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EXISTENCE AND UNIQUENESS OF MATRIX PADE APPROXIMANTS*[1)]

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Abstract

For the problems of the left and right matrix Padé approximations, we give the necessary and sufficient conditions for the existence of their solutions. If the left Padé approximant exists, then we prove that its uniqueness is equivalent to the existence of right Padé approximants, and we further give the exact result about the dimension of the linear space ${}^L R^*(m,n)$ formed from the left Padé approximants.

§1. Introduction

Let

$$f(z)=\sum_{i=0}^{\infty} c_i z^i, \quad c_i \in \mathbf{C}^{d\times d},$$

$$H_k=\{\sum_{i=0}^{k} a_i z^i : a_i \in \mathbf{C}^{d\times d}\},$$

where $\mathbf{C}^{d\times d}$ consists of all $d\times d$ complex matrices with $d>0$. We define right Padé approximants ${}^R[m/n]_f = {}^R P\,{}^R Q^{-1}$ by

$$\begin{aligned} f(z)\,{}^R Q(z) - {}^R P(z) &= O(z^{m+n+1}),\\ {}^R Q(0) &= I \end{aligned} \tag{1.1}$$

and left-handed Padé approximants ${}^L[m/n]_f = {}^L Q^{-1}\,{}^L P$ by

$$\begin{aligned} {}^L Q(z) f(z) - {}^L P(z) &= O(z^{m+n+1}),\\ {}^L Q(0) &= I \end{aligned} \tag{1.2}$$

where $({}^R P, {}^R Q), ({}^L P, {}^L Q) \in H_m \times H_n$, and $I \in \mathbf{C}^{d\times d}$ is a unit matrix.

The approach to matrix Padé approximants adopted here follows that of Bessis [1]. For other approaches and generalizations to a non-commutative algebra, we refer to [1] and [2]. For their applications in many domains such as the theoretical physics, the realization problem in system theory, and many other problems such as algebraic properties, computations and convergence of matrix Padé approximants, we refer to the references of [3]. However, the most basic problems, i.e., the existence and uniqueness for matrix Padé approximants, have not yet been investigated completely.

In this paper, the questions concerning the existence, and uniqueness, or nonuniqueness, for matrix Padé approximants are discussed. Some interesting results are established by careful analysis.

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§2. Existence

We shall first quote the following notations:

$$H(i,j,k)=\begin{bmatrix} c_i & c_{i-1} & \cdots & c_{i-j}\\ c_{i+1} & c_i & \cdots & c_{i-j+1}\\ \cdots & \cdots & \cdots & \cdots\\ c_{i+k} & c_{i+k-1} & \cdots & c_{i+k-j}\end{bmatrix},\ H_T(i,j,k)=\begin{bmatrix} c_i^T & c_{i-1}^T & \cdots & c_{i-j}^T\\ c_{i+1}^T & c_i^T & \cdots & c_{i-j+1}^T\\ \cdots & \cdots & \cdots & \cdots\\ c_{i+k}^T & c_{i+k-1}^T & \cdots & c_{i+k-j}^T\end{bmatrix},$$

where c_i^T is the transpose of c_t , the coefficients of $f(z)$, and define $c_t = 0$, if $t < 0$.

Lemma 2.1. *If $c_t \in \mathbf{C}^{d\times d}$, then*

$$\operatorname{rank} H(i,j,k) = \operatorname{rank} H_T(i+k-j,k,j),$$

where rank denotes the rank of a matrix.

Proof. Since

$$\begin{bmatrix} & & I\\ & O\ \ I & \\ I & \cdot\ O & \end{bmatrix} H^T(i,j,k) \begin{bmatrix} & & I\\ & O\ \ I & \\ I & O & \end{bmatrix} = H_T(i+k-j,k,j),$$

by the relation *rank* $H(i,j,k) =$ *rank* $H^T(i,j,k)$, the lemma is valid.

Now we establish the existence results.

