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SEMI–COARSENING IN MULTIGRID SOLUTION OF STEADY INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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Abstract

We present a semi–coarsening procedure, i. e., coarsening in one space direction, to improve the convergence rate of the multigrid solver presented in [5] for solving the 2D steady Navier–Stokes equations in primitive variables when the aspect ratio of grid cells is not equal to 1, i. e., when $h_x/h_y \gg 1$ or $\ll 1$, where h_x is the grid step in x direction and h_y the grid step in y direction, x and y represent the Cartesian coordinates.

Introduction

In numerical simulation of fluid flows we encounter frequently situations in which physical quantities (pressure, velocity, etc.) change at different scales in different space directions. When there is a main flow direction, where the change of physical quantities is much smaller along the flow direction than in the direction orthogonal to the main flow, different grid steps in different directions are often used. For dealing with these problems, it is essential to have a solver of the discrete system whose convergence rate is not very sensitive to the ratio of grid steps.

In [5], we have presented a multigrid solver for solving the 2D steady Navier–Stokes equations in primitive variables on rectangular regions. It is based on second–order upwind differencing for the discretization of the convection terms and the SCGS relaxation procedure (this procedure was originally proposed by S.P. Vanka[3] as smoothing operator for his multigrid solver based on hybrid differencing) and has been observed to have good convergence rate for Reynolds numbers up to 10000*.

If we denote by h_x the grid step in x direction and h_y the grid step in y direction, the convergence rate of the above M.G. solver depends on the ratio $\rho \stackrel{\text{def}}{=} h_x/h_y$. The best convergence rate is obtained when $\rho = 1$ while the convergence rate is significantly slowed down when $\rho \ll 1$ or $\rho \gg 1$, as can be seen through Figure 1, which shows the total residual of the approximate solution with regard to the number of multigrid iterations. In this example, the computational region is the rectangle $(0, A) \times (0, B)$ and the grid is the 32×32 uniform grid, so $\rho = A/B$. The four curves are obtained with $A = 1$ and $B = 1$, 4, 8, 16, respectively and the following test solution :

$$u(x,y)=A\sin(\frac{x}{A})\cos(\frac{y}{B}),$$
$$v(x,y)=-B\cos(\frac{x}{A})\sin(\frac{y}{B}),$$
$$p(x,y)=\frac{x}{A}\frac{y}{B}.$$

The V-cycle is used with 2 pre–relaxations, 1 post–relaxation in the multigrid solver and the relaxation parameter $\beta = 0.8$. The multigrid procedure converges when $\rho\ geq 1/4$ but

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the convergence is very slow when $\rho = 1/4$ and it even diverges when $\rho \leq 1/8$ (we can get convergence when $\rho \geq 1/8$ by using smaller relaxation parameter β but the convergence is always very slow).

