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CONSTRUCTION AND ANALYSIS OF A NEW ENERGY -ORTHOGONAL UNCONVENTIONAL PLATE ELEMENT*

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Abstract

The paper describes an interpolation procedure of formulating shape functions for a new energy-orthogonal plate element. Sample problems using the new element show satisfactory numerical results.

§1. Introduction

Bergan et al.[1,2] have recently proposed the "free formulation" scheme of unconventional finite element methods. The element stiffness matrix consists of two separate parts: $K = K_{rc} + K_h$, where K_{rc} corresponds to constant strain modes of shape functions and is independent of any form of high order modes, while K_h is determined by high order modes based on a conventional energy consideration[3]. The TRUNC element developed by Argyris et al.[3] is an example of Bergan's free formulation scheme, which is proved to be convergent for arbitrary mesh partitions[4]. Reference [5] provides a mathematical explanation of the free formulation scheme. It is observed that the scheme actually leads to a nonconforming element method with a specific form of interpolation of shape functions. Reference [6] gives a detailed mathematical analysis for Bergan's energy-orthogonal element based on the free formulation [2]. Its convergence together with error estimates are derived and a modification of Bergan's element with better convergence properties is proposed.

Bergan's free formulation scheme has been stated in [1, 2] by mechanical considerations. The derivation of the stiffness matrix K_{rc} corresponding to constant strain modes, however, appears somewhat difficult of access. While the analysis in [5] shows that the matrix K_{rc} is identical with the matrix resulting from the constant strain modes of Zienkiewicz's incompatible cubic element, the reason for choice of this particular matrix as K_{rc} regardless of any form of high order modes is still not clear at least from a view-point of mathematics.

The purpose of this paper is to present a modified scheme of free formulation in accordance with a simple convergence requirement of nonconforming finite elements. The element stiffness matrix formulated by this modified scheme is again consisting of two separate parts, one corresponds to constant strain modes and the other to high order modes of shape functions. However, the stiffness matrix K_{rc} now is simply derived from the convergence requirement. It seems a more direct way of derivation of K_{rc} than that in Bergan's scheme. The treatment of high order modes leaves the same as before, using the conventional method. Starting from the shape function space of Bergan's energy-orthogonal element, the modified scheme provides a new energy-orthogonal element. Numerical experiments show that this new element gives more accurate results than Bergan's. The convergence proof as

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well as the error estimates are derived. Along the line of this paper a general nine parameter unconventional element, not necessarily energy-orthogonal, may be constructed, which will be analyzed in another paper.

§2. Formulation of Shape Functions and the Element Stiffness Matrix

Given a triangle K with the vertices $p_i = (x_i, y_i)$, the area Δ and the diameter $h_K \le h$, we denote by λ_i the area coordinates for the triangle K and put

$$\xi_1 = x_2 - x_3,\quad \xi_2 = x_3 - x_1,\quad \xi_3 = x_1 - x_2,\quad \eta_1 = y_2 - y_3,\quad \eta_2 = y_3 - y_1,\quad \eta_3 = y_1 - y_2,$$

$$F_i^2 = \xi_i^2 + \eta_i^2,\quad t_i = F_i^2/\Delta,\quad r_i = (\xi_j\xi_k + \eta_j\eta_k)/\Delta,\quad e_{ij} = \frac{r_i}{t_j},$$

$$i, j, k = 1, 2, 3,\quad j, k \ne i,\quad j \ne k.$$

The nodal parameters are the function values and the two first derivatives of w at the vertices p_i, which are denoted by

$$w = (w_1, w_{1x}, w_{1y}, w_2, w_{2x}, w_{2y}, w_3, w_{3x}, w_{3y})^T. \tag{2.1}$$

The space of shape functions under consideration is of the form

$$P(K) = \text{span}\,\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, N_7, N_8, N_9\}, \tag{2.2}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_3\lambda_1$ are constant strain modes and N_7, N_8, N_9 are high order modes. Every function $w \in P(K)$ may be written in the form

$$w = \bar{w} + w' \tag{2.3}$$

with

$$\bar{w} = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1,$$

$$w' = b_7N_7 + b_8N_8 + b_9N_9,$$

representing a constant strain term and a high order term, respectively.

In order to determine $\bar{w}$ we use an interpolation technique like the treatment of Morley's element[7]. Let the function value of $\bar{w}$ at the vertex p_i be identical with that of w at the same vertex, and the normal derivative of $\bar{w}$ at the middle point of one side be identical with the average of normal derivatives of w at the two end points of the side, i.e.

$$\bar{w}(p_i) = w_i,$$

$$\frac{\partial \bar{w}}{\partial n_i}(p_{jk}) = \frac{1}{2}\Big[\Big(\frac{\partial w}{\partial n_i}\Big)_j + \Big(\frac{\partial w}{\partial n_i}\Big)_k\Big] = \frac{1}{2F_i}\Big[-(w_{jx} + w_{kx})\eta_i + (w_{jy} + w_{ky})\xi_i\Big], i = 1, 2, 3, \tag{2.4}$$

where n_i denotes the unit outward normal vector of the side p_jp_k, opposite to the vertex p_i, and p_{jk} is the middle point of p_ip_k.

The interpolation conditions (2.4) uniquely determine the six coefficients $a_i, i = 1, \cdots, 6$, of $\bar{w}$ as follows:

$$
\begin{aligned}
a_i &= w_i, \quad i = 1,2,3, \\
a_4 &= e_{12}w_1 + e_{21}w_2 - (e_{12}+e_{21})w_3 + \frac{\eta_2}{t_2}w_{1x} + \frac{\eta_1}{t_1}w_{2x} \\
&\quad + \left(\frac{\eta_1}{t_1}+\frac{\eta_2}{t_2}\right)w_{3x} - \frac{\xi_2}{t_2}w_{1y} - \frac{\xi_1}{t_1}w_{2y} - \left(\frac{\xi_1}{t_1}+\frac{\xi_2}{t_2}\right)w_{3y}, \\
a_5 &= -(e_{23}+e_{32})w_1 + e_{23}w_2 + e_{32}w_3 + \left(\frac{\eta_2}{t_2}+\frac{\eta_3}{t_3}\right)w_{1x} + \frac{\eta_3}{t_3}w_{2x} \\
&\quad + \frac{\eta_2}{t_2}w_{3x} - \left(\frac{\xi_2}{t_2}+\frac{\xi_3}{t_3}\right)w_{1y} - \frac{\xi_3}{t_3}w_{2y} - \frac{\xi_2}{t_2}w_{3y}, \\
a_6 &= e_{13}w_1 - (e_{13}+e_{31})w_2 + e_{31}w_3 + \frac{\eta_3}{t_3}w_{1x} + \left(\frac{\eta_3}{t_3}+\frac{\eta_1}{t_1}\right)w_{2x} \\
&\quad + \frac{\eta_1}{t_1}w_{3x} - \frac{\xi_3}{t_3}w_{1y} - \left(\frac{\xi_3}{t_3}+\frac{\xi_1}{t_1}\right)w_{2y} - \frac{\xi_1}{t_1}w_{3y}.
\end{aligned}
\tag{2.5}
$$

(2.5) may be written in a matrix form

$$
\mathbf{a} = H_{rc}\mathbf{w} \quad , \quad \mathbf{a} = (a_1, a_2, \cdots, a_6)^T, \tag{2.6}
$$

where H_{rc} is a 6 by 9 coefficient matrix of (2.5), the interpolation matrix of $\bar{w}$, by which the constant strain term $\bar{w}$ is completely determined. In particular, if w is a quadratic polynomial, then $\bar{w} = w$.