Theorem 2.1. *Let $f(z)=\sum_{i=0}^{\infty} c_i z^i$, $c_i \in \mathbf{C}^{d\times d}$. Then*

(i) ${}^R[m/n]_f$ *exists if and only if*

$$\operatorname{rank} H(m,n-1,n-1) = \operatorname{rank} H(m+1,n,n-1). \tag{2.1}$$

(ii) ${}^L[m/n]_f$ *exists if and only if*

$$\operatorname{rank} H(m,n-1,n-1) = \operatorname{rank} H(m,n-1,n). \tag{2.2}$$

(iii) *Both ${}^R[m/n]_f$ and ${}^L[m/n]_f$ exist, if $H(m,n-1,n-1)$ is nonsingular* (see [5]).

Proof. (i) Let ${}^RP(z)=\sum_{i=0}^{m} {}^Ra_i z^i$, ${}^RQ(z)=\sum_{i=0}^{n} {}^Rb_i z^i$, ${}^Ra_i$, ${}^Rb_i \in \mathbf{C}^{d\times d}$. Then by equating the coefficients of z^i in (1.1) for $i=0,1,\ldots,m+n$, one has

$$\begin{bmatrix} {}^Ra_0\\ {}^Ra_1\\ \vdots\\ {}^Ra_m\end{bmatrix} = H(0,n,m)\begin{bmatrix} {}^Rb_0\\ {}^Rb_1\\ \vdots\\ {}^Rb_n\end{bmatrix} \tag{2.3}$$

and

$$H(m,n-1,n-1)\begin{bmatrix} {}^Rb_1\\ {}^Rb_2\\ \vdots\\ {}^Rb_n\end{bmatrix} = \begin{bmatrix} -c_{m+1}\\ -c_{m+2}\\ \vdots\\ -c_{m+n}\end{bmatrix}. \tag{2.4}$$

Therefore ${}^R[m/n]_f$ exists if and only if (2.4) is solvable. Because (2.1) is the necessary and sufficient condition for the solvability of (2.4), the conclusion (i) is thus proved.

(ii) Let ${}^LP(z) = \sum_{i=0}^m {}^La_iz^i$, ${}^LQ(z) = \sum_{i=0}^n {}^Lb_iz^i$. Then by a similar approach, we have

$$\begin{bmatrix} {}^La_0^T \\ {}^La_1^T \\ \vdots \\ {}^La_m^T \end{bmatrix} = H_T(0,n,m) \begin{bmatrix} {}^Lb_0^T \\ {}^Lb_1^T \\ \vdots \\ {}^Lb_n^T \end{bmatrix} \tag{2.5}$$

and

$$H_T(m,n-1,n-1) \begin{bmatrix} {}^Lb_1^T \\ {}^Lb_2^T \\ \vdots \\ {}^Lb_n^T \end{bmatrix} = \begin{bmatrix} -c_{m+1}^T \\ -c_{m+2}^T \\ \vdots \\ -c_{m+n}^T \end{bmatrix}. \tag{2.6}$$

It follows that ${}^L[m/n]_f$ exists if and only if

$$\text{rank } H_T(m,n-1,n-1) = \text{ rank } H_T(m+1,n,n-1).$$

From Lemma 2.1 we get the assertion (ii).

(iii) If $H(m,n-1,n-1)$ is nonsingular, then both (2.1) and (2.2) are valid. Thus ${}^R[m/n]_f$ and ${}^L[m/n]_f$ exist simultaneously.

In the case of ordinary Padé approximation (i.e., $d=1$), we always have

$$\text{rank } H(m+1,n,n-1) = \text{rank } H(m,n-1,n).$$

At present, the equality may not be valid in some cases. This means that there is a possibility that only one side matrix Padé approximant exists. The following is an example.

Let $c_0 = c_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, c_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, m = 1$ and $n = 2$. Then rank $H(m,n-1,n-1)$ = rank $(m,n-1,n) = 2$, rank $H(m+1,n,n-1) = 3$. Therefore ${}^L[1/2]_f$ exists; ${}^R[1/2]$ does not.

From the proof of Theorem 2.1, we get immediately the following

Corollary 2.1. If c_i are symmetric matrices for $i = 0,1,\ldots,m+n$, then both ${}^R[m/n]_f$ and ${}^L[m/n]_f$ exist or neither exists.

