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COMPUTATION OF MINIMAL AND QUASI–MINIMAL SUPPORTED BIVARIATE SPLINES *

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When multivariate splines are needed to approximate the solution of a certain problem, to generate surfaces or solids, to analyze discrete data, etc., with specified grid and degree of smoothness, it is important to seek locally supported (ls) elements whose polynomial pieces have the lowest total degree. In addition, it is usually desirable to select those with minimal supports (ms). In [1] and [2], de Boor and Höllig studied ms splines in the 3-and 4-direction meshes and pointed out that their study of the 4-direction mesh is not complete. In [3], we have shown that even in the 4-direction mesh, it may happen that there are too few ms bivariate splines to generate all the ls ones. Hence, the notion of quasi-minimal supported (qms) splines was introduced. In this paper, the uniqueness problem is discussed, and recurrence relations as well as computational schemes for both the 3-direction and 4-direction meshes will be given. One interesting property of ms and qms splines is that all of them in the same space are needed to form a partition of unity. We will also characterize those spaces which are spanned by these ls functions.

As usual, $\Delta^{(1)}$ and $\Delta^{(2)}$ will denote the 3-and 4-direction meshes in R^2 with integral grid points, respectively, and $S_m^k(\Delta^{(i)})$, $i = 1, 2$, the spaces of functions in C^k whose restricions on the triangular cells are polynomials of degree m.

The number of "independent" locally supported functions in $S_m^k(\Delta^{(i)})$ will be called the locally supported spline cardinality of $S_m^k(\Delta^{(i)})$, denoted by # lss of $S_m^k(\Delta^{(i)})$, $i = 1, 2$. It is well known (cf. [6]) that

$$\# \text{ lss of } S_m^k(\Delta^{(1)}) = d_m^k(3) = (m - k - [\frac{k+1}{2}])_+(m - 2k + [\frac{k+1}{2}]) \tag{1}$$

and

$$\# \text{ lss of } S_m^k(\Delta^{(2)}) = d_m^k(4) = \frac{1}{2}(m - k - [\frac{k+1}{3}])_+(3m - 5k + 1 + [\frac{k+1}{3}]) \tag{2}$$

where $[x]$ denotes, as usual, the integer part of x.

In order that $S_m^k(\Delta^{(i)})$ may be useful for approximation purposes, we must have positive #lss. In addition, given the smoothness condition C^k, the lowest degree m is desirable. Following de Boor and Höllig [2], we set

$$d_i(k) = \text{the smallest } \; m \; \text{ such that } d_m^k(i+2) > 0.$$

From (1) and (2), we obtain

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k	$2r$-1	$2r$
$d_1(k)$	$3r$	$3r+1$
#lss	2	1

3-direction mesh $\Delta^{(1)}$
r=0,1, $\cdots$

k	$3r$	$3r+1$	$3r+2$
$d_2(k)$	$4r+1$	$4r+2$	$4r+4$
#lss	2	1	3

4-direction mesh $\Delta^{(2)}$
r=0,1, $\cdots$

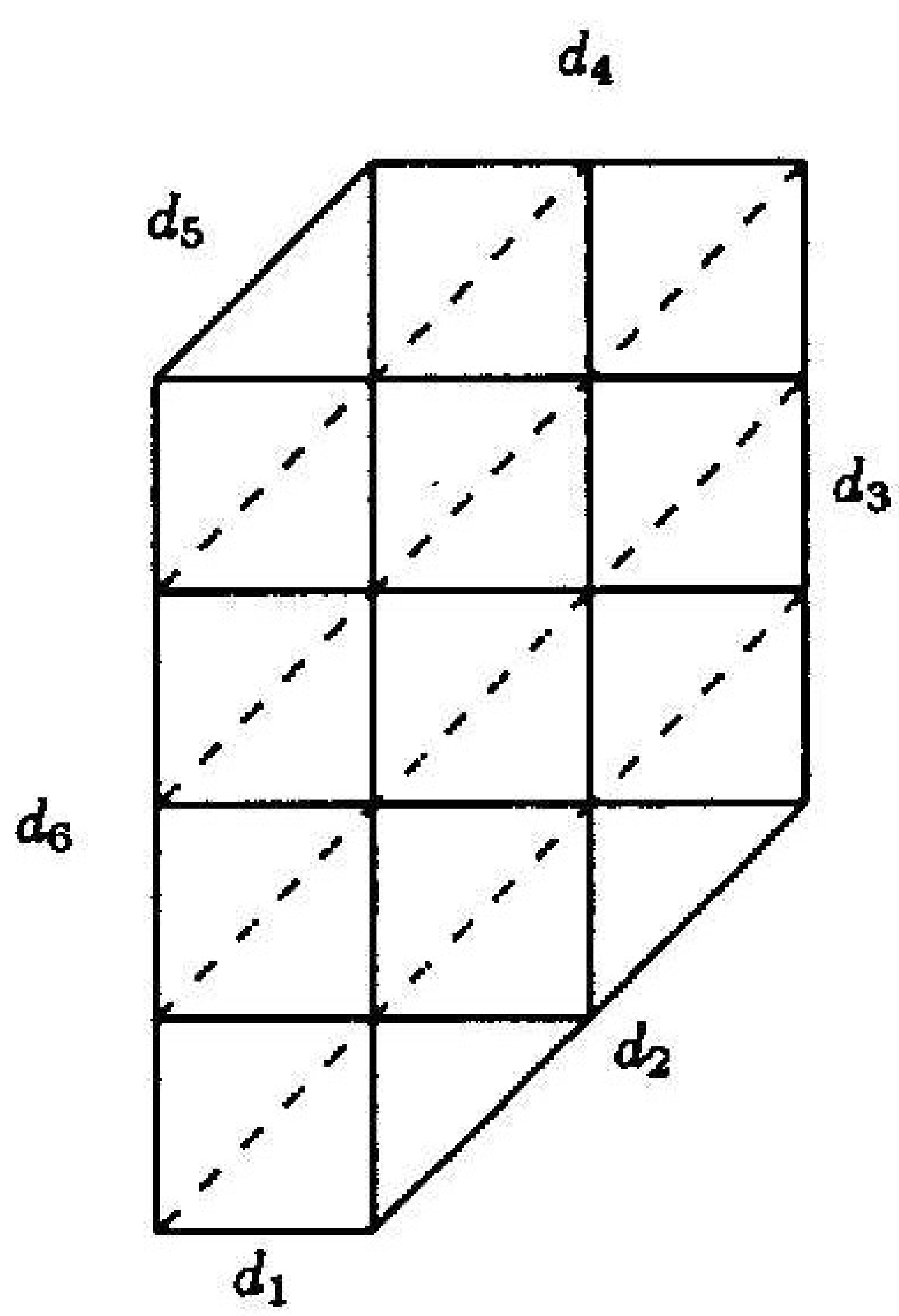
If g is an ls function in $S^k_{d_1(k)}(\Delta^{(1)})$ whose support is a convex polygon, we will denote its support by