The interpolation conditions (2.4) resemble those of Morley's element. Indeed, it will be shown that imposing these conditions on $\bar{w}$ is an essential step to make the element convergent.

Notice that the determination of $\bar{w}$ is independent of any form of the high order term w' that is quite different from the usual procedure of formulation of shape functions.

The high order term w' is determined by a usual interpolation process. For instance, if the three high order modes are chosen to be those of Bergan's energy-orthogonal element, i.e.

$$
N_7 = (\lambda_1 - \lambda_2)^3, \quad N_8 = (\lambda_2 - \lambda_3)^3, \quad N_9 = (\lambda_3 - \lambda_1)^3,
$$

and if we let

$$
\begin{aligned}
w = {} & b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_1\lambda_2 + b_5\lambda_2\lambda_3 + b_6\lambda_3\lambda_1 + b_7(\lambda_1 - \lambda_2)^3 \\
& + b_8(\lambda_2 - \lambda_3)^3 + b_9(\lambda_3 - \lambda_1)^3 = \bar{w} + w',
\end{aligned}
$$

then, using the nine nodal parameters $\mathbf{w}$, the coefficients $b_i, i = 1, 2, \cdots, 9$, can be uniquely determined. The three last coefficients in w' are as follows:

$$
\begin{aligned}
b_7 &= -\frac{2}{9}(w_1 - w_2) + \frac{1}{54}\Big[(6\xi_3 - \xi_1)w_{1x} + (6\xi_3 - \xi_2)w_{2x} - \xi_3 w_{3x}\Big] \\
&\quad + \frac{1}{54}\Big[(6\eta_3 - \eta_1)w_{1y} + (6\eta_3 - \eta_2)w_{2y} - \eta_3 w_{3y}\Big], \\
b_8 &= -\frac{2}{9}(w_2 - w_3) + \frac{1}{54}\Big[-\xi_1 w_{1x} + (6\xi_1 - \xi_2)w_{2x} + (6\xi_1 - \xi_3)w_{3x}\Big] \\
&\quad + \frac{1}{54}\Big[-\eta_1 w_{1y} + (6\eta_1 - \eta_2)w_{2y} + (6\eta_1 - \eta_3)w_{3y}\Big], \\
b_9 &= -\frac{2}{9}(w_3 - w_1) + \frac{1}{54}\Big[(6\xi_2 - \xi_1)w_{1x} - \xi_2 w_{2x} + (6\xi_2 - \xi_3)w_{3x}\Big] \\
&\quad + \frac{1}{54}\Big[(6\eta_2 - \eta_1)w_{1y} - \eta_2 w_{2y} + (6\eta_2 - \eta_3)w_{3y}\Big].
\end{aligned}
\tag{2.7}
$$

Writing (2.7) in a matrix form gives

$$
b' = H_h \boldsymbol{w} \quad , \quad b' = (b_7, b_8, b_9)^T, \tag{2.8}
$$

where H_h is a 3 by 9 coefficient matrix of (2.7), the interpolation matrix of w', which defines the high order term w'. Again, if w is a quadratic polynomial, then $b_7 = b_8 = b_9 = 0$, i.e. $w' = 0$.

Combining (2.6) and (2.8) together in one matrix equation, we have

$$
t = H\boldsymbol{w}, \tag{2.9}
$$

where

$$
H = \begin{pmatrix} H_{rc} \\ H_h \end{pmatrix} \quad , \quad t = \begin{pmatrix} a \\ b' \end{pmatrix}.
$$

Evaluation shows that

$$
\det H = -\frac{4\Delta^3}{243},
$$

hence H is nonsingular. Therefore, formula (2.9) defines an interpolation operator Π_K on K such that

$$
\Pi_K : \quad w \in H^3(K) \to \Pi_K w \in P(K). \tag{2.10}
$$

More precisely, every function $w \in H^3(K)$ is associated with a set of parameters $\boldsymbol{w} = (w_1, w_{1x}, w_{1y}, w_2, w_{2x}, w_{2y}, w_3, w_{3x}, w_{3y})^T$, which defines a set of coefficients t by (2.9). Then a cubic polynomial $\Pi_K w \in P(K)$ is derived from (2.3). Conversely, for a given cubic polynomical $\Pi_K w$ with the coefficient set t, since the matrix H in (2.9) is nonsingular, there exists a corresponding set of parameters $\boldsymbol{w}$, by which a cubic polynomial $w \in P(K)$ may be uniquely determined. The nodal parameters of w at the vertices of K are identical with $\boldsymbol{w}$. Therefore, $w \in P(K)$ and $\Pi_K w \in P(K)$ are in one-to-one correspondence.

Notice that, instead of a unified interpolation procedure in formulation of shape functions as in a conventional finite element method, here, like Bergan's scheme, we have used two different interpolations independent of each other to formulate the constant strain term $\bar{w}$ and the high order term w' of the shape function w. Thus we obtain an unconventional method of formulation of shape functions. It differs, however, from Bergan's scheme in the derivation of the constant strain term using a rather simple convergence requirement.

Now, taking $\Pi_K w$ as the shape function on K and w as its associated nodal parameters, the usual procedure of formulation of an element stiffness matrix gives

$$K_e = H^T K_q H = H_{rc}^T K_{qrc} H_{rc} + H_{rc}^T K_{qrch} H_h + H_h^T K_{qrch}^T H_{rc} + H_h^T K_{qh} H_h, \qquad (2.11)$$

where

$$K_q = \begin{pmatrix} K_{qrc} & K_{qrch} \\ K_{qrch}^T & K_{qh} \end{pmatrix}, \quad K_{qrc} = \int_K B_{rc}^T D B_{rc} dx dy,$$

$$K_{qrch} = \int_K B_{rc}^T D B_h dx dy, \quad K_{qh} = \int_K B_h^T D B_h dx dy,$$

$$B_{rc} = \begin{pmatrix} \partial_{xx} N_1, \cdots, \partial_{xx} N_6 \\ \partial_{yy} N_1, \cdots, \partial_{yy} N_6 \\ 2\partial_{xy} N_1, \cdots, 2\partial_{xy} N_6 \end{pmatrix}, \quad B_h = \begin{pmatrix} \partial_{xx} N_7, \cdots, \partial_{xx} N_9 \\ \partial_{yy} N_7, \cdots, \partial_{yy} N_9 \\ 2\partial_{xy} N_7, \cdots, 2\partial_{xy} N_9 \end{pmatrix},$$

$$D \triangleq \begin{pmatrix} 1 & \sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & \frac{1-\sigma}{2} \end{pmatrix},$$

$N_1 = \lambda_1, N_2 = \lambda_2, N_3 = \lambda_3, N_4 = \lambda_1\lambda_2, N_5 = \lambda_2\lambda_3, N_6 = \lambda_3\lambda_1$, $0 < \sigma < \frac{1}{2}$ is the Poisson ratio. It is worth mentioning that the values of the nodal parameter set w do not, in general, agree with the function values as well as the two first derivatives of the shape function $\Pi_K w$ at the vertices of K.