§3. Structure

Let

$${}^L\mathbf{R}(m,n) = \{(P,Q) \in H_m \times H_n \backslash \{0\} : Qf - P = O(z^{m+n+1})\},$$
$${}^R\mathbf{R}(m,n) = \{(P,Q) \in H_m \times H_n \backslash \{0\} : fQ - P = O(z^{m+n+1})\}.$$

We attempt to characterize the structure of ${}^L\mathbf{R}(m,n)$. First we have

Lemma 3.1. *Let* $({}^LP, {}^LQ) \in {}^L\mathbf{R}(m,n)$, $({}^RP, {}^RQ) \in {}^R\mathbf{R}(m,n)$. *Then*

$${}^LP\,{}^RQ = {}^LQ\,{}^RP. \tag{3.1}$$

Proof. Since

$${}^LQf - {}^LP = O(z^{m+n+1}), \tag{3.2}$$
$$f\,{}^RQ - {}^RP = O(z^{m+n+1}), \tag{3.3}$$

by subtracting (3.3) multiplied on the left by ${}^LQ$ from (3.2) multiplied on the right by ${}^RQ$, we get

$$ {}^LQ\,{}^RP - {}^LP\,{}^RQ = O(z^{m+n+1}). \tag{3.4}$$

The left-hand side of (3.4), which is a matrix polynomial of degree $m+n$ at most, can not vanish $m+n+1$ times without being identically zero, thus the lemma is true.

Now we define integers:

$$u = min\{\partial(P) : (P,Q) \in {}^L\mathbf{R}(m,n)\},$$
$$v = min\{\partial(Q) : (P,Q) \in {}^L\mathbf{R}(m,n)\},$$

where $\partial(P)$ denotes the degree of the matrix polynomial P, and we define $\partial(0) = -\infty$.

Lemma 3.2. *Let $f(z) = \sum_{i=0}^{\infty} c_i z^i$, and c_0 be invertible. Then there exists a $(P^*, Q^*) \in {}^L\mathbf{R}(m,n)$, such that*

$$\partial(P^*) = u, \quad \partial(Q^*) = v,$$

provided there exists a $(P,Q) \in {}^R\mathbf{R}(m,n)$, such that the leading coefficients of P and Q are invertible.

Proof. We first note that u and v are nonnegative. By the definitions of u and v, there are $(P_i, Q_i) \in {}^L\mathbf{R}(m,n)$, $i = 1,2$, such that

$$\partial(Q_1) = v, \quad \partial(P_1) \geq u,$$
$$\partial(Q_2) \geq v, \quad \partial(P_2) = u.$$

From Lemma 3.1, it follows that

$$P_iQ = Q_iP, \quad i = 1,2.$$

Thus

$$\partial(P_i) + \partial(Q) = \partial(Q_i) + \partial(P), \quad i = 1,2,$$

and hence

$$\partial(P_1) + \partial(Q_2) = \partial(Q_1) + \partial(P_2).$$

Therefore $\partial(P_1) = u$, $\partial(Q_2) = v$.

The following theorem characterizes the structure of ${}^L\mathbf{R}(m,n)$.

Theorem 3.1. *If there exist $(P^*, Q^*) \in {}^L\mathbf{R}(m,n)$ and $({}^RP, {}^RQ) \in {}^R\mathbf{R}(m,n)$, such that*

$$\partial(P^*) = u, \quad \partial(Q^*) = v,$$

and the leading coefficients of P^*, Q^*, ${}^RP$ and ${}^RQ$ are invertible, then for any $(P,Q) \in {}^L\mathbf{R}(m,n)$, there exists a matrix polynomial q, such that

$$(P,Q) = q(P^*, Q^*). \tag{3.5}$$

Proof. By division with remainder ([4], p.248), P and Q could be expressed as

$$P = qP^* + r, \quad Q = q_1Q^* + r_1, \tag{3.6}$$

where $\partial(r) < \partial(P^*)$, $\partial(r_1) < \partial(Q^*)$. From Lemma 3.1 and (3.6), one has

$$qP^{*\,R}Q + r\,{}^RQ = q_1Q^{*\,R}P + r_1\,{}^RP,$$

and hence

$$(q-q_1)(P^{*\,R}Q) = r_1\,{}^RP - r\,{}^RQ. \tag{3.7}$$

Since

$$\partial(P^*) + \partial({}^RQ) = \partial(Q^*) + \partial({}^RP),$$
$$\partial(r_1\,{}^RP) \le \partial(r_1) + \partial({}^RP) < \partial(Q^*) + \partial({}^RP),$$
$$\partial(r\,{}^RQ) \le \partial(r) + \partial({}^RQ) < \partial(P^*) + \partial({}^RQ).$$

(3.7) can not hold if $q - q_1 \neq 0$. Hence $q_1 = q$.