$$\text{supp } g = \{d_1, \cdots, d_6\}$$

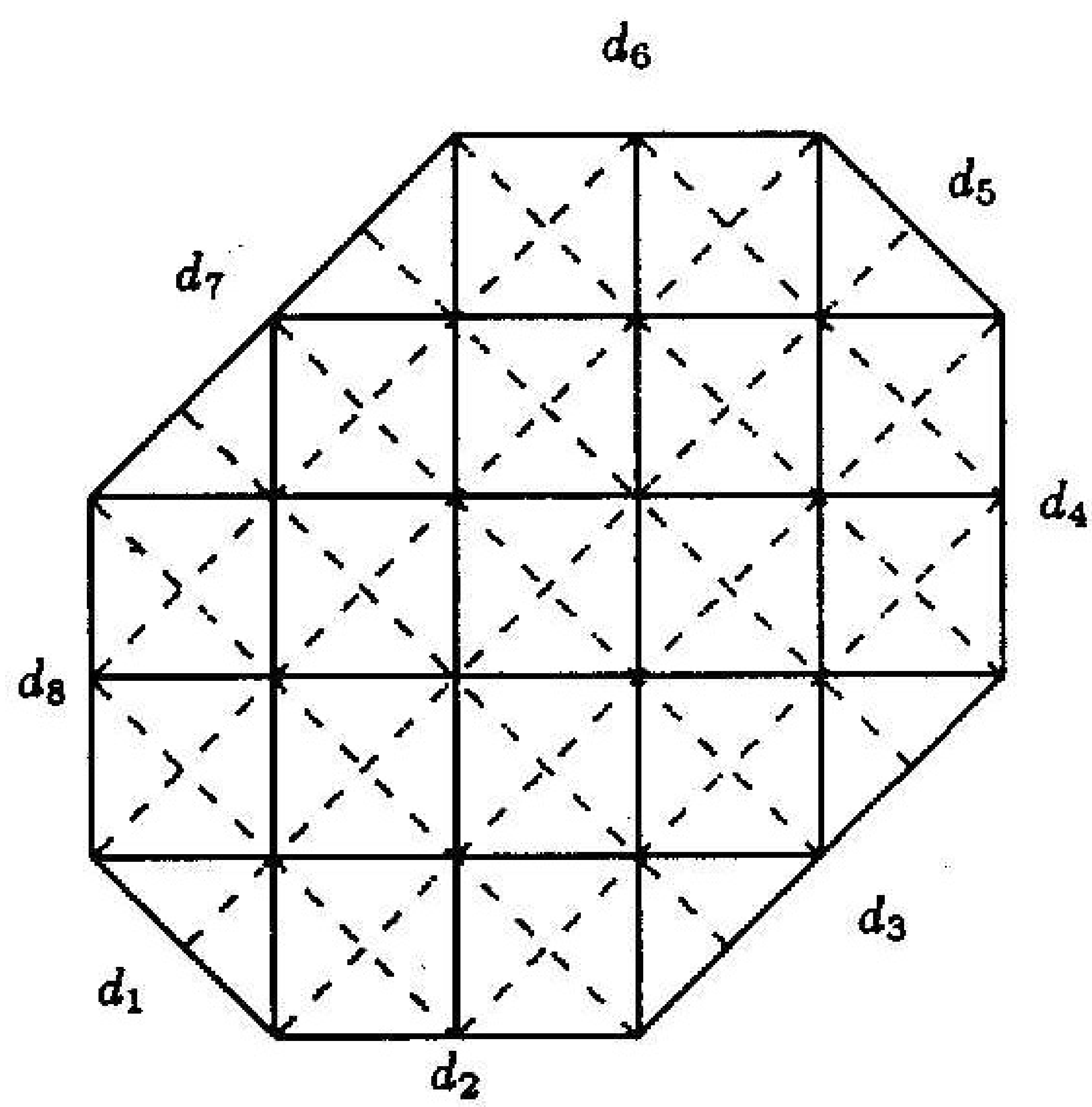
where $d_1, \cdots, d_6$ are nonnegative integers, indicating the number of units (i.e. horizontal or vertical edges, or diagonals) of the partition $\Delta^{(1)}$ in the " directions " $e_1, e_3, e_2, -e_1, -e_3, -e_2$, respectively, where $e_1 = (1,0), e_2 = (0,1), e_3 = (1,1)$. If f is an ls function in $S^k_{d_2(k)}(\Delta^{(2)})$ whose support is a convex polygon, it is clear that none of its vertices lie on a grid point determined by only two grid lines, and its support will be denoted by