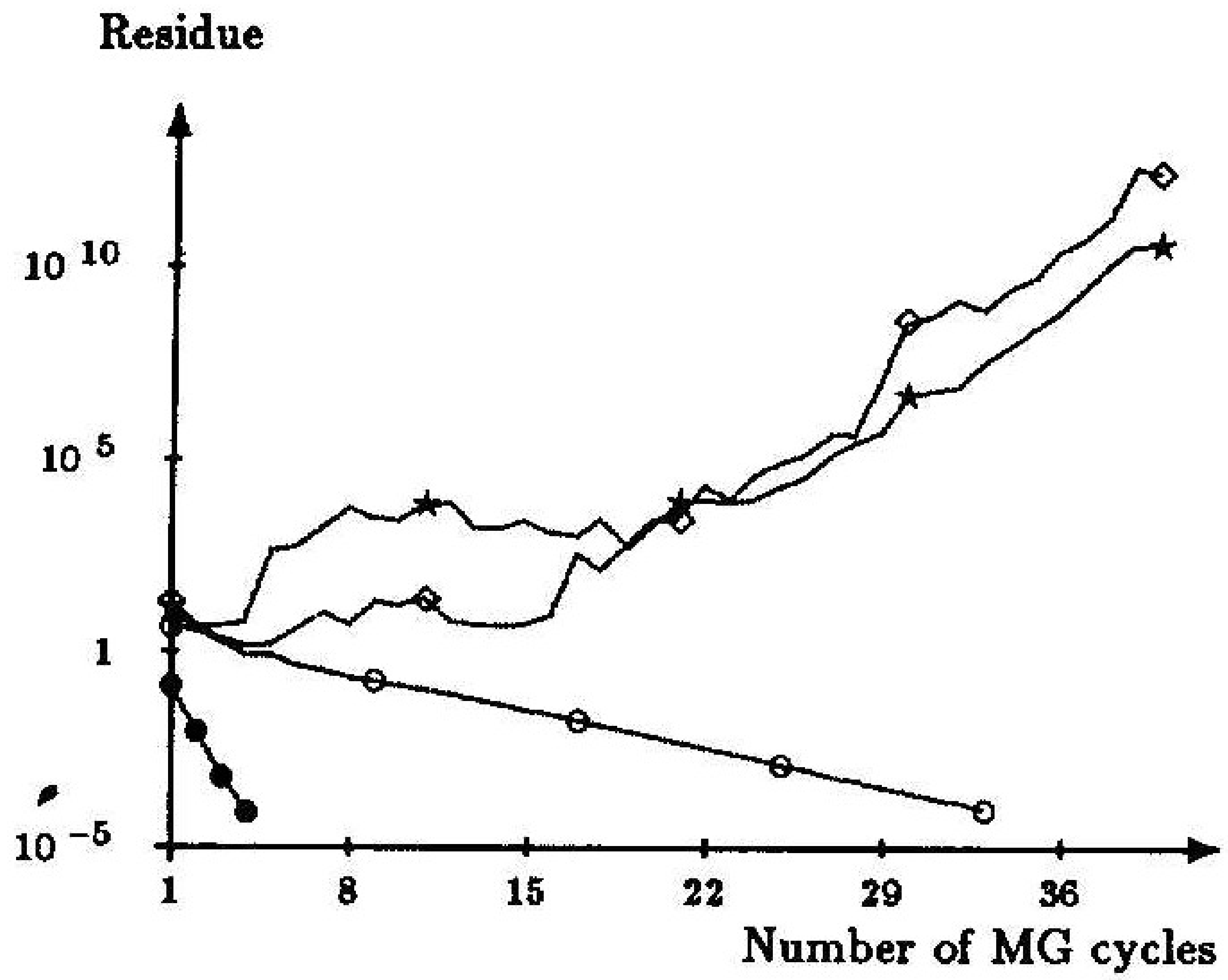

Figure 1: Convergence rate of the MG solver for the test problem ($R = 100$) with different values of ρ :
$\bullet : \rho = 1$ $\circ : \rho = 1/4$ $\star : \rho = 1/8$ $\diamond : \rho = 1/16$

The convergence rate of multigrid solvers depends essentially on the smoothing properties of the relaxation procedure (also called smoothing operator). The SCGS relaxation has better smoothing properties when grid cells are nearly square. An easy way to improve the convergence rate when $\rho \gg 1$ or $\rho \ll 1$ is to use coarse grids obtained by increasing only one grid step in one space direction, instead of increasing the grid steps in all directions, in the multigrid procedure (this approach was also proposed by Hackbusch[2] for solving anisotropic problems). The object of this study is to investigate the efficiency of the semi-coarsening in multigrid solution of steady Navier–Stokes equations.

In the present paper, we will first recall briefly the multigrid solver presented in [5]. Then we give details of the implementation of the coarsening procedure and corresponding numerical results.

Remark. Vanka has also done some numerical experiments with the second–order upwind differencing. Contrast to our conclusions given in [1] and [5], he has observed very slow convergence of his multigrid solver combined with the second–order upwind differencing and no improvement in the precision of approximation with regard to the hybrid scheme, when Reynolds number is greater than or equal to 600 (see [4]). There are several differences between his scheme and the ours which may be the cause of slow convergence and poor precision of his scheme :

1. He wrote the convection terms discretized by second–order upwinding finite differences as the corresponding first–order upwinding discretization multiplied by a constant plus a

correction term. He put the correction terms at the right hand side of the equation at *each iteration, i.e., he computed them using known values. So his solution procedure is* in some sense a preconditioned Richardson scheme using the first-order upwinding scheme as preconditioning. In our scheme the multigrid procedure and the SCGS relaxation are applied directly to the discrete system obtained by second-order differencing.