Decomposing the shape function $\Pi_K w$ into two parts

$$\Pi_K w = \hat{\Pi}_K w + \Pi'_K w \qquad (2.12)$$

with $\hat{\Pi}_K w$ representing the constant strain term and $\Pi'_K w$ the high order term, since $\hat{\Pi}_K w$ and $\Pi'_K w$ are energy-orthogonal on K and thus the off-diagonal submatrices K_{qrch} and K_{qrch}^T in K_q vanish, we obtain the element stiffness matrix as follows:

$$K_e = H_{rc}^T K_{qrc} H_{rc} + H_h^T K_{qh} H_h = K_{rc} + K_h, \qquad (2.13)$$

$$K_q = \begin{pmatrix} K_{qrc} & 0 \\ 0 & K_{qh} \end{pmatrix},$$

where K_{rc} corresponds to $\hat{\Pi}_K w$ and K_h to $\Pi'_K w$; there are no coupling terms in K_e.

Application of this new element has been done for sample plate bending problems. Numerical results (see Tables 1–4) show its advantages over Bergan's energy-orthogonal element.

§3. Convergence Analysis

Theorem 1. *For every function $w \in H^3(K)$ the following interpolation inequalities hold:*

$$|w - \Pi_K w|_{m,K} \le C h_K^{3-m} |w|_{3,K}, \quad m = 0,1,2, \tag{3.1}$$

$$|\Pi'_K w|_{m,K} \le C h_K^{3-m} |w|_{3,K}, \tag{3.2}$$

$$|\Pi'_K w|_{m,K} \le C h_K^{3-m} |\Pi_K w|_{3,K}, \quad 0 \le m \le 3, \tag{3.3}$$

$$|w - \Pi_K w|_{m,K} \le C h_K^{3-m} |w|_{3,K} \tag{3.4}$$

where C is a generic constant independent of w and h_K.

Proof. In view of the definition the operator Π_K is related to the normal derivatives of the shape function at the middle points of the sides of K, hence Π_K is not an affine family[8]. However, using the same argument as for nonconforming Morley's element in [7], it can be shown that Π_K is an almost affine family. For this, we introduce a new interpolation operator

$$\tilde{\Pi}_K : w \in H^3(K) \to \tilde{\Pi}_K w \in P_2(K),$$

where $P_2(K)$ is the quadratic polynomial space on K, such that

$$\tilde{\Pi}_K w(p_i) = w_i,$$

$$[\overrightarrow{D\tilde{\Pi}_K w} \cdot \vec{m}_i] \equiv \frac{1}{F_i}\int_{p_j p_k} \overrightarrow{D\tilde{\Pi}_K w} \cdot \vec{m}_i ds = \frac{1}{F_i}\int_{p_j p_k} \overrightarrow{Dw} \cdot \vec{m}_i ds, \tag{3.5}$$

$$i,j,k = 1,2,3, \quad j,k \ne i, \quad j \ne k,$$

with

$$\overrightarrow{Dg} = (\partial_x g, \partial_y g)^T, \quad \vec{m}_i = \overrightarrow{p_i p_{jk}}.$$

Evidently, $\tilde{\Pi}_K$ is an affine family and $\tilde{\Pi}_K w = w$ for every $w \in P_2(K)$. According to the interpolation theory[8], we have

$$|w - \tilde{\Pi}_K w|_{m,K} \le C h_K^{3-m} |w|_{3,K}, \quad m = 0,1,2, \tag{3.6}$$

$$|w - \tilde{\Pi}_K w|_{1,\infty,K} \le C h_K |w|_{3,K}. \tag{3.7}$$

Both $\Pi_K w$ and $\tilde{\Pi}_K w$ are quadratic polynomials and, moreover, by the interpolation conditions,

$$\Pi_K w(p_i) = \tilde{\Pi}_K w(p_i) = w_i.$$

Therefore (see [7])

$$\begin{aligned} \Pi_K w - \tilde{\Pi}_K w = {} & \Big(\frac{\partial \Pi_K w}{\partial n_1} - \frac{\partial \tilde{\Pi}_K w}{\partial n_1}\Big)(p_{23})\psi_1 + \Big(\frac{\partial \Pi_K w}{\partial n_2} - \frac{\partial \tilde{\Pi}_K w}{\partial n_2}\Big)(p_{31})\psi_2 \\ & + \Big(\frac{\partial \Pi_K w}{\partial n_3} - \frac{\partial \tilde{\Pi}_K w}{\partial n_3}\Big)(p_{12})\psi_3, \end{aligned} \tag{3.8}$$

where

$$\psi_i = -\frac{2\Delta}{F_i}\lambda_i(1-\lambda_i), \quad i = 1,2,3.$$

It is easily seen that

$$|\psi_i|_{m,K} \le Ch_K^{2-m}, \quad m = 0, 1, 2. \tag{3.9}$$

By the interpolation conditions

$$\frac{\partial \Pi_K w}{\partial n_i}(p_{jk}) = \frac{1}{2}\Big[\Big(\frac{\partial w}{\partial n_i}\Big)_j + \Big(\frac{\partial w}{\partial n_i}\Big)_k\Big],$$

and since $\frac{\partial \tilde{\Pi}_K w}{\partial n_i}$ is a linear function on $p_j p_k$,

$$\frac{\partial \tilde{\Pi}_K w}{\partial n_i}(p_{jk}) = \frac{1}{2}\Big[\Big(\frac{\partial \tilde{\Pi}_K w}{\partial n_i}\Big)_j + \Big(\frac{\partial \tilde{\Pi}_K w}{\partial n_i}\Big)_k\Big],$$

applying inequality (3.7) gives

$$\begin{aligned}\Big|\Big(\frac{\partial \Pi_K w}{\partial n_i} - \frac{\partial \tilde{\Pi}_K w}{\partial n_i}\Big)(p_{jk})\Big| &= \frac{1}{2}\Big|\Big(\frac{\partial (w - \tilde{\Pi}_K w)}{\partial n_i}\Big)_j + \Big(\frac{\partial (w - \tilde{\Pi}_K w)}{\partial n_i}\Big)_k\Big| \\ &\le C|w - \tilde{\Pi}_K w|_{1,\infty,K} \le Ch_K |w|_{3,K}.\end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we obtain

$$|\Pi_K w - \tilde{\Pi}_K w|_{m,K} \le Ch_K^{3-m}|w|_{3,K}, \quad m = 0, 1, 2.$$

Then, the triangular inequality and inequality (3.6) yield

$$\begin{aligned}|w - \Pi_K w|_{m,K} &\le |w - \tilde{\Pi}_K w|_{m,K} + |\tilde{\Pi}_K w - \Pi_K w|_{m,K} \\ &\le Ch_K^{3-m}|w|_{3,K}, \quad m = 0, 1, 2,\end{aligned}$$

which is inequality (3.1).