$$\text{supp} f = \{d_1, \cdots, d_8\}$$

where $d_1, \cdots, d_8$ are nonnegative integers, indicating the number of units of the partition $\Delta^{(2)}$ in the "directions" $e_4, e_1, e_3, e_2, -e_4, -e_1, -e_3, -e_2$, respectively, where $e_4 = (1,-1)$. These are shown in the followig figures.



3-direction mesh



4-direction mesh

Following [4], we denote by g^r_i and f^r_i the ls functions of $S^k_{d_1(k)}(\Delta^{(1)})$ and $S^k_{d_2(k)}(\Delta^{(2)})$, respectively, with minimal or quasi-minimal supports. In addition, we will call g^0_i and f^0_i the initial ones.

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Space	$S_0^{-1}(\Delta^{(1)})$		$S_1^0(\Delta^{(1)})$
ms spline	g_1^0	g_2^0	g_3^0
support	$\{1,0,1,0,1,0\}$	$\{0,1,0,1,0,1\}$	$\{1,1,1,1,1,1\}$

Space	$S_1^0(\Delta^{(2)})$		$S_2^1(\Delta^{(2)})$
ms spline	f_1^0	f_2^0	f_3^0
support	$\{0,1,0,1,0,1,0,1\}$	$\{1,0,1,0,1,0,1,0\}$	$\{1,1,1,1,1,1,1,1\}$

.

Space	$S_4^2(\Delta^{(2)})$		
ls element	f_4^0, ms	f_5^0, ms	f_6^0, qms
support	$\{1,1,1,1,1,1,1,1\}$	$\{0,2,0,2,0,2,0,2\}$	$\{2,0,2,0,2,0,2,0\}$

Let

$$g_i^r = \underbrace{g_3^0 \star g_3^0 \star \cdots \star g_3^0 \star}_{r} \quad g_i^0, i = 1,2,3 \tag{3}$$

and

$$f_i^r = \underbrace{f_3^0 \star f_3^0 \star \cdots \star f_3^0}_{r} \star f_i^0, i = 1,2,\cdots,6. \tag{4}$$

We remark that g_i^r and $f_i^r, i = 1,2,3$, were discussed by de Boor and Höllig [1,2] and $f_i^r, i = 4,5,6$ were formulated in our earlier work [3]. We summarize these results in the following tables.

Space	$S_{3r}^{2r-1}(\Delta^{(1)})$		$S_{3r+1}^{2r}(\Delta^{(1)})$
ms element	g_1^r	g_2^r	g_3^r
support	$\{r+1,r,r+1,r,r+1,r\}$	$\{r,r+1,r,r+1,r,r+1\}$	$\{r+1,r+1,r+1,r+1,r+1,r+1\}$

Space	$S_{4r+1}^{3r}(\Delta^{(2)})$		$S_{4r+2}^{3r+1}(\Delta^{(2)})$
ms element	f_1^r	f_2^r	f_3^r
support	$\{r,r+1,r,r+1,r,r+1,r,r+1\}$	$\{r+1,r,r+1,r,r+1,r,r+1,r\}$	$\{r+1,r+1,r+1,r+1,r+1,r+1,r+1,r+1\}$

Space	$S_{4r+4}^{3r+2}(\Delta^{(2)})$		
ls element	f_4^r, ms	f_5^r, ms	f_6^r, qms
support	$\{r+1,r+1,r+1,r+1,r+1,r+1,r+1,r+1\}$	$\{r,r+2,r,r+2,r,r+2,r,r+2\}$	$\{r+2,r,r+2,r,r+2,r,r+2,r\}$

In addition, the " uniqueness " of $f_i^r, i = 4,5,6$, has been established in [3]. By using the same technique, we may also conclude the " uniqueness" of the other ms functions. Here we say that the " linearly independent" ms functions $f_1, \cdots, f_p$ in a spline space S are " unique" , if $g(\cdot) = cf_i(\cdot - \underline{j})$ for some $i = 1, \cdots, p$, some $\underline{j} \in Z^2$, and some constant $c \neq 0$

whenever g is an ms function with convex support. We remark that when the definition of minimal support was given in [1,2], the notion " uniqueness" was used but no further discussion was included. In fact, the problem of uniqueness is still open if the convexity assumption is not imposed.