2. He imposed the zero normal derivative condition to the pressure on construct the prolongation operator at points near the boundaries. In our scheme, the pressure on points near the boundaries are prolonged by extrapolations.

3. The third difference is in the discretization of the convection terms used on points near the boundaries. Vanka used the first-order upwinding differencing while we pass to centered differencing when points outside the computational region have to be used. We think that this difference should not slow down the convergence rate, but it can have influences on the precision of the discrete solution.

§1. Discretization of the Governing Equations and the Multigrid Solver

We use the 2D incompressible Navier-Stokes equations in their primitive formulation :

$$\begin{cases} -\frac{1}{R}\Delta u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = f_1, \\ -\frac{1}{R}\Delta v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = f_2, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases}$$

where (u, v) represents the velocity, p the pressure and (f_1, f_2) the external force. $R > 0$ is the Reynolds number.

These equations are discretized on staggered grids in which the values of u are taken at the centers of the two vertical sides of each grid cell, the values of v at the centers of the two horizontal sides of each cell and the values of p at the center of each cell.

The linear parts of the equations are approximated by usual second-order centered differencing. For the nonlinear terms, second-order upwind differencing is used. For example, $u\frac{\partial u}{\partial x}|_{(x_0,y_0)}$ is discretized by :

$$u(x_0, y_0)\frac{3u(x_0, y_0) - 4u(x_0 - h_x, y_0) + u(x_0 - 2h_x, y_0)}{2h_x} \qquad \text{if } u(x_0, y_0) \geq 0,$$

or by

$$-u(x_0, y_0)\frac{3u(x_0, y_0) - 4u(x_0 + h_x, y_0) + u(x_0 + 2h_x, y_0)}{2h_x} \qquad \text{if } u(x_0, y_0) < 0,$$

where h_x is the grid step in x direction.

The resulting nonlinear system is solved by an iterative method called SCGS iteration (Symmetrical Collective Gauss-Seidel iteration), proposed by S. P. Vanka[3]. In this method, five unknowns on a grid cell (comprising two horizontal velocities, two vertical velocities and one pressure) and the five corresponding discrete equations (four momentum equations and one continuity equation) are relaxed simultaneously by solving a system of linear equations with five unknowns (this linear system is obtained by linearizing locally the corresponding equations). To ensure and accelerate the convergence, the velocities are underrelaxed by introducing an underrelaxation factor $\beta \in (0,1]$. One SCGS relaxation consists of one scanning of all grid cells in a certain order. This iteration procedure has been shown to be very efficient for solving the discrete system obtained by hybrid differencing by Vanka in [3], and also for solving the discrete system obtained by second-order upwind differencing by our numerical experiences reported in [5].

To use multigrid techniques to accelerate the convergence rate of the above iteration procedure, a sequence of coarse grids is constructed by doubling the grid steps h_x and h_y each time. The approximate solution on a grid is relaxed by several SCGS iterations, then it is transferred to a coarser grid using an interpolation operator called restriction operator which is constructed by linear interpolations. Another transfer operator, called prolongation operator, from a grid to a finer one constructed by linear interpolations or extrapolations, is used to correct the approximate solution of the finer grid using the solution obtained on the coarser grid. For details of multigrid methods, see, for example, Hackbusch[2].

The SCGS iteration combined with the multigrid techniques provides a rapidly convergent solver for Navier–Stokes equations in primitive variables discretized by hybrid or first or second-order upwinding finite differences, when grid cells are nearly square (i. e., when $h_x/h_y \sim 1$). Detailed implementation of the MG solver and corresponding numerical results obtained for the driven cavity problem can be found in [5].

