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A TRILAYER DIFFERENCE SCHEME FOR ONE-DIMENSIONAL PARABOLIC SYSTEMS*

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In order to obtain the numerical solution for a one-dimensional parabolic system, an unconditionally stable difference method is investigated in [1]. If the number of unknown functions is M, for each time step only M times of calculation are needed. The rate of convergence is $O(\tau + h^2)$. On the basis of [1], an alternating calculation difference scheme is presented in [2]; the rate of convergence is $O(\tau^2 + h^2)$. The difference schemes in [1] and [2] are economic ones. For the α-th equation, only U_α is an unknown function; the others, $U_\beta(\beta = 1, 2, \cdots, \alpha - 1, \alpha + 1, \cdots, M)$, are given evaluated either in the last step or in the present step. So the practical calculation is quite convenient.

The purpose of this paper is to derive a trilayer difference scheme for one-dimensional parabolic systems. It is shown that the scheme is also unconditionally stable and the rate of convergence is $O(\tau^2 + h^2)$.

§1

On the domain $D\{0 < x < 1, 0 < t \le T\}$, we consider the partial differential equations

$$\frac{\partial}{\partial t}u_\alpha(x,t) = \sum_{\beta=1}^{M} \frac{\partial}{\partial x}\Big[K_{\alpha\beta}(x,t)\frac{\partial}{\partial x}u_\beta(x,t)\Big], \alpha = 1, \cdots, M, \tag{1}$$

with the initial and boundary conditions

$$u_\alpha(x,0) = u_\alpha^0(x), u_\alpha(0,t) = 0, u_\alpha(1,t) = 0, \alpha = 1, \cdots, M. \tag{2}$$

Suppose that the coefficients of equations (1) satisfy the following conditions:

K1°. $K_{\alpha\beta}(x,t) = K_{\beta\alpha}(x,t)$;

K2°. there exist positive constants $\sigma_1 > 0, \sigma_2 > 0$, such that

$$\sigma_1 \sum_{\alpha=1}^{M} \xi_\alpha^2 \le \sum_{\alpha,\beta=1}^{M} K_{\alpha\beta}(x,t)\xi_\alpha\xi_\beta \le \sigma_2 \sum_{\alpha=1}^{M} \xi_\alpha^2, \quad (x,t) \in D,$$

for any M-dimensional real vectors $\vec{\xi} \in I\!R^m$;

K3°. the coefficients $K_{\alpha\beta}$ of equations (1) are sufficiently smooth on the domain $D\{0 \le x \le 1, 0 \le t \le T\}$ and especially, there exists a constant $K > 0$, so that $|K_{\alpha\beta}(x,t)| < K, \left|\frac{K_{\alpha\beta}(x,t+\tau) - K_{\alpha\beta}(x,t)}{\tau}\right| < K$.

Since the coefficients satisfy condition K2°, equations (1) belong to the parabolic system. In addition, we assume that there exist unique sufficiently smooth solutions of equations (1) with initial and boundary conditions (2).

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We solve the problem (1)–(2) by the difference method. Divide the intervals [0, 1] and $[0,T]$ into J and N points respectively. The space step is $h = 1/J$ and the time step is $\tau = T/N$. Let $\omega_h = \{x_j = jh | j = 0,1,\cdots,J\}$ and $\omega_\tau = \{t^n = n\tau | n = 0,1,\cdots,N\}$. The set of all net points on the domain $\bar{D}$ is denoted by $\bar{\Omega} = \omega_h \times \omega_\tau$, and $\Omega = \bar{\Omega} \cap D$.

Let $U(x,t)$ and $V(x,t)$ be the discrete functions, defined on the set $\bar{\Omega}$. Introduce the following notations:

$$U^n \equiv U_j^n = U(jh, n\tau),$$

$$\begin{aligned} U_{\bar{x}}^n &= U_{\bar{x},j}^n = \frac{1}{h}(U_j^n - U_{j-1}^n), \quad U_x^n = U_{x,j}^n = \frac{1}{h}(U_{j+1}^n - U_j^n), \\ U_{\bar{t}}^n &= U_{\bar{t},j}^n = \frac{1}{\tau}(U_j^n - U_j^{n-1}), \quad U_{\overset{\circ}{t}}^n = U_{\overset{\circ}{t},j}^n = \frac{1}{2\tau}(U_j^{n+1} - U_j^{n-1}). \end{aligned} \tag{3}$$

Define the following scalar products and norms:

$$(U^n, V^n) = \sum_{j=1}^{J-1} U_j^n V_j^n h, \qquad (U^n, V^n] = \sum_{j=1}^{J} U_j^n V_j^n h,$$
$$\|U^n\| = \sqrt{(U^n, U^n)}, \quad \|U^n]| = \sqrt{(U^n, U^n]}, \quad \|U^n\|_\infty = \max_{x \in \omega_h} |U(x, t^n)|.$$