Theorem 1. *The functions f_1^r and f_2^r are " unique" ms elements in $s_{4r+1}^{3r}(\Delta^{(2)})$, f_3^r the " unique" ms function in $S_{4r+2}^{3r+1}(\Delta^{(2)})$, g_1^r, g_2^r the " unique " ms functions in $S_{3r}^{2r-1}(\Delta^{(1)})$, and g_3^r the " unique" ms function in $S_{3r+1}^{2r}(\Delta^{(1)})$.*

Sketch of Proof. We first consider $\Delta^{(2)}$. Let

$$A_1(x,y) = \begin{cases} -y & \text{for } y \geq 0 \text{ and } x+y \leq 0, \\ x & \text{for } x+y > 0 \text{ and } x \leq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$A_2(x,y) = \begin{cases} y+x & \text{for } x+y > 0 \text{ and } x \leq 0, \\ y-x & \text{for } x > 0 \text{ and } y \geq x, \\ 0 & \text{otherwise;} \end{cases}$$

$$D(x,y) = \begin{cases} 2y^2 & \text{for } y \geq 0 \text{ and } x+y \leq 0, \\ 2y^2-(x+y)^2 & \text{for } x+y > 0 \text{ and } x \leq 0, \\ (x-y)^2 & \text{for } x > 0 \text{ and } y \geq x, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A_1, A_2 \in S_1^0(\Delta^{(2)})$, $D \in S_2^1(\Delta^{(2)})$. Following [3], we consider the integral operators

$$J_i = \int_0^\infty f((x,y)+te_i)dt, \quad i = 1,4$$

and

$$J_i = \int_{-\infty}^0 f((x,y)+te_j)dt, \quad j = 2,3$$

and set $J = J_4J_3J_2J_1$. Then

$$J^rA_i|_{y\geq 0 \text{ and } x+y\leq 0} = y^{3r+i}p_i,$$

$$J^rA_i|_{x>0 \text{ and } y\geq x} = (x-y)^{3r+3-i}\bar{p}_i,$$

$$J_rD|_{y\geq 0 \text{ and } x+y\leq 0} = y^{3r+2}q,$$

$$J^rD|_{x>0 \text{ and } y\geq x} = (x-y)^{3r+2}\bar{q}$$

for some appropriate polynomials $p_i, \bar{p}_i, q$ and $\bar{q}$, and the collections

$$\{J_i^rA_i(\cdot - \underline{j}) : \underline{j} \in Z^2, i = 1,2\}$$

and

$$\{J^rD(\cdot - \underline{j}) : \underline{j} \in Z^2\}$$

are bases of $S^{3r}_{4r+1}(\Delta^{(2)})$ and $S^{3r+1}_{4r+2}(\Delta^{(2)})$, respectively. Hence, for any f in $S^{3r}_{4r+1}(\Delta^{(2)})$ with supp $f = \{d_1, \cdots, d_8\}$, by using the first basis we obtain minimal pairs (cf. [3]) $(d_i, d_{i+1}) = (r, r+1)$ or $(r+1, r), i = 1, \cdots, 8, d_9 \equiv d_1$, and

$$\begin{cases} d_1 + d_2 \geq 2r+1, d_2 + d_3 \geq 2r+1, \cdots, d_8 + d_1 \geq 2r+1, \\ d_1 + d_2 + d_3 = d_5 + d_6 + d_7, \\ d_7 + d_8 + d_1 = d_3 + d_4 + d_5. \end{cases} \tag{5}$$

If $d_1 + d_2 + d_3 \geq 3r+3$ and $d_7 + d_8 + d_1 \geq 3r+3$, then we have

$$\text{supp} f \supsetneqq \{r, r+1, r, r+1, r, r+1, r, r+1\} = \text{ supp } f_1^r$$

or

$$\text{supp } f \supsetneqq \{r+1, r, r+1, r, r+1, r, r+1, r\} = \text{supp } f_2^r$$

so that f is not ms. Consequently, if f is ms, then we must have $d_1 + d_2 + d_3 \leq 3r+2$, or $d_7 + d_8 + d_1 \leq 3r+2$, or both. In this case, using the same argument as given in [3], we have

$f \equiv cf_1^r(\cdot - \underline{j})$ or $f \equiv cf_2^r(\cdot - \underline{j})$ for some $c \neq 0$ and some $\underline{j} \in Z^2$.