§2. Alternative Coarsening Procedure

Since the convergence rate of the multigrid solver based on the SCGS iteration depends heavily on the grid step ratio $\rho = h_x/h_y$, we can construct coarse grids by doubling only the smaller of h_x and h_y instead of doubling them simultaneously to make ρ as close to 1 as possible on coarse grids to ameliorate the convergence rate of the multigrid procedure. The use of semi-coarsening in one space direction needs more computational work in each MG cycle because there are more level of grids and more grid points on coarse grids, but it is largely compensated by the gain obtained in the convergence rate.

To describe our coarsening procedure, let the computational region be the rectangle $(0,A) \times (0,B)$ and let A/N and B/M be the steps of the finest grid with $N, M \in \mathbb{N}$. Suppose $N = N_0 2^J$ and $M = M_0 2^K$ where N_0 and M_0 are odd or equal to 2. Suppose also that the current grid steps are $h_x = A/(N_0 2^j)$ and $h_y = B/(M_0 2^k)$, then the next coarse grid is determined by the following rules :

$$\left[\begin{array}{lcl} j = 0 \text{ and } k = 0 & \Rightarrow & \text{This is the coarsest grid} \\ j > 0 \text{ and } k = 0 & \Rightarrow & \text{Coarsening in x direction} \\ j = 0 \text{ and } k > 0 & \Rightarrow & \text{Coarsening in y direction} \\ j > 0 \text{ and } k > 0 & \Rightarrow & \left[\begin{array}{lcl} h_y > \delta h_x & \Rightarrow & \text{Coarsening in x direction} \\ h_x > \delta h_y & \Rightarrow & \text{Coarsening in y direction} \\ h_x \le \delta h_y \text{ and } h_y \le \delta h_x & \Rightarrow & \text{Coarsening in both directions} \end{array}\right. \end{array}\right.$$

where δ is a control parameter greater than 1. In our numerical experiments, it is set to $\sqrt{2}$.

§3. Numerical Results

We have solved the same test problem used in the introduction using the coarsening procedure described in the previous section; we will call it *semi-coarsening procedure* while multigrid procedures in which no semi-coarsening is used will be referred to as *total-coarsening procedures.*

Figure 2 and Figure 3 show the convergence history for Reynolds number $R = 100$ and 400 respectively. The relaxation parameter β is equal to 0.8 for $R = 100$ and 0.6 for $R = 400$.

Because the number of pre-relaxations and post-relaxations is fixed in the multigrid procedure, the CPU time needed for one multigrid iteration is nearly constant. In Table 1, we list the CPU time for performing one multigrid iteration (on an IBM/PC) and the number of multigrid iterations to get the residual less than 1×10^{-4} for $R = 100$.

We conclude with the remark that with the semi-coarsening procedure about 50% more CPU time is needed for each iteration, but the number of iterations is magically reduced, so it provides an efficient multigrid solver for steady Navier-Stokes equations on grids with reasonable aspect ratio of grid cells.

Table 1. CPU time per iteration and number of iterations to have the residual $< 1 \times 10^{-4}$ ($R = 100$)

	Total-coarsening		Semi-coarsening	
	CPU time/it.	# of iterations	CPU time/it.	# of iterations
$h_y : h_x = 1$	233 seconds	4	–	–
$h_y : h_x = 4$	234 seconds	33	327 seconds	5
$h_y : h_x = 8$	236 seconds	diverged	346 seconds	6
$h_y : h_x = 16$	236 seconds	diverged	357 seconds	13