Now we have

$$O(z^{m+n+1}) = Qf - P = q(Q^*f - P^*) + r_1f - r$$

and hence $(r, r_1) \in {}^L\mathbf{R}(m,n)$. This is a contradiction. Therefore $r_1 = r = 0$, and the theorem is thus proved.

§4. Uniqueness

Assume problem (1.2) is solvable, and $Q^{*^{-1}}P^*$ is one of its solutions. Let

$${}^L\mathbf{R}^*(m,n) = \{QQ^{*^{-1}}P^* - P : (P,Q) \text{ satisfies } (1.2)\}.$$

Then we have

Lemma 4.1. *Let* $Q^{*-1}P^* = \sum_{i=0}^{\infty} \tilde{c}_i z^i$. *Then for any* $R \in {}^L\mathbf{R}^*(m,n)$, *there exist* $\beta_i \in \mathbf{C}^{d\times d}$ *such that*

$$H_T(m, n-1, n-1)\begin{bmatrix}\beta_1^T\\ \beta_2^T\\ \vdots\\ \beta_n^T\end{bmatrix} = 0, \tag{4.1}$$

$$R(z) = z^{m+n+1}\sum_{k=0}^{\infty} E_k z^k, \tag{4.2}$$

where

$$E_k = \sum_{i=1}^{n} \beta_i \tilde{c}_{m+n+k+1-i}. \tag{4.3}$$

Proof. Let

$$R = QQ^{*^{-1}}P^* - P,$$

$$\Delta Q = \sum_{i=1}^{n} \beta_i z^i = Q - Q^*,$$

$$\Delta P = \sum_{i=1}^{m} \alpha_i z^i = P - P^*.$$

Then by (2.5) and (2.6),

$$H_T(m, n-1, n-1)B_T = 0, \quad A_T = H_T(0, n-1, m-1)B_T, \tag{4.4}$$

and

$$R = \Delta Q Q^{*^{-1}} P^* - \Delta P, \tag{4.5}$$

where

$$B_T = \begin{bmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{bmatrix}, \quad A_T = \begin{bmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_m^T \end{bmatrix}.$$

It follows from (1.2) that

$$\tilde{c}_i = c_i, \quad for\ i = 0, 1, \ldots, m+n. \tag{4.6}$$

Therefore

$$\begin{aligned} R(z) &= (\sum_{i=1}^{n} \beta_i z^i)(\sum_{i=0}^{\infty} \tilde{c}_i z^i) - \sum_{k=1}^{m} \alpha_k z^k = \sum_{k=1}^{\infty}(\sum_{i=1}^{n} \beta_i \tilde{c}_{k-i}) z^k - \sum_{k=1}^{m} \alpha_k z^k \\ &= \sum_{k=1}^{m}(\sum_{i=1}^{n} \beta_i c_{k-i} - \alpha_k) z^k + \sum_{k=m+1}^{m+n} (\sum_{i=1}^{n} \beta_i c_{k-i}) z^k + \sum_{k=m+n+1}^{\infty} (\sum_{i=1}^{n} \beta_i \tilde{c}_{k-i}) z^k. \end{aligned}$$

Using (4.4), we have

$$\begin{aligned} R(z) &= z^{m+n+1} \sum_{k=0}^{\infty}(\sum_{i=1}^{n} \beta_i \tilde{c}_{m+n+k+1-i}) z^k \\ &= z^{m+n+1} \sum_{k=0}^{\infty} E_k z^k. \end{aligned}$$

Thus the lemma is proved.