As regards inequalities (3.2) and (3.3), results of [6] show that coefficients b_7, b_8, b_9 of $\Pi'_K w$ satisfy

$$|b_i| \le Ch_K^2 |w|_{3,K}, \quad |b_i| \le Ch_K^2 |\Pi'_K w|_{3,K} = Ch_K^2 |\Pi_K w|_{3,K},$$

and

$$|N_i|_{m,K} \le Ch_K^{1-m}, \quad 0 \le m \le 3, \quad i = 7, 8, 9,$$

which imply the validity of (3.2) and (3.3).

From (3.1) and (3.2) inequality (3.4) follows immediately.

Now let V_h be the finite element space on $\Omega = \cup K$. On each triangle K the shape function of V_h is the interpolation operator $\Pi_K w$ with w as its associated nodal parameters, vanishing at vertices on the boundary $\partial\Omega$. We apply Stummel's generalized patch test to establish the convergence property of the finite element space V_h. Following [9], for a fourth order problem the generalized patch test consists in verifying that as $h \to 0$, the relations

(i) $T_l(\psi, w_h) = \sum_K \int_{\partial K} \psi w_h n_l ds \to 0, \quad l = 1, 2,$

(ii) $T_{rl}(\psi, w_h) = \sum_K \int_{\partial K} \psi \frac{\partial w_h}{\partial x_r} n_l ds \to 0, \quad r, l = 1, 2$

hold for every bounded sequence $w_h \in V_h$ and for all test functions $\psi \in C_0^\infty(\Omega)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$ in the case of Dirichlet boundary conditions), where n_l are the components of the unit outward normal vector on ∂K and x_r are the cartesian coordinates of $\mathbb{R}^2$.

Theorem 2. *The finite element space V_h defined above passes the generalized patch test.*

Proof. Every function $w_h \in V_h$ may be decomposed into two parts

$$w_h = \bar{w}_h + w_h',$$

where $\bar{w}_h$ is a piecewise quadratic polynomial and w_h' a piecewise cubic one. Then

$$T_l(\psi, w_h) = T_l(\psi, \bar{w}_h) + T_l(\psi, w_h').$$

By the definition $\bar{w}_h$ is continuous at the vertices of K, hence the piecewise linear interpolation $P_1\bar{w}_h$ on Ω is a continuous function, vanishing on the boundary $\partial\Omega$. The remainder term

$$R_1\bar{w}_h = \bar{w}_h - P_1\bar{w}_h$$

satisfies

$$T_l(\psi, \bar{w}_h) = T_l(\psi, R_1\bar{w}_h) = \sum_K \int_{\partial K} \psi R_1\bar{w}_h n_l ds.$$

Application of Schwarz inequality and the interpolation theory leads to

$$\left| \int_{\partial K} \psi R_1\bar{w}_h n_l ds \right| \le \left(\int_{\partial K} \psi^2 ds \right)^{\frac12} \left(\int_{\partial K} (R_1\bar{w}_h)^2 ds \right)^{\frac12} \le C h_K \|\psi\|_{1,K} |\bar{w}_h|_{2,K}.$$

Since $\bar{w}_h$ and w_h' are energy orthogonal on K,

$$|w_h|_{2,K}^2 = |\bar{w}_h|_{2,K}^2 + |w'|_{2,K}^2$$

and so

$$|\bar{w}_h|_{2,K} \le |w_h|_{2,K},$$

therefore

$$|T_l(\psi, w_h)| \le \sum_K \left| \int_{\partial K} \psi R_1\bar{w}_h n_l ds \right| \le C h \|\psi\|_1 |w_h|_{2,h}, \tag{3.11}$$

where

$$|w_h|_{2,h} = \left(\sum_K |w_h|_{2,K}^2 \right)^{\frac12}.$$

As for

$$T_l(\psi, w_h') = \sum_K \int_{\partial K} \psi w_h' n_l ds,$$

using inequality (3.3), the imbedding theorem and the inverse inequality, we obtain

$$\left| \int_{\partial K} \psi w_h' n_l ds \right| \le \left(\int_{\partial K} \psi^2 ds \right)^{\frac12} \left(\int_{\partial K} (w_h')^2 ds \right)^{\frac12}$$
$$\le C h_K^{-1} \|\psi\|_{1,K} (|w_h'|_{0,K} + h_K |w_h'|_{1,K}) \le C h_K^2 \|\psi\|_{1,K} |w_h|_{3,K} \le C h_K \|\psi\|_{1,K} |w_h|_{2,K}$$

and

$$|T_l(\psi, w_h')| \le C h \|\psi\|_1 |w_h|_{2,h}. \tag{3.12}$$

Inequalities (3.11) and (3.12) imply satisfaction of condition (i) of the generalized patch test.

To verify condition (ii), we decompose again $T_{rl}(\psi, w_h)$ into two parts $T_{rl}(\psi, \bar{w}_h)$ and $T_{rl}(\psi, w_h')$. According to the definition of $\bar{w}_h$, the mean values of $\frac{\partial \bar{w}_h}{\partial n}$ and $\frac{\partial \bar{w}_h}{\partial s}$ are continuous at interelement sides F and vanish when $F \subset \partial\Omega$. For every function $g \in L^2(F)$, let

$$P_0^F g = \frac{1}{|F|}\int_F g ds$$

be the mean value operator of g on F, and the remainder term $R_0^F g = g - P_0^F g$. Then

$$T_{rl}(\psi, \bar{w}_h) = \sum_K \sum_{F\subset\partial K} \int_F \psi \frac{\partial \bar{w}_h}{\partial x_r} n_l ds$$

$$= \sum_K \sum_{F\subset\partial K} \int_F \psi R_0^F\left(\frac{\partial \bar{w}_h}{\partial x_r}\right) n_l ds = \sum_K \sum_{F\subset\partial K} \int_F R_0^F \psi R_0^F\left(\frac{\partial \bar{w}_h}{\partial x_r}\right) n_l ds.$$

The interpolation theory gives

$$\left|\int_F R_0^F \psi R_0^F\left(\frac{\partial \bar{w}_h}{\partial x_r}\right) n_l ds\right| \le \left(\int_F (R_0^F\psi)^2 ds\right)^{\frac{1}{2}} \left(\int_F \left(R_0^F\left(\frac{\partial \bar{w}_h}{\partial x_r}\right)\right)^2 ds\right)^{\frac{1}{2}}$$

$$\le Ch_K|\psi|_{1,K}|\bar{w}_h|_{2,K} \le Ch_K|\psi|_{1,K}|w_h|_{2,K}$$

and

$$|T_{rl}(\psi, \bar{w}_h)| \le Ch|\psi|_1|w_h|_{2,h}. \tag{3.13}$$

Further, a partial integration yields

$$T_{rl}(\psi, w_h') = \sum_K \int_{\partial K} \psi \frac{\partial w_h'}{\partial x_r} n_l ds = \sum_K \int_K \psi \frac{\partial^2 w_h'}{\partial x_r \partial x_l} d\sigma - \sum_K \int_K \frac{\partial \psi}{\partial x_l}\frac{\partial w_h'}{\partial x_r} d\sigma. \tag{3.14}$$