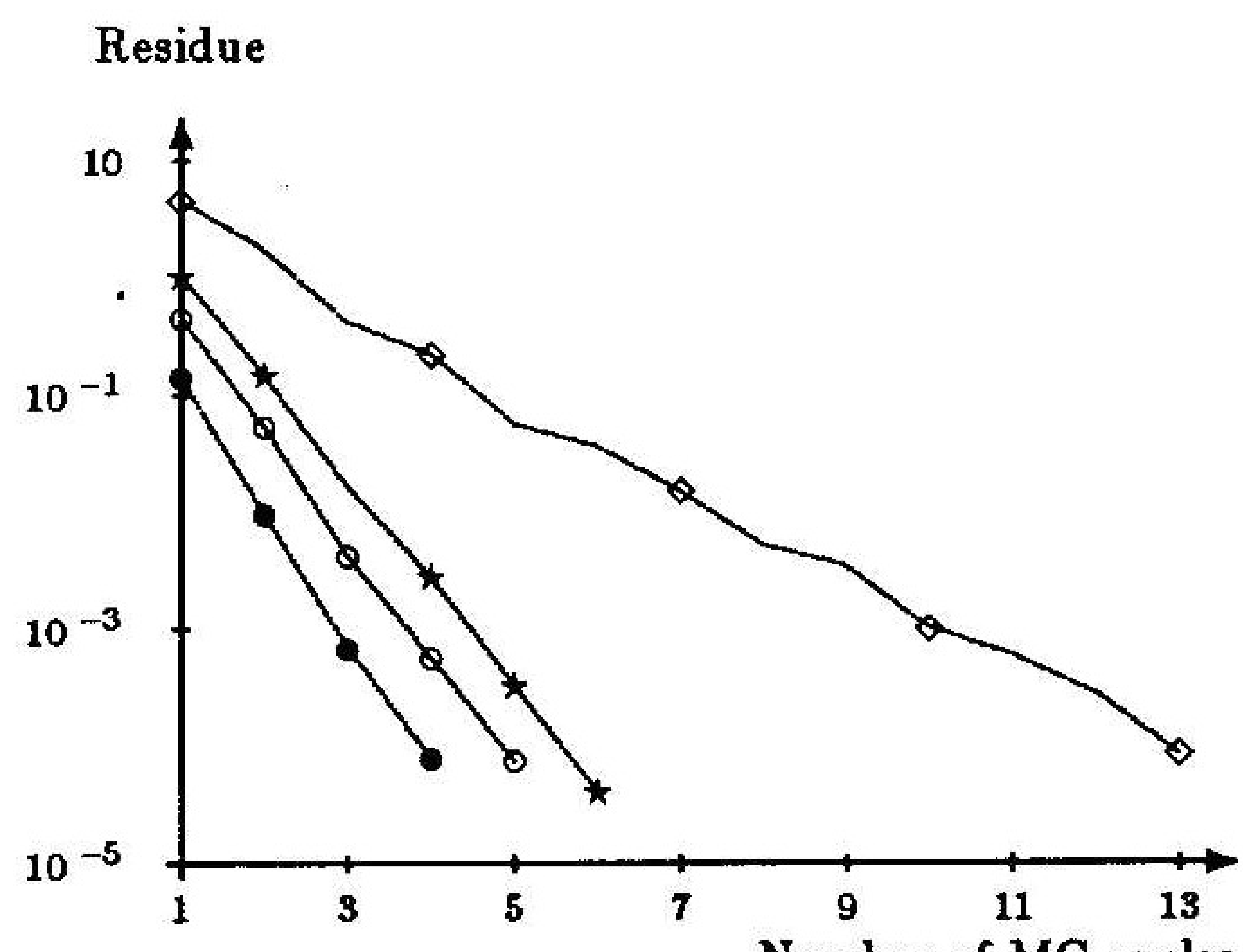

Figure 2: Convergence rate of the MG solver with semi-coarsening for the test problem ($R = 100$) with different values of ρ :
$\bullet : \rho = 1$ $\circ : \rho = 1/4$ $\star : \rho = 1/8$ $\diamond : \rho = 1/16$