For any $f \in S^{3r+1}_{4r+2}(\Delta^{(2)})$ with supp $f = \{d_1, \cdots, d_8\}$, by using the second basis we also obtain minimal pairs $(d_1, d_{i+1}) = (r+1, r+1)$, $i = 1, \cdots, 8, d_9 \equiv d_1$, and

$$\begin{cases} d_1 + d_2 \geq 2r+2, d_2 + d_3 \geq 2r+2, \cdots, d_8 + d_1 \geq 2r+2, \\ d_1 + d_2 + d_3 = d_5 + d_6 + d_7, \\ d_7 + D_8 + d_1 = d_3 + d_4 + d_5. \end{cases}$$

If $d_1 + d_2 + d_3 \geq 3r+4$ and $d_7 + d_8 + d_1 \geq 3r+4$, then

$$\text{supp} f \supsetneqq \{r+1, r+1, r+1, r+1, r+1, r+1, r+1, r+1\} = \text{supp} f_3^r$$

so that f is not ms. Hence, if f is ms, we may conclude that $d_1 + d_2 + d_3 \leq 3r+3$, or $d_7 + d_8 + d_1 \leq 3r+3$, or both, and it follows as in [3] that $f \equiv cf_3^r(\cdot - \underline{j})$ for some $c \neq 0$ and $\underline{j} \in Z^2$.

To study the 3-direction mesh, we set

$$E_1(x,y) = \begin{cases} 1 & \text{for } x \leq 0 \text{ and } y > 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$E_2(x,y) = \begin{cases} 1 & \text{for } x > 0 \text{ and } y \geq x, \\ 0 & \text{otherwise;} \end{cases}$$

$$H(x,y) = \begin{cases} y & \text{for } x \leq 0 \text{ and } y > 0, \\ y - x & \text{for } x > 0 \text{ and } y \geq x, \\ 0 & \text{otherwise} \end{cases}$$

where $E_1, E_2 \in S_0^{-1}(\Delta^{(1)})$, $H \in S_1^0(\Delta^{(1)})$, and consider

$$\bar{J} = J_3 J_2 J_1.$$

Then $\{J^r E_i(\cdot - \underline{j}) : \underline{j} \in Z^2, i = 1,2\}$ is a basis of $S_{3r}^{2r-1}(\Delta^{(1)})$, and $\{J^r H(\cdot - \underline{j}) : \underline{j} \in Z^2\}$ a basis of $S_{3r+1}^{2r}(\Delta^{(1)})$. Hence, for any f in $S_{3r}^{2r-1}(\Delta^{(1)})$ with supp $f = \{d_1, \cdots, d_3\}$, the first basis yields minimal pairs: $(d_i, d_{i+1}) = (r, r+1)$ or $(r+1, r), i = 1, \cdots, 6, d_7 \equiv d_1$, but instead of (5), we have

$$\begin{cases} d_1 + d_2 \geq 2r+1, d_2 + d_3 \geq 2r+1, \cdots, d_6 + d_1 \geq 2r+1, \\ d_1 + d_2 = d_4 + d_5, \\ d_2 + d_3 = d_5 + d_6. \end{cases}$$

Similarly, for the space $S_{3r+1}^{2r}(\Delta^{(1)})$, the minimal pairs are $(d_i, d_{i+1}) = (r+1, r+1), i = 1, \cdots, 6, d_7 \equiv d_1$, as a consequence of using the second basis, and again instead of (5) we have

$$\begin{cases} d_1 + d_2 \geq 2r+2, d_2 + d_3 \geq 2r+2, \cdots, d_6 + d_1 \geq 2r+2, \\ d_1 + d_2 = d_4 + d_5, \\ d_2 + d_3 = d_5 + d_6. \end{cases}$$

This will complete the proof of the theorem.

From the property of box splines (cf. [8]), we have the following

Proposition 1. Set $B_{V_1} = g_3^{r-1}$ and $B_{V_2} = f_3^{r-1}, r \geq 1$, where $V_1 = \{e_1 : r, e_2 : r, e_3 : r\}, V_2 = \{e_1 : r, e_2 : r, e_3 : r, e_4 : r\}$. Then for any $v_1 \in V_1$ and $v_2 \in V_2$,

$$D_{v_1} g_i^r(\cdot) = (g_i^0 \star B_{V_1 \setminus v_1})(\cdot) - (g_i^0 \star Vb_{V_1} \setminus v_1)(\cdot - v_1),$$
$$D_{v_2} f_i^r(\cdot) = (f_2^0 \star B_{V_2 \setminus v_2})(\cdot) - (f_i^0 \star B_{V_2 \setminus v_2})(\cdot - v_2).$$

Proof.

$$\begin{aligned} D_{v_1} g_i^r(\cdot) &= D_{v_1}(g_i^0 \star B_{V_1})(\cdot) \\ &= D_{v_1} \int\int_{R_2} B_{V_1}(\cdot - (s,t)) g_i^0(s,t) ds dt \\ &= \int\int_{R^2} g_i^0(s,t)[B_{V_1 \setminus v_1}(\cdot - (s,t)) - B_{V_1 \setminus v_1}(\cdot - v_1 - (s,t))] ds dt \\ &= (g_i^0(s,t) \star B_{V_1 \setminus v_1})(\cdot) - (g_i^0 \star B_{V_1 \setminus v_1})(\cdot - v_1). \end{aligned}$$

Similarly, we can prove the formula for $D_{v_2} f_i^r(\cdot)$.