In view of the energy orthogonality of $\bar{w}_h$ and w_h' on K we have

$$\int_K \psi \frac{\partial^2 w_h'}{\partial x_r \partial x_l} d\sigma = \int_K R_0\psi \frac{\partial^2 w_h'}{\partial x_r \partial x_l} d\sigma,$$

where

$$R_0\psi = \psi - P_0\psi, \qquad P_0\psi = \frac{1}{\Delta}\int_K \psi d\sigma.$$

By virtue of the interpolation theory and inequality (3.3) it follows that

$$\left|\int_K \psi \frac{\partial^2 w_h'}{\partial x_r \partial x_l} ds\right| \le \left(\int_K (R_0\psi)^2\right)^{\frac{1}{2}} \left(\int_K \left(\frac{\partial^2 w_h'}{\partial x_r \partial x_l}\right)^2 d\sigma\right)^{\frac{1}{2}}$$

$$\le |R_0\psi|_{0,K}|w_h'|_{2,K} \le Ch_K|\psi|_{1,K}|w_h|_{2,K},$$

and

$$\left|\sum_K \int_K \psi \frac{\partial^2 w_h'}{\partial x_r \partial x_l} ds\right| \le Ch|\psi|_1|w_h|_{2,h}. \tag{3.15}$$

Applying inequality (3.3) and the inverse inequality to the second term on the right side of (3.14) gives

$$\Big|\sum_K \int_K \frac{\partial \psi}{\partial x_l}\frac{\partial w_h'}{\partial x_r}d\sigma\Big| \le \sum_K |\psi|_{1,K}|w_h'|_{1,K} \le C\sum_K h_K^2|\psi|_{1,K}|w_h|_{3,K} \le Ch|\psi|_1|w_h|_{2,h},$$

which together with (3.14) and (3.15) show

$$|T_{rl}(\psi, w_h')| \le Ch|\psi|_1|w_h|_{2,h}. \tag{3.16}$$

Combining (3.13) and (3.16), we obtain

$$|T_{rl}(\psi, w_h)| \le Ch|\psi|_1|w_h|_{2,h}, \quad r,l=1,2, \tag{3.17}$$

which implies condition (ii) of the generalized patch test.

According to Stummel's theory [9], success in the generalized patch test together with the approximability condition via Theorem 1 ensures the convergence of the finite element space V_h for general fourth order elliptic problems.

§4. Error Estimates

Consider the plate bending problem with the clamped boundary conditions

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The weak form of the problem (4.1) is to find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^2(\Omega), \tag{4.2}$$

where

$$a(u,v) = \int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]d\sigma,$$

$$(f,v) = \int_\Omega f v d\sigma.$$

For simplicity we assume that the domain Ω is a polygon. Divide Ω into a regular family of triangular elements K satisfying the inverse assumption. Taking V_h as the finite element space on Ω, we consider the finite element approximation of the problem (4.2): to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, P_1 \bar{v}_h) \quad \forall v_h \in V_h, \tag{4.3}$$

where

$$a_h(u,v) = \sum_K \int_K [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]d\sigma.$$

Theorem 3. *Let $u \in H^3(\Omega) \cap H_0^2(\Omega)$ and $u_h \in V_h$ be the solution of (4.3) and (4.3), respectively. Then*

$$|u - u_h|_{2,h} \le Ch|u|_3, \tag{4.4}$$

$$|u - u_h|_{1,h} \le Ch^2(|u|_3 + |f|_0), \quad \text{if } \Omega \text{ is convex}, \tag{4.5}$$

where

$$|\cdot|_{1,h}^2 \equiv \sum_K |\cdot|_{1,K}^2.$$

Proof. According to the Strang lemma,

$$|u-u_h|_{2,h} \le C\Big(\inf_{v_h\in V_h}|u-v_h|_{2,h} + \sup_{w_h\in V_h}\frac{|E_h(u,w_h)|}{|w_h|_{2,h}}\Big), \tag{4.6}$$

where the consistency error functional

$$E_h(u,w_h) = a_h(u,w_h) - (f, P_1\bar{w}_h).$$

The first term on the right side of (4.6) is simply estimated by Theorem 1:

$$\inf_{v_h\in V_h}|u-v_h|_{2,h} \le \Big(\sum_K |u-\Pi_K u|_{2,K}^2\Big)^{\frac{1}{2}} \le Ch|u|_3. \tag{4.7}$$

The estimate of the second term, i.e. the consistency error estimate, rests on a careful calculation of $E_h(u,w_h)$. By Green's formula,

$$\begin{aligned} a_h(u,w_h) = &\sum_K \int_{\partial K}\Big[\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial s^2}\Big]\frac{\partial w_h}{\partial n}ds + (1-\sigma)\sum_K\int_{\partial K}\frac{\partial^2 u}{\partial n\partial s}\frac{\partial w_h}{\partial s}ds \\ &-\sum_K\int_K \nabla\Delta u\cdot\nabla w_h d\sigma = E_1(u,w_h)+E_2(u,w_h)+\bar{E}_3(u,w_h), \end{aligned} \tag{4.8}$$

so that

$$E_h(u,w_h) = E_1(u,w_h)+E_2(u,w_h)+E_3(u,w_h), \tag{4.9}$$

where

$$E_1(u,w_h) = \sum_K\int_{\partial K}[\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial s^2}]\frac{\partial w_h}{\partial n}ds,$$

$$E_2(u,w_h) = (1-\sigma)\sum_K\int_{\partial K}\frac{\partial^2 u}{\partial n\partial s}\frac{\partial w_h}{\partial s}ds,$$

$$E_3(u,w_h) = \bar{E}_3(u,w_h) - (f, P_1\bar{w}_h).$$

Since the piecewise linear interpolation $P_1\bar{w}_h$ is continuous in Ω and vanishes on $\partial\Omega$, then $P_1\bar{w}_h \in H_0^1(\Omega)$. In view of the assumption $u\in H^3(\Omega)$, $\Delta^2 u \in H^{-1}(\Omega)$. Therefore, the scalar product $(\Delta^2 u, P_1\bar{w}_h)$ makes sense and Green's formula yields

$$(f, P_1\bar{w}_h) = (\Delta^2 u, P_1\bar{w}_h) = -\int_\Omega \nabla\Delta u\cdot\nabla P_1\bar{w}_h d\sigma,$$

and so

$$E_3(u,w_h) = \sum_K\int_K \nabla\Delta u\cdot\nabla(P_1\bar{w}_h - w_h)d\sigma. \tag{4.10}$$

