n-1}}{(n-1)!},$$

$$g_i^{(m,\nu)}(x,d) = (-1)^{m} \frac{(x-x_i)_+^{m-1}}{(m-1)!} \frac{(y_i-d)^{\nu}}{\nu!}, \qquad \nu = 0, \cdots, n-1,$$

$$g_i^{(\mu,n)}(b,y) = (-1)^{n} \frac{(y-y_i)_+^{n-1}}{(n-1)!} \frac{(x_i-b)^{\mu}}{\mu!}, \qquad \mu = 0, \cdots, m-1.$$

It is easy to verify that $g_i(x,y)$ satisfy these conditions.

Since u is an arbitrary function, it is easy to see $\sigma(x,y) - G(x,y) \in N(T)$. Thus, we get the expression of $\sigma(x,y)$ in this theorem.

From interpolation conditions $\sigma(x_i, y_i) = z_i, i = \overline{1,N}$, we have

$$A\lambda + Bc = z. \tag{3.6}$$

Combining (3.2) with (3.6), we complete the proof of the theorem.

The theorem can be generalized as:

Theorem 9 (Construction Theorem). *Suppose that $\sigma(x,y)$ is a bivariate polynomial interpolt tion natural spline solution. Then it can be expressed in the form*

$$\sigma(x,y) = \sum_{i=1}^{N} \sum_{\alpha \in I_i} \sum_{\beta \in J_i} \lambda_i^{\alpha\beta} g_i^{\alpha\beta}(x,y) + \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} c_{\mu\nu} x^{\mu} y^{\nu}$$

where

$$\begin{aligned} g_i^{\alpha\beta}(x,y) &= (-1)^{m+n-\alpha-\beta} \frac{(x-x_i)_+^{2m-\alpha-1}}{(2m-\alpha-1)!} \frac{(y-y_i)_+^{2n-\beta-1}}{(2n-\beta-1)!} \\ &+ \sum_{\nu=\beta}^{n-1} (-1)^{m-\alpha} \frac{(x-x_i)_+^{2m-\alpha-1}}{(2m-\alpha-1)!} \frac{(y-d)^{\nu}}{\nu!} \left[\frac{(y_i-d)^{\nu-\beta}}{(\nu-\beta)!} - (-1)^{n-\beta} \frac{(d-y_i)^{2n-\nu-\beta-1}}{(2n-\nu-\beta-1)!} \right] \\ &+ \sum_{\mu=\alpha}^{m-1} (-1)^{n-\beta} \frac{(y-y_i)_+^{2n-\beta-1}}{(2n-\beta-1)!} \frac{(x-b)^{\mu}}{\mu!} \left[\frac{(x_i-b)^{\mu-\alpha}}{(\mu-\alpha)!} - (-1)^{m-\alpha} \frac{(b-x_i)^{2m-\mu-\alpha-1}}{(2m-\mu-\alpha-1)!} \right] \end{aligned}$$

and coefficients

$$c = [c_{\mu\nu}]^T_{\mu=\overline{0,m-1}, \nu=\overline{0,n-1}}, \qquad \lambda = [\lambda_i^{\alpha\beta}]^T_{\alpha \in I_i, \beta \in J_i, i=\overline{1,N}}$$

are determined by the following linear algebraic system

$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ c \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}$$

with $p\times p$ matrix $A=(a_{ij}^{\mu\nu,\alpha\beta})_{\mu\in I_i,\nu\in J_i,\alpha\in I_j,\beta\in J_j,i,j=\overline{1,N}}$ and p rows and mn columns matrix $B=(b_{i,\mu\nu}^{\alpha\beta})_{\alpha\in I_i,\beta\in J_i,i=\overline{1,N},\mu=\overline{0,m-1},\nu=\overline{0,n-1}}$ where

$$a_{ij}^{\mu\nu,\alpha\beta}=g_j^{\alpha\beta(\mu,\nu)}(x_i,y_i),\quad b_{i,\mu\nu}^{\alpha\beta}=(x^\mu y^\nu)^{(\alpha,\beta)}\Big|_{(x_i,y_i)}.$$

B^* is the transpose matrix of B and 0 is a zero matrix. The proof is omitted.