From Proposition 1 and a formula in [8], we also have

Proposition 2. If $x = \sum_{V_1} t_v v$ and $r \geq 1$, then

$$(\#V_1 - 2) g_i^r(\cdot) = \sum_{V_1} [t_v (g_i^0 \star B_{V_1 \setminus v})(\cdot) + (1 - t_v)(g_i^0 \star B_{V_1} \setminus v)(\cdot - v)].$$

If $x = \sum_{V_2} t_v v$ and $r \geq 1$, then

$$(\#V_2 - 2) f_i^r(\cdot) = \sum_{V_2} [t_v (f_i^0 \star B_{V_2 \setminus v})(\cdot) + (1 - t_v)(f_i^0 \star B_{V_2 \setminus v})(\cdot - v)].$$

In order to consider computational schemes for g_i^r and f_i^r, we need to decompose each convolution in (3) and (4) with g_3^0 and f_3^0 by using integral operators

$$(I_i f)(x,y) = \int_{-1}^{0} f((x,y)+te_i)dt, \quad i = 1,2,3,4 \tag{6}$$

as follows.

Lemma 1. (i) $g_3^r * g = (I_3 I_2 I_1)^r g, \quad g \in S_{d_1(k)}^k(\Delta^{(1)})$, (ii) $f_3^r * f = (I_4 I_3 I_2 I_1)^r f, f \in S_{d_2(k)}^k(\Delta^{(2)})$.

Proof. First, note that by setting

$$s(x) = (e^{ix} - 1)/(ix)$$

we have

$$(\widehat{g_3^r * f})(x,y) = \hat{f}(x,y)s(x)s(y)s(x+y).$$

On the other hand,

$$(I_1 f)(x,y) = \int_{x-1}^{x} f(t,y)dt = \int_{-\infty}^{\infty} f(t,y)\chi(x-t)dt \tag{7}$$

where χ is the characteristic function of [0, 1]. Thus

$$(\widehat{I_1 f})(x,y) = \hat{f}(x,y)s(x).$$

Similarly, we have

$$(\widehat{I_2 f})(x,y) = \hat{f}(x,y)s(y),$$
$$(\widehat{I_3 f})(x,y) = \hat{f}(x,y)s(x+y),$$
$$(\widehat{I_4 f})(x,y) = \hat{f}(x,y)s(x-y).$$

This completes the proof of Lemma 1.

Next, consider

$$g_i^{r;m,n,u} = (I_3^u I_2^n I_1^m) g_i^r \quad and \quad f_i^{r;m,u,v} = (I_4^v I_3^u I_2^n I_1^m) f_i^r$$

where

$$I_4^v = \underbrace{I_4 \cdots I_4}_{v}, \quad I_3^u = \underbrace{I_3 \cdots I_3}_{u}, \quad I_2^n = \underbrace{I_2 \cdots I_2}_{n}, \quad I_1^m = \underbrace{I_1 \cdots I_1}_{m},$$

and it is obvious that

$$g_i^{r+1} = g_i^{r;1,1,1,} = I_3 g_i^{r;1,1,0} = (I_3 I_2) g_i^{r;1,0,0} = (I_3 I_2 I_1) g_i^r,$$

$$f_i^{r+1} = f_i^{r;1,1,1,1} = I_4 f_i^{r;1,1,1,0} = (I_4 I_3) f_i^{r;1,1,0,0} = (I_4 I_3 I_2) f_i^{r;1,0,0,0} = (I_4 I_3 I_2 I_1) f_i^r.$$

Then, we have the following

Proposition 3. For the 3-direction mesh,

$$g_i^{r;m,n,u} \in S_{d_{1,i}+l_1}^{k_{1,i}+\mu_1}(\Delta^{(1)})$$

where $k_{1,1} = k_{1,2} = -1, k_{1,3} = 0, d_{1,1} = d_{1,2} = 0, d_{1,3} = 1$, and $\mu_1 = 2r + m + n + u - \max(m, n, u), l_1 = 3r + m + n + u$. For the 4-direction mesh,

$$f_i^{r;m,n,u,v} \in S_{d_{2,i}+l_2}^{k_{2,i}+\mu_2}(\Delta^{(2)})$$

where $k_{2,1} = k_{2,2} = 0, k_{2,3} = 1, k_{2,4} = k_{2,5} = k_{2.6} = 2, d_{2,1} = d_{2,2} = 1, d_{2,3} = 2, d_{2,4} = d_{2,5} = d_{2,6} = 4$, and