Let $R_i = Q_i Q^{*^{-1}} P^* - P_i \in {}^L\mathbf{R}^*(m,n)$ for $i = 1, 2$, and $R_1 \neq R_2$. Then by Lemma 4.1, there exist $E_k^{(1)}, E_k^{(2)} \in \mathbf{C}^{d \times d}$, such that

$$R_i(z) = z^{m+n+1} \sum_{k=0}^{\infty} E_k^{(i)} z^k, \quad i = 1, 2.$$

Since $R_1 \neq R_2$, there is some k_0 such that $E_k^{(1)} = E_k^{(2)}$ for $k < k_0$ and $E_{k_0}^{(1)} \neq E_{k_0}^{(2)}$. Let

$$Q_1^{-1} P_1 - Q_2^{-1} P_2 = \sum_{k=0}^{\infty} e_k z^k.$$

Then by

$$\begin{aligned} Q_1^{-1} P_1 - Q_2^{-1} P_2 &= (Q^{*^{-1}} P^* - Q_2^{-1} P_2) - (Q^{*^{-1}} P^* - Q_1^{-1} P_1) \\ &= Q_2^{-1} R_2 - Q_1^{-1} R_1 \\ &= z^{m+n+1}(Q_2^{-1} \sum_{k=0}^{\infty} E_k^{(2)} z^k - Q_1^{-1} \sum_{k=0}^{\infty} E_k^{(1)} z^k), \end{aligned}$$

we have

$$e_i = 0, \quad \text{for} \quad i = 0, 1, \cdots, m+n+k_0,$$
$$e_{m+n+k_0+1} \neq 0.$$

This fact implies that the different elements in ${}^L\mathbf{R}^*(m,n)$ correspond to the different left Padé approximants. Therefore, it is important for us to introduce and characterise the set ${}^L\mathbf{R}^*(m,n)$. To establish results about ${}^L\mathbf{R}^*(m,n)$, we introduce the following

Lemma 4.2. *Let* $A \in \mathbf{C}^{p\times s}, B \in \mathbf{C}^{q\times s}$. *Let*

$$\text{rang}(B, A) = \{y \in \mathbf{C}^q : y = Bx,\ x \in N(A)\}.$$

Then

$$\dim \text{rang}(B, A) = \text{rank } C - \text{rank } A,$$

where $\dim \text{rang}(B,A)$ *denotes the dimension of the linear space* $\text{rang}(B,A)$,

$$N(A) = \{x \in \mathbf{C}^s : Ax = 0\},$$

and

$$C = \begin{pmatrix} A \\ B \end{pmatrix}.$$

Proof. Since $N(C) \subset N(A)$, we may choose $x_1, x_2, \ldots, x_s \in \mathbf{C}^s$ so that
i) $\{x_1, x_2, \ldots, x_k\}$ is a base for N(C),

ii) $\{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{k+l}\}$ is a base for N(A), and

iii) $\{x_1, x_2, \ldots, x_s\}$ is a base for $\mathbf{C}^s$.

From the relation

$$\text{rank } D + \dim N(D) = q, \quad D \in \mathbf{C}^{p\times q},$$

we have that $Cx_{k+1}, \ldots, Cx_s$ are linearly independent. Therefore $Cx_{k+1}, \ldots, Cx_{k+l}$ are linearly independent also, and hence

$$\begin{aligned}\dim \text{rang}(B, A) &= \dim \text{rang}(C, A) = (k+1) - k \\ &= (s - \text{rank } A) - (s - \text{rank } C) = \text{rank } C - \text{rank } A.\end{aligned}$$

Theorem 4.1. *Let* $Q^{*-1}P^* = \sum_{i=0}^{\infty} \tilde{c}_i z^i$. *Then* ${}^L\mathbf{R}^*(m,n)$ *is a linear space and*

$$\begin{aligned}\dim {}^L\mathbf{R}^*(m,n) &= d\sum_{k=0}^{\infty}[\text{rank } \tilde{H}(m+k+1, n+k, n-1) - \text{rank } \tilde{H}(m+k, n-1+k, n-1)] \\ &= d\,[\lim_{k\to\infty} \text{rank } \tilde{H}(m+k+1, n+k, n-1) - \text{rank } \tilde{H}(m, n-1, n-1)],\end{aligned}$$

where

$$\tilde{H}(i,j,k) = \begin{bmatrix} \tilde{c}_i & \tilde{c}_{i-1} & \cdots & \tilde{c}_{i-j} \\ \tilde{c}_{i+1} & \tilde{c}_i & \cdots & \tilde{c}_{i-j+1} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{c}_{i+k} & \tilde{c}_{i+k-1} & \cdots & \tilde{c}_{i+k-j} \end{bmatrix}.$$