If we consider the following instead of problem P. Problem P′: Find $\sigma(x,y)\in X$ such that

$$\|T\sigma\|_Y^2=\min_{x\in I_z}\|Tx\|_Y^2$$

where

$$\|Tx\|_Y^2=\iint_R(u^{(m,n)}(x,y))^2dxdy+\sum_{\nu=0}^{n-1}\int_a^b(u^{(m,\nu)}(x,c))^2dx+\sum_{\mu=0}^{m-1}\int_c^d(u^{(\mu,n)}(a,y))^2dy,$$

then like Theorem 8, we have

Theorem 10. *If $\sigma(x,y)$ is a simple interpolation natural spline solution of* Problem P′, *then*

$$\sigma(x,y)=\sum_{i=1}^N\lambda_ig_i(x,y)+\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}c_{\mu\nu}x^\mu y^\nu$$

where

$$\begin{aligned}g_i(x,y)&=(-1)^{m+n}\frac{(x_i-x)_+^{2m-1}}{(2m-1)!}\frac{(y_i-y)_+^{2n-1}}{(2n-1)!}\\&+\sum_{\nu=0}^{n-1}(-1)^m\frac{(x_i-x)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^\nu}{\nu!}\left[\frac{(y_i-c)^\nu}{\nu!}-(-1)^{n-\nu}\frac{(y_i-c)^{2n-\nu-1}}{(2n-\nu-1)!}\right]\\&+\sum_{\mu=0}^{m-1}(-1)^n\frac{(y_i-y)_+^{2n-1}}{(2n-1)!}\frac{(x-a)^\mu}{\mu!}\left[\frac{(x_i-a)^\mu}{\mu!}-(-1)^{m-\mu}\frac{(x_i-a)^{2m-\mu-1}}{(2m-\mu-1)!}\right]\end{aligned}$$

and the coefficients $\lambda=[\lambda_i]_{i=\overline{1,N}}^T$, $c=[c_{\mu\nu}]_{\mu=\overline{0,m-1},\nu=\overline{0,n-1}}^T$ are determined by a linear algebraic system

$$\begin{bmatrix}A&B\\B^*&0\end{bmatrix}\begin{bmatrix}\lambda\\c\end{bmatrix}=\begin{bmatrix}z\\0\end{bmatrix}$$

with a given real number set z, matrices $A=(a_{ij})_{N\times N}=(g_j(x_i,y_i))$, $B=(b_{i,\mu\nu})_{N\times mn}=(x_i^\mu,y_i^\nu)$, $B^*=B^T$ and zero matrix 0.

The following is a simple example of the interpolation problem: Find the solution $\sigma(x,y)$ of Problem P′ satisfying $\sigma(1,1)=1,\sigma(1,2)=2,\sigma(2,1)=2,m=n=1$ in a rectangular domain $[0,3]\times[0,3]$.

By Theorem 10, the solution is

$$\sigma(x,y) = 3-(1-x)_+(1-y)_+ + \frac{1}{2}(1-x)_+(2-y)_+ + \frac{1}{2}(2-x)_+(1-y)_+$$

$$+\frac{1}{2}(1-x)_+ + \frac{1}{2}(1-y)_+ - (2-x)_+ - (2-y)_+$$

$$= 1+\frac{1}{2}(x-1)_+ - \frac{1}{2}(x-2)_+ + \frac{1}{2}(y-1)_+ - \frac{1}{2}(y-2)_+$$

$$+\frac{1}{2}x(y-1)_+ - \frac{1}{2}x(y-2)_+ + \frac{1}{2}y(x-1)_+ - \frac{1}{2}y(x-2)_+$$

$$-(x-1)_+(y-1)_+ + \frac{1}{2}(x-1)_+(y-2)_+ + \frac{1}{2}(x-2)_+(y-1)_+.$$

References

[1] L. L. Schumaker, Fitting surfaces to scattered data, in Approximation Theory II, Ed. G. G. Lorentz, C. K. Chui & L. L. Schumaker, Academic Press, 1976, 203–268.

[2] R. E. Barnhill, Representation and approximation of surfaces, In Mathematical Software III, Ed. J. R. Rice, Academic Press, 1977, 69–120.

[3] R. Franke, Scattered data interpolation: Tests of some methods, *Mathematics of Computation*, **38** (1982), 181–220.

[4] W. Böhm, G. Farin and J. Kahmann, A survey of curve and surface methods in CAGD, *Computer Aided Geometric Design*, **1** (1984), 1–60.

[5] C. de Boor, Multivariate Approximation, in State of the Art in Numerical Analysis, A. Iserles and M. Powell (eds.), Institute Mathematics Application, Essex. 1987.

[6] W. Dahmen and C. A. Micchelli, Recent progress in multivariate splines, in Approximation Theory IV, Ed. C. K. Chui, L. L. Schumaker and J. D. Ward, Academic Press, 1983, 27–121.

[7] Li Yue-sheng, Multivariate optimal interpolation to scattered data throughout a rectangle I–with Continuous Boundary Conditions, CAT# 55, Center for Approx. Theory, Texas A&M Univ., 1984.

[8] Li Yue-sheng, Multivariate optimal interpolation to scattered data throughout a rectangle II–with Discrete Boundary Conditions, CAT# 56, center for Approx. Theory, Texas A&M Univ., 1984.

[9] P. J. Laurent, Approximation et Optimisation, Hermann, Paris, 1972, 153–251.

[10] V. A. Vasilenko, Spline Functions: Theory, Algorithm and Program, Sciences Press, USSR, 1983.