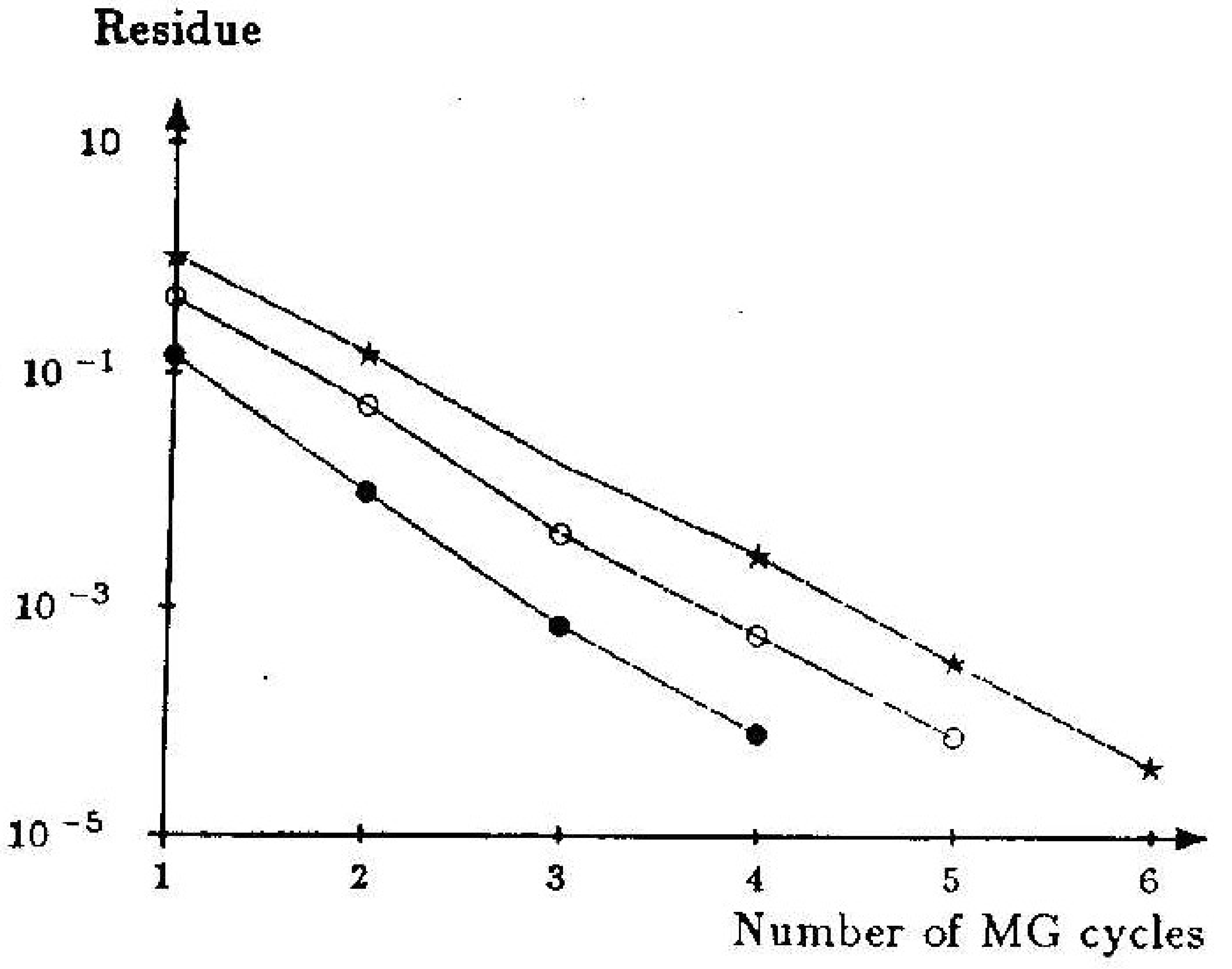

Figure 3: Convergence rate of the MG solver with semi–coarsening for the test problem ($R = 400$) with different values of ρ :
$\bullet : \rho = 1$ $\circ : \rho = 1/4$ $\star : \rho = 1/8$ $\diamond : \rho = 1/16$

References

[1] Ch. H. Bruneau, C. Jouron & L. B. Zhang Multigrid Solvers for Steady Navier–Stokes Equations in a Driven Cavity 11th International Conference on Numerical Methods in Fluid Dynamics, Williamsburg, Virginia, 1988.

[2] Hackbusch Multi–grid Methods and Applications Springer–Verlag, 1985.

[3] S. P. Vanka Block implicit multigrid aolutions of Navier-Stokes equations in primitive variables *J. of Comp. Phys.*, 65, 1985.

[4] S. P. Vanka Second–order upwind differencing in a recirculating Flow *AIAA Journal*, Vol. 25, No. 11, 1987.

[5] L. B. Zhang A Second-order upwinding finite difference scheme for the steady Navier-Stokes equations in primitive variables in a driven cavity with a multigrid solver submitted to *Mathematical Modelling and Numerical Analysis, RAIRO.*