Proof. (i) From the proof of Lemma 4.1, we have

$$^{L}\mathbf{R}^*(m,n)=\{R:R=z^{m+n+1}\sum_{i=1}^{n}\beta_i(\sum_{k=0}^{\infty}\tilde{c}_{m+n+k+1-i}z^k),\ \tilde{H}_T(m,n-1,n-1)B_T=0\}.$$

Therefore $^{L}\mathbf{R}^*(m,n)$ is a linear space.

(ii) Let

$$S_k=\{E^T\in\mathbf{C}^{d\times d}:E^T=\tilde{H}_T(m+n+k,n-1,0)B_T,\ \tilde{H}_T(m,n-1,n-1+k)B_T=0\},$$

$E^T_{k,1},\cdots,E^T_{k,l_k}$ be the base of S_k (l_k may be zero) and

$$B^T_{k,1},\cdots,B^T_{k,l_k}\in\{B_T:\tilde{H}_T(m,n-1,n-1)B_T=0\},$$

such that

$$E^T_{k,j}=\tilde{H}_T(m+n+k,n-1,0)B^T_{k,j},\quad j=1,2,\cdots,l_k.$$

Then for any

$$R(z)=z^{m+n+1}\sum_{k=0}^{\infty}E_kz^k=z^{m+n+1}\sum_{k=0}^{\infty}[\tilde{H}_T(m+n+k,n-1,0)B_T]^Tz^k$$

in $^{L}R^*(m,n)$, there exist constants $\alpha_{i,j}$ such that

$$E^T_k=\tilde{H}_T(m+n+k,n-1,0)\sum_{i=0}^{\infty}\sum_{j=1}^{l_i}\alpha_{i,j}B^T_{i,j}. \tag{4.7}$$

In fact we can prove by induction that there exist $\alpha_{i,j}$ such that

$$E^T_k=\tilde{H}_T(m+n+k,n-1,0)\sum_{i=0}^{k}\sum_{j=1}^{l_i}\alpha_{i,j}B^T_{i,j}. \tag{4.8}$$

For $i=0$, since $\{E^T_{0,j}\}$ is a base of S_0, there exist $\alpha_{0,j}$, $j=1,\cdots,l_0$, such that

$$E^T_0=\sum_{j=0}^{l_0}\alpha_{0,j}E^T_{0,j}=\tilde{H}_T(m+n,n-1,0)\sum_{j=1}^{l_0}\alpha_{0,j}B^T_{0,j}.$$

Suppose $\alpha_{i,j}$ have been determined for $i=0,1,\cdots,k-1$, and $j=1,\cdots,l_i$. Then by

$$\begin{aligned}E^T_k&=E^T_k-\tilde{H}_T(m+n+k,n-1,0)\sum_{i=0}^{k-1}\sum_{j=1}^{l_i}\alpha_{i,j}B^T_{i,j}\\&+\tilde{H}_T(m+n+k,n-1,0)\sum_{i=0}^{k-1}\sum_{j=1}^{l_i}\alpha_{i,j}B^T_{i,j}\\&=\tilde{H}_T(m+n+k,n-1,0)(B_T-\sum_{i=0}^{k-1}\sum_{j=1}^{l_i}\alpha_{i,j}B^T_{i,j})\\&+\tilde{H}_T(m+n+k,n-1,0)\sum_{i=0}^{k-1}\sum_{j=1}^{l_i}\alpha_{i,j}B^T_{i,j},\end{aligned}$$

and

$$\tilde{H}_T(m+n+l,n-1,0)(B_T-\sum_{i=0}^{k-1}\sum_{j=1}^{l_i}\alpha_{i,j}B_{i,j}^T)=E_l^T-E_l^T=0, \quad l=0,1,\cdots,k-1,$$

we have

$$\tilde{H}_T(m+n+k,n-1,0)(B_T-\sum_{i=0}^{k-1}\sum_{j=1}^{l_i}\alpha_{i,j}B_{i,j}^T)\in S_k.$$