Using the interpolation theory and inequality (3.3), we find

$$\begin{aligned}|E_3(u,w_h)| &\le \sum_K \left|\int_K \nabla\Delta u\cdot\nabla(P_1\bar{w}_h - w_h)d\sigma\right| \le |\Delta u|_1|P_1\bar{w}_h - w_h|_{1,h}\\ &\le |\Delta u|_1(|P_1\bar{w}_h-\bar{w}_h|_{1,h}+|w'_h|_{1,h}) \le Ch|u|_3(|\bar{w}_h|_{2,h}+|w_h|_{2,h})\\ &\le Ch|u|_3|w_h|_{2,h}. \end{aligned}\tag{4.11}$$

On the other hand, setting $\psi = \Delta u-(1-\sigma)\dfrac{\partial^2 u}{\partial s^2}$ and $\psi = \dfrac{\partial^2 u}{\partial n\partial s}$, respectively, into inequality (3.17) gives

$$|E_i(u,w_h)| \le Ch|u|_3|w_h|_{2,h}, \quad i=1,2. \tag{4.12}$$

Combining (4.11) and (4.12) we conclude that

$$|E_h(u,w_h)| \le Ch|u|_3|w_h|_{2,h}, \tag{4.13}$$

which together with (4.6) and (4.7) proves inequality (4.4).

Now we are going to prove inequality (4.5). Let Π_h be the piecewise cubic interpolation operator on Ω, whose restrictions to each triangle K are Π_K, defined by (2.9) and (2.10). By virtue of the decomposition of Π_K into $\bar{\Pi}_K$ and Π'_K, the operator Π_h may also be decomposed correspondingly into

$$\Pi_h = \bar{\Pi}_h + \Pi'_h.$$

Let us set $e = u - u_h$. Then $P_1\bar{\Pi}_h e \in H_0^1(\Omega)$ and $g = -\Delta P_1\bar{\Pi}_h e \in H^{-1}(\Omega)$. Consider the auxiliary variational problem : to find $\varphi \in H_0^2(\Omega)$ such that

$$a(\varphi, v) = (g, v) \quad \forall v \in H_0^2(\Omega). \tag{4.14}$$

According to a regularity theory of solutions, when Ω is a convex polygon, the following *a priori* estimate holds:

$$\|\varphi\|_3 \le C\|g\|_{-1}.$$

By the definition,

$$\|g\|_{-1} = \sup_{\substack{v\in H_0^1(\Omega)\\ v\neq 0}} \frac{(g,v)}{\|v\|_1}. \tag{4.15}$$

For every function $v \in H_0^1(\Omega)$ Green's formula gives

$$(g,v) = -(\Delta P_1\bar{\Pi}_h e, v) = \int_\Omega \nabla P_1\bar{\Pi}_h e\cdot\nabla v d\sigma,$$

so that

$$|(g,v)| \le |P_1\bar{\Pi}_h e|_1|v|_1, \quad (g, P_1\bar{\Pi}_h e) = \int_\Omega \nabla P_1\bar{\Pi}_h e\cdot\nabla P_1\bar{\Pi}_h e d\sigma = |P_1\bar{\Pi}_h e|_1^2,$$

and

$$\|\varphi\|_3 \le C\|g\|_{-1} \le C|P_1\bar{\Pi}_h e|_1. \tag{4.16}$$

On the other hand,

$$|P_1\tilde{\Pi}_h e|_1^2 = (g, P_1\tilde{\Pi}_h e) = (\Delta^2\varphi, P_1\tilde{\Pi}_h e) = -\int_\Omega \nabla\Delta\varphi \cdot \nabla P_1\tilde{\Pi}_h e d\sigma$$
$$= \sum_K \int_K \nabla\Delta\varphi \cdot \nabla(\tilde{\Pi}_h e - P_1\tilde{\Pi}_h e)d\sigma - \sum_K \int_K \nabla\Delta\varphi \cdot \nabla\tilde{\Pi}_h e d\sigma = I_1 + I_2. \quad (4.17)$$

The first term I_1 on the right side of (4.17) is bounded from above:

$$|I_1| \le \left|\sum_K \int_K \nabla\Delta\varphi\cdot\nabla(\tilde{\Pi}_h e - P_1\tilde{\Pi}_h e)d\sigma\right| \le \sum_K |\Delta\varphi|_{1,K}|\tilde{\Pi}_h e - P_1\tilde{\Pi}_h e|_{1,K} \le Ch|\varphi|_3|\tilde{\Pi}_h e|_{2,h}.$$

Application of inequalities (3.1) and (4.4) gives

$$|\tilde{\Pi}_h e|_{2,h} \le |\tilde{\Pi}_h u - u|_{2,h} + |u - u_h|_{2,h} + |u_h - \tilde{\Pi}_h u_h|_{2,h} \le Ch(|u|_3 + |u_h|_{3,h}),$$

but

$$|u_h|_{3,h} \le |u - \Pi_h u|_{3,h} + |\Pi_h u - u_h|_{3,h} + |u|_3 \le C|u|_3 + C\sum_K h_K^{-1}|\Pi_h u - u_h|_{2,K}$$
$$\le C|u|_3 + C\sum_K h_K^{-1}(|\Pi_h u - u|_{2,K} + |u - u_h|_{2,K}) \le C|u|_3. \quad (4.18)$$

Therefore

$$|\tilde{\Pi}_h e|_{2,h} \le Ch|u|_3, \quad (4.19)$$

and so

$$|I_1| \le Ch|\varphi|_3|u|_3. \quad (4.20)$$

In view of (4.8) the second term I_2 on the right side of (4.17) has the form

$$I_2 = \bar{E}_3(\varphi, \tilde{\Pi}_h e) = a_h(\varphi, \tilde{\Pi}_h e) - E_1(\varphi, \tilde{\Pi}_h e) - E_2(\varphi, \tilde{\Pi}_h e). \quad (4.21)$$

Using (4.12) and (4.19), we have immediately

$$|E_i(\varphi, \tilde{\Pi}_h e)| \le Ch|\varphi|_3|\tilde{\Pi}_h e|_{2,h} \le Ch^2|\varphi|_3|u|_3, \quad i = 1, 2. \quad (4.22)$$

The first term on the right side of (4.21) may be written as

$$a_h(\varphi, \tilde{\Pi}_h e) = a_h(\varphi, \tilde{\Pi}_h e - e) + a_h(\varphi - \Pi_h\varphi, e) + a_h(\Pi_h\varphi, e) = J_1 + J_2 + J_3.$$

By (4.8)

$$J_1 = a_h(\varphi, \tilde{\Pi}_h e - e) = E_1(\varphi, \tilde{\Pi}_h e - e) + E_2(\varphi, \tilde{\Pi}_h e - e) + \bar{E}_3(\varphi, \tilde{\Pi}_h e - e).$$

Application of (4.12) and (4.18) yields

$$|E_i(\varphi, \tilde{\Pi}_h e - e)| \le Ch|\varphi|_3|\tilde{\Pi}_h e - e|_{2,h} \le Ch^2|\varphi|_3|e|_{3,h} \le Ch^2|\varphi|_3|u|_3,$$

$$|\bar{E}_3(\varphi, \tilde{\Pi}_h e - e)| \le \sum_K \left|\int_K \nabla\Delta\varphi \cdot \nabla(\tilde{\Pi}_h e - e)d\sigma\right| \le Ch^2|\varphi|_3|u|_3.$$