$$\mu_2 = 3r + m + n + u + v - \max(m, n, u, v),$$

$$l_2 = 4r + m + n + u + v.$$

Proof. We only give the proof for $g_i^{r;m,n,u}$. Since

$$g_i^{r;m,n,u} = (I_3^u I_2^n I_1^m) g_i^r = (I_3^u I_2^n I_1^m)(g_i^0 \star B_{V_1}) = g_i^0 \star (I_3^u I_2^n I_1^m) B_{V_1} = g_2^0 \star B_V$$

where $V = V_1 \cup \{e_1 : m, e_2 : n, e_3; u\} = \{e_1 : r + m, e_2 : r + n, e_3 : r + u\}$, all the partial derivatives of $g_i^{r;m,n,u}$ of order $k_{1,i} + \mu_1$ are continuous. In addition, from Lemma 1, we know that the total degree of $g_i^0 \star B_V$ is that of g_i^0 plus $\#V$, which is $d_{1,i} + l_1$.

We now consider the following computational schemes:

$$\begin{cases} \frac{\partial}{\partial x} g_i^{r;1,0,0}(\cdot - (k,l)) = g_i^r(\cdot - (k,l)) - g_i^r(\cdot - (k+1,l)), \\ \frac{\partial}{\partial y} g_i^{r;1,1,0}(\cdot - (k,l)) = g_i^{r;1,0,0}(\cdot - (k,l)) - g_i^{r;1,0,0}(\cdot - (k,l+1)), \\ \frac{\partial}{\partial x} g_i^{r+1}(\cdot - (k,l)) = g_i^{r;1,1,0}(\cdot - (k,l)) - g_i^{r;1,1,0}(\cdot - (k+1,l+1)), \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial x} f_i^{r;1,0,0,0}(\cdot - (k,l)) = f_i^r(\cdot - (k,l)) - f_i^r(\cdot - (k+1,l)), \\ \frac{\partial}{\partial y} f_i^{r;1,1,0,0}(\cdot - (k,l)) = f_i^{r;1,0,0,0}(\cdot - (k,l)) - f_i^{r;1,0,0,0}(\cdot - (k,l+1)), \\ \frac{\partial}{\partial x} f_i^{r;1,1,1,0}(\cdot - (k,l)) = f_i^{r;1,1,0,0}(\cdot - (k,l)) - f_i^{r;1,1,0,0}(\cdot - (k+1,l+1)), \\ \frac{\partial}{\partial x} f_i^{r+1}(\cdot - (k,l)) = f_i^{r;1,1,1,0}(\cdot - (k,l)) - f_i^{r;1,1,1,0}(\cdot - (k+1,l-1)). \end{cases}$$

From these formulas, we can use the computation procedure in [5] to determine the Beziér nets of the polynomial pieces of g_i^r, f_i^r for all r, i.

From the properties of the initial ls functions and the generating formulas (3) and (4), we have the following

Theorem 2 (Partition of unity). For each $r = 0, 1, \cdots$, we have

$$\sum_{\underline{j} \in Z^2} g_3^r(\cdot - \underline{j}) \equiv 1,$$

$$\sum_{\underline{j} \in Z^2} (g_1^r(\cdot - \underline{j}) + g_2^r(\cdot - \underline{j})) \equiv 1,$$

$$\sum_{\underline{j} \in Z^2} f_3^r(\cdot - \underline{j}) \equiv 1,$$

$$\sum_{\underline{j} \in Z^2} (f_1^r(\cdot - \underline{j}) + f_2^r(\cdot - \underline{j})) \equiv 1,$$

$$\sum_{\underline{j} \in Z^2} (f_4^r(\cdot - \underline{j}) + f_5^r(\cdot - \underline{j}) + f_6^r(\cdot - \underline{j})) \equiv 1.$$

Next, we study the basis problem for the spaces $S^k_{d_i(k)}(\Delta^{(i)})$. Let $G_1 = \{d_{1,1}, \cdots, d_{1,6}\}$ and $G_2 = \{d_{2,1}, \cdots, d_{2,8}\}$ be closed convex sets in R^2. From the results in [1], [2] and [3], we know that the collections $\{g^r_i(\cdot - \underline{j}) : i = 1,2, \underline{j} \in Z^2 \cap G_1\}$, $\{g^r_3(\cdot - \underline{j}) : \underline{j} \in Z^2 \cap G_1\}$, $\{f^r_i(\cdot - \underline{j}) : i = 1,2, \underline{j} \in Z^2 \cap G_2\}$, $\{f^r_3(\cdot - \underline{j}), \underline{j} \in Z^2 \cap G_2\}$ and $\{f^r_i(\cdot - \underline{j}) : i = 4,5,6, \underline{j} \in Z^2 \cap G_2\}$ span the spaces $\{f \in f \in S^{2r-1}_{3r}(\Delta^{(1)})$: supp $f \subset G_1\}$, $\{f \in S^{2r}_{3r+1}(\Delta^{(1)})$: supp $f \subset G_1\}$, $\{f \in S^{3r}_{4r+1}(\Delta^{(2)})$: supp $f \subset G_2\}$, $\{f \in S^{3r+1}_{4r+2}(\Delta^{(2)})$: supp $f \subset G_2\}$ and $\{f \in S^{3r+2}_{4r+4}(\Delta^{(2)})$: supp $f \subset G_2\}$, respectively. Thus we have the following

Proposition 4. Every box spline f in $S^k_{d_i(k)}(\Delta^{(i)})$ can be written as a linear combination of the translates of ms (and/or qms) splines in the same space whose supports are contained in supp f.