Then there exist $\alpha_{k,j}$ such that

$$\tilde{H}_T(m+n+k,n-1,0)(B_T-\sum_{i=0}^{k-1}\sum_{j=1}^{l_i}\alpha_{i,j}B_{i,j}^T)$$
$$=\sum_{j=1}^{l_k}\alpha_{k,j}E_{k,j}^T=\tilde{H}_T(m+n+k,n-1,0)\sum_{j=1}^{l_k}\alpha_{k,j}B_{k,j}^T.$$

Hence (4.8) is proved. (4.7) follows from (4.8).

From (4.7) we have

$$\dim {}^L\mathbf{R}^*(m,n)=\sum_{k=0}^{\infty}\dim S_k,$$

so we need only to consider the dimension of the linear space S_k.

Now we note that

$$X^k=\{(x_1,x_2,\ldots,x_k)\in \mathbf{C}^{d\times k}:x_i\in X\subset \mathbf{C}^d\}$$

is a $k\cdot\dim X$ dimensional linear subspace of $\mathbf{C}^{d\times k}$, provided X is a linear subspace of $\mathbf{C}^d$. Therefore, from the fact that

$$S_k=\{\text{rang}(\tilde{H}_T(m+n+k,n-1,0),\tilde{H}_T(m,n-1,n-1+k))\}^d,$$

and Lemma 4.2, we have

$$\dim S_k=d\cdot[\text{rank }\tilde{H}_T(m,n-1,n+k)-\text{rank }\tilde{H}_T(m,n-1,n-1+k)].$$

Then the theorem is proved by using Lemma 2.1.

Now we establish the uniqueness theorem.

Theorem 4.2. *Suppose* ${}^L[m/n]_f$ *exists. Then the following statements are equivalent.*

(i) ${}^L[m/n]_f$ *is unique,*

(ii) ${}^R[m/n]_f$ *exists,*

(iii) *There exists a* $(P,Q)\in {}^R\mathbf{R}(m,n)$, *such that* $\det Q\neq 0$,

(iv) *For any* $(P,Q)\in {}^L\mathbf{R}(m,n)$,

$$Q\,{}^L[m/n]_f-P=0. \tag{4.9}$$

Proof. a) (i) $\Rightarrow$ (*ii*). If ${}^L[m/n]_f$ is unique, then by Theorem 4.1 we have

$$\text{rank } \tilde{H}(m+1,n,n-1) - \text{rank } \tilde{H}(m,n-1,n-1)] = 0.$$

Then from (4.6), relation (2.2) is valid; this implies the existence of ${}^R[m/n]_f$.

b) (ii) $\Rightarrow$ (iii). Let ${}^R[m/n]_f = PQ^{-1}$. Then $(P,Q) \in {}^R\mathbf{R}(m,n)$ and $detQ \neq 0$.

c) (iii) $\Rightarrow$ (iv). Let $({}^RP, {}^RQ) \in {}^R\mathbf{R}(m,n)$, and $det^RQ \neq 0$. Then for any $(P,Q) \in {}^L\mathbf{R}(m,n)$, by Lemma 3.1 we have

$$\begin{aligned}(Q\,{}^L[m/n]_f - P)\,{}^RQ &= QQ^{*^{-1}}P^*\,{}^RQ - P\,{}^RQ \\ &= QQ^{*^{-1}}Q^*\,{}^RP - Q\,{}^RP = Q\,{}^RP - Q\,{}^RP = 0,\end{aligned}$$

where ${}^L[m/n]_f = Q^{*^{-1}}P^*$. Hence (4.9) holds.

d) (iv) $\Rightarrow$ (i). From (4.9), one has ${}^L\mathbf{R}^*(m,n) = \{0\}$. Then ${}^L[m/n]_f$ is unique.

Remark. The results obtained in §3 and §4 can be established for right matrix Padé approximants in a similar manner.

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