Therefore

$$|J_1| \le Ch^2|\varphi|_3|u|_3.$$

Further,

$$|J_2| = |a_h(\varphi - \Pi_h\varphi, e)| \le C|\varphi - \Pi_h\varphi|_{2,h}|e|_{2,h} \le Ch^2|\varphi|_3|u|_3.$$

The last term

$$J_3 = a_h(\Pi_h\varphi, e) = a_h(u, \Pi_h\varphi) - a_h(u_h, \Pi_h\varphi) = E_h(u, \Pi_h\varphi)$$
$$= E_1(u, \Pi_h\varphi - \varphi) + E_2(u, \Pi_h\varphi - \varphi) + E_3(u, \Pi_h\varphi),$$

where the property $E_i(u, \varphi) = 0, i = 1, 2$, is used. By (4.12)

$$|E_i(u, \Pi_h\varphi - \varphi)| \le Ch|u|_3|\Pi_h\varphi - \varphi|_{2,h} \le Ch^2|u|_3|\varphi|_3.$$

It is easily verified that

$$(f, \varphi - P_1\varphi) + \sum_K \int_K \nabla\Delta u \cdot \nabla(\varphi - P_1\varphi)d\sigma = 0,$$

hence

$$E_3(u, \Pi_h\varphi) = (f, \varphi - P_1\varphi)$$
$$+ \sum_K \int_K \nabla\Delta u \cdot \nabla(\varphi - \Pi_h\varphi - P_1(\varphi - \Pi_h\varphi))d\sigma - \sum_K \int_K \nabla\Delta u \cdot \nabla P_1\Pi_h'\varphi d\sigma.$$

Since

$$|(f, \varphi - P_1\varphi)| \le |f|_0|\varphi - P_1\varphi|_0 \le Ch^2|f|_0|\varphi|_2,$$

$$\left|\sum_K \int_K \nabla\Delta u \cdot \nabla(\varphi - \Pi_h\varphi - P_1(\varphi - \Pi_h\varphi))d\sigma\right| \le Ch|u|_3|\varphi - \Pi_h\varphi|_{2,h} \le Ch^2|u|_3|\varphi|_3,$$

$$\left|\sum_K \int_K \nabla\Delta u \cdot \nabla P_1\Pi_h'\varphi d\sigma\right| \le |\Delta u|_1|P_1\Pi_h'\varphi|_{1,h}$$
$$\le C|u|_3(|\Pi_h'\varphi - P_1\Pi_h'\varphi|_{1,h} + |\Pi_h'\varphi|_{1,h}) \le Ch^2|u|_3|\varphi|_3,$$

it follows that

$$|E_3(u, \Pi_h\varphi)| \le Ch^2(|u|_3 + |f|_0)\|\varphi\|_3, \quad |J_3| \le Ch^2(|u|_3 + |f|_0)\|\varphi\|_3.$$

Combining all inequalities for $J_i, i = 1, 2, 3$, we have

$$|a_h(\varphi, \Pi_h e)| \le Ch^2(|u|_3 + |f|_0)\|\varphi\|_3,$$

which together with (4.22) gives

$$|I_2| \le Ch^2(|u|_3 + |f|_0)\|\varphi\|_3. \tag{4.24}$$

Substituting (4.20) and (4.24) into (4.17) and using inequality (4.16), we obtain

$$|P_1\Pi_h e|_1 \le Ch^2(|u|_3 + |f|_0).$$

Finally, the triangular inequality and (4.18), (4.19) imply

$$|e|_{1,h} \le |e - \Pi_h e|_{1,h} + |\Pi_h e - P_1\Pi_h e|_{1,h} + |P_1\Pi_h e|_{1,h}$$

$$\le Ch^2|e|_{3,h} + Ch|\Pi_h e|_{2,h} + Ch^2(|u|_3 + |f|_0) \le Ch^2(|u|_3 + |f|_0).$$

Inequality (4.5) is thus proved.

Remark. In the finite element equation (4.3) the right side $(f, P_1\bar{v}_h)$ is used instead of the standard form (f, v_h). This kind of modification appeared in [10] for an analysis of Morley's element. If we consider the usual finite element equation

$$a_h(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h, \tag{4.3'}$$

the following theorem can be proved.

Theorem 3'. *Let $u \in H^3(\Omega) \cap H_0^2(\Omega)$ and $u_h \in V_h$ be the solution of (4.2) and (4.3'), respectively. Then*

$$|u - u_h|_{2,h} \le Ch(|u|_3 + h|f|_0),$$

$$|u - u_h|_{1,h} \le Ch^2(|u|_3 + h|f|_0), \quad \text{if } \Omega \text{ is convex.}$$

§5. Applications

We apply this new energy-orthogonal element, denoted by the SZ element in Tables 1–4, to sample plate bending problems. Let us consider a square plate with the side length $= 1$, Poisson's ratio $\sigma = 0,3$, the bending stiffness $D = \dfrac{Et^3}{12(1-\sigma^2)} = 1$ and with two loading cases, namely a unit uniform distributed loading and a unit vertical loading. The boundary of the square plate is assumed to be either simply supported or clamped. Because of symmetry a quarter of the square plate is calculated using two mesh patterns (Fig. 1). The results are compared to those of Bergan's energy-orthogonal element. In Tables 1–4 the deflections and the moments are normalized by a factor 10^2 and 10^3 respectively. The percentage in brackets after each figure indicates the relative error to the true solution.