We now discuss the problem of global basis. Let $S(\Delta, \Omega) = \{f \in S(\Delta)$: supp $f \cup \Omega \neq \emptyset\}$.

Theorem 3. Let Ω be a rectangular region. Then

$$\text{span } \{g^r_1(\cdot - \underline{j}), g^r_2(\cdot - \underline{k})\} = S^{2r-1}_{3r}(\Delta^{(1)}, \Omega) \Leftrightarrow r = 0, 1,$$
$$\text{span } \{g^r_3(\cdot - \underline{j})\} = S^{2r}_{3r+1}(\Delta^{(1)}, \Omega) \Leftrightarrow r = 0$$

and

$$\text{span } \{f^r_1(\cdot - \underline{j}), f^r_2(\cdot - \underline{k})\} = S^{3r}_{4r+1}(\Delta^{(2)}, \Omega) \Leftrightarrow r = 0, 1,$$
$$\text{span } \{f^r_3(\cdot - \underline{j})\} = S^{3r+1}_{4r+2}(\Delta^{(2)}, \Omega) \Leftrightarrow r = 0,$$
$$\text{span } \{f^r_4(\cdot - \underline{j}), f^r_5(\cdot - \underline{k}), f^r_6(\cdot - \underline{i})\} = S^{3r+2}_{4r+4}(\Delta^{(2)}, \Omega) \Leftrightarrow r = 0, 1.$$

Proof. The necessary conditions can be proved by using the fact that dim $S^k_{d_i(k)}(\Delta^{(i)}, \Omega)$ does not exceed the cardinality of the corresponding spanning set.

Analogously to a result in [3], we can prove the sufficient conditions by noting that

$$S^k_{d_i(k)}(\Delta^{(i)}, \Omega) = \Pi_{d_i(k)} + P + B$$

where $\Pi_{d_i(k)}$ denotes the space of polynomials of degree $d_i(k)$, P the linear span of the corresponding truncated powers (or cone splines), and B the linear span of the correspinding box splines (cf. [1], [7]).

For $i = 1$, if

$$d_1(k) > \frac{3k}{2}, \tag{8}$$

then $\Pi_{d_1(k)}$ and $P \subset B$, and by using Proposition 3, $S^k_{d_1(k)}(\Delta^{(1)}, \Omega)$ is contained in the linear span of the corresponding ms splines For $i = 2$, if

$$d_2(k) > \frac{4k}{3}, \tag{9}$$

then $\Pi_{d_2(k)}$ and $P \subset B$, and from Proposition 3, $S^k_{d_2(k)}(\Delta^{(2)}, \Omega)$ is contained in the linear span of the corresponding ms (and /or qms) splines.

If $k = 2r - 1$, then $d_1(k) = 3r$ and (8) follows provided $r = 0, 1$. If $k = 2r$, then $d_1(k) = 3r + 1$, so that for the special case $r = 0$, inequality (8) holds. For the 4-direction mesh, in each of the three cases (i) $k = 3r$ and $r = 0, 1$, (ii) $k = 3r + 1$ and $r = 0$; (iii) $k = 3r + 2$ and $r = 0, 1$, inequality (9) also holds. This completes the proof of the theorem.

References

[1] C. de Boor and K. Höllig, Bivariate box splines and smooth pp functions on a three-direction mesh, *J. Comp. Appl. Math.*, **9** (1983), 13-28.

[2] C. de Boor and K. Höllig, Minimal support for bivariate splines, *Approx. Theory and Its Appl.*, **3** (1987), 11-23.

[3] C. K. Chui and T. X. He, On minimal and quasi-minimal supported bivariate splines, *J. Approx. Theory*, **52** (1988), 217-238.

[4] C. K. Chui, T. X. He, and R. H. Wang, The C^2 quartic spline space on a four-directional mesh, *Approx. Theory and Its Appl.*, **3** (1987), 32-36.

[5] C. K. Chui and M. J. Lai, Computation of box splines and B-splines on triangulations of nonuniform rectangular partitions, *Approx. Theory and Its Appl.*, **3** (1987), 37-62.

[6] C. K. Chui and R. H. Wang, Multivariate splines spaces, *J. Math. Anal. and Its Appl.*, **94** (1983), 197-221.

[7] W. Dahmen and C. A. Micchelli, On the optimal approximation rates from criss-cross finite element spaces, *J. Comp. Appl. Math.*, **10** (1984), 257-273.

[8] K. Höllig, Box Splines, in " Approximation Theory " , C. K. Chui, L. L. Schumaker, and J. D. Ward, eds., Academic Press, 1986, 71-96.