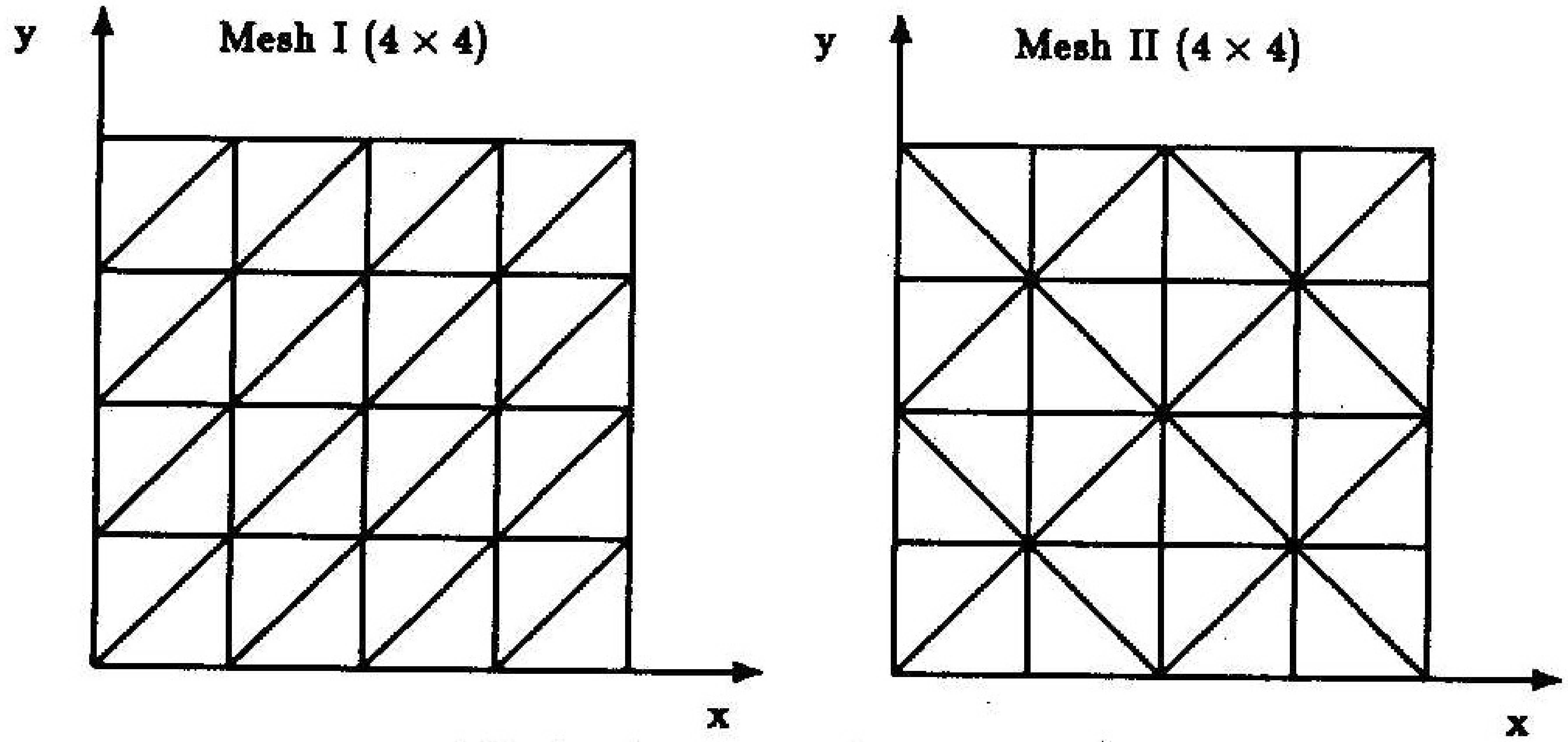

Fig 1. A quarter of the square plate

Table 1. Clamped square plate. Mesh I

	size	SZ	Bergan	theory
Deflection $w(0, 0)$	4×4	1.305 (3.16%)	1.325 (4.74%)	
(Distrib. load)	8×8	1.276 (0.87%)	1.280 (1.19%)	1.265
Deflection $w(0, 0)$	4×4	5.502 (1.96%)	5.868 (4.57%)	
(Concent. load)	8×8	5.590 (0.3%)	5.694 (1.46%)	5.612
Moment $M_x(0,0)$	4×4	2.355 (2.89%)	2.540 (10.9%)	
(Distrib. load)	8×8	2.300 (0.44%)	2.353 (2.72%)	2.290
Moment $M_x(\frac{1}{2},0)$	4×4	-5.219 (1.67%)	-5.633 (10.32%)	
(Distrib. load)	8×8	-5.187 (1.05%)	-5.445 (6.07%)	-5.133
Moment $M_x(\frac{1}{2},0)$	4×4	-12.855 (2.21%)	-13.556 (7.78%)	
(Concent load)	8×8	-12.712 (1.07%)	-13.146 (4.53%)	-12.577

Table 2. Clamped square plate. Mesh II

	size	SZ	Bergan	theory
Deflection $w(0, 0)$	4×4	1.293 (2.21%)	1.315 (3.90%)	
(Distrib. load)	8×8	1.273 (0.63%)	1.278 (1.02%)	1.265
Deflection $w(0, 0)$	4×4	5.528 (1.50%)	5.870 (4.60%)	
(Concent. load)	8×8	5.603 (0.16%)	5.695 (1.49%)	5.612
Moment $M_x(0,0)$	4×4	2.426 (5.94%)	2.525 (10.25%)	
(Distrib. load)	8×8	2.326 (1.57%)	2.352 (2.69%)	2.290
Moment $M_x(\frac{1}{2},0)$	4×4	-3.846 (25.07%)	-3.838 (25.23%)	
(Distrib. load)	8×8	-4.416 (13.97%)	-4.423 (13.83%)	-5.133
Moment $M_x(\frac{1}{2},0)$	4×4	-10.607 (15.66%)	-10.387 (17.41%)	
(Concent load)	8×8	-11.401 (9.35%)	-11.356 (9.71%)	-12.577

Table 3. Simply supported square plate. Mesh I

	size	SZ	Bergan	theory
Deflection $w(0, 0)$	4×4	4.108 (1.13%)	4.126 (1.58%)	
(Distrib. load)	8×8	4.075 (0.32%)	4.079 (0.41%)	4.602
Deflection $w(0, 0)$	4×4	11.741 (1.12%)	11.848 (2.13%)	
(Concent. load)	8×8	11.575 (0.22%)	11.681 (0.69%)	11.601
Moment $M_{xy}(0,0)$	4×4	4.775 (0.29%)	5.004 (4.49%)	
(Distrib. load)	8×8	4.776 (0.27%)	4.841 (1.09%)	4.789
Moment $M_{xy}(1,1)$	4×4	3.434 (5.73%)	3.490 (7.44%)	
(Distrib. load)	8×8	3.311 (1.94%)	3.335 (2.67%)	3.248
Moment $M_{xy}(1,1)$	4×4	6.256 (2.64%)	6.360 (4.35%)	
(Concent load)	8×8	6.136 (0.67%)	6.164 (1.14%)	6.095

Table 4. Simply supported square plate. Mesh II.

	size	SZ	Bergan	theory
Deflection $w(0,0)$	4×4	4.116 (1.33%)	4.128 (1.61%)	
(Distrib. load)	8×8	4.075 (0.32%)	4.081 (0.46%)	4.062
Deflection $w(0,0)$	4×4	11.553 (0.41%)	11.881 (2.41%)	
(Concent. load)	8×8	11.598 (0.03%)	11.692 (0.79%)	11.601
Moment $M_{xy}(0,0)$	4×4	4.945 (3.26%)	5.034 (5.12%)	
(Distrib. load)	8×8	4.828 (0.81%)	4.854 (1.37%)	4.789
Moment $M_{xy}(1,1)$	4×4	3.544 (9.11%)	3.523 (8.64%)	
(Distrib. load)	8×8	3.351 (3.17%)	3.346 (3.01%)	3.248
Moment $M_{xy}(1,1)$	4×4	6.450 (5.82%)	6.413 (5.21%)	
(Concent load)	8×8	6.183 (1.44%)	6.180 (1.39%)	6.095

Conclusions. 1. It is evident from Tables 1–4 that the SZ element gives better results than those of Bergan's.

2. Both Mesh I and II are convergent and Mesh I is preferable.

3. The SZ element seems to be a good nine parameter plate element with clear formations and satisfactory numerical accuracy.

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