If $U_0^n = U_J^n = 0$ in the interval $0 \le x \le 1$, then there is Green's difference formula

$$(U^n, V_x^n) = -(U_{\bar{x}}^n, V^n] \tag{4}$$

and the relations [3]

$$\|U^n\| \le \|U^n\|_\infty \le \frac{1}{2}\|U_{\bar{x}}^n]|. \tag{5}$$

For the problem (1)–(2), finite difference equations may be constructed in various ways. If we use explicit difference schemes, we have to consider the restriction of the stability condition and require small computational steps. If we adopt fully implicit difference schemes, the iterative computation leads to a huge amount of calculation; so we must consider economic schemes, which both are unconditionally stable and require small amount of calculation. For instance, the following difference scheme is investigated in [1]:

$$U_{\alpha,\bar{t}}^{n+1} = \sum_{\beta=1}^{\alpha-1} \left(a_{\alpha\beta}^{n+1} U_{\beta,\bar{x}}^{n+1}\right)_x + \left(\theta a_{\alpha\alpha}^{n+1} U_{\alpha,\bar{x}}^{n+1} + (1-\theta) a_{\alpha\alpha}^n U_{\alpha,\bar{x}}^n\right)_x + \sum_{\beta=\alpha+1}^{M} \left(a_{\alpha\beta}^n U_{\beta\bar{x}}^n\right)_x,$$
$$(x,t) \in \Omega, \alpha = 1,\cdots,M. \tag{6}$$

where $0.5 \le \theta \le 1$ is an arbitrarily selected parameter, and

$$a_{\alpha\beta}^n = a_{\alpha\beta,j}^n = K_{\alpha\beta}\big((j - \frac{1}{2})h, n\tau\big), \quad \alpha, \beta = 1, \cdots, M.$$

In [2], an alternating calculation difference scheme is considered:

$$U_{\alpha,\bar{t}}^{2n+1} = \sum_{\beta=1}^{\alpha-1} (a_{\alpha\beta}^{2n+1} U_{\beta,\bar{x}}^{2n+1})_x + \frac{1}{2}(a_{\alpha\alpha}^{2n+1} U_{\alpha,\bar{x}}^{2n+1} + a_{\alpha\alpha}^{2n} U_{\alpha,\bar{x}}^{2n})_x + \sum_{\beta=\alpha+1}^{M} (a_{\alpha\beta}^{2n} U_{\beta,\bar{x}}^{2n})_x, \tag{7}$$
$$(x,t) \in \Omega, \alpha = 1, \cdots, M,$$

$$U_{\alpha,\bar{t}}^{2n+2}=\sum_{\beta=1}^{\alpha-1}(a_{\alpha\beta}^{2n+1}U_{\beta,\bar{x}}^{2n+1})_x+\frac{1}{2}(a_{\alpha\alpha}^{2n+1}U_{\alpha,\bar{x}}^{2n+1}+a_{\alpha\alpha}^{2n+2}U_{\alpha,\bar{x}}^{2n+2})_x+\sum_{\beta=\alpha+1}^{M}(a_{\alpha\beta}^{2n+2}U_{\beta,\bar{x}}^{2n+2})_x,$$
$$(x,t)\in\Omega,\alpha=M,\cdots,1. \tag{8}$$

Equations (7) and (8) are used for calculations for odd and even time steps respectively.

In this paper we consider another unconditionally stable and highly accurate difference scheme, of the following form:

$$U_{\alpha,\overset{\circ}{t}}^{n}=\mu(U_\alpha^{n+1}-2U_\alpha^n+U_\alpha^{n-1})_{\bar{x}x}+\sum_{\beta=1}^{M}(a_{\alpha\beta}^n U_{\beta,\bar{x}}^n)_x,$$
$$(x,t)\in\Omega,\quad \alpha=1,\cdots,M, \tag{9}$$

$$U_{\alpha,j}^0=u_\alpha^0(jh),j=1,\cdots,J-1,\ U_{\alpha,0}^n=U_{\alpha,J}^n=0,\quad n=0,1,\cdots,N,\quad \alpha=1,\cdots,M, \tag{10}$$

where

$$\mu>\frac{1}{4}\sigma_2.$$

§2

In this section we consider the error estimation of the discrete solutions for difference scheme (9)–(10). Let $Z_{\alpha}(x,t)$ be the difference between the solutions of the difference equations (9)–(10) and those of the differential equations (1) with additional conditions (2), namely

$$Z_\alpha(x,t)=U_\alpha(x,t)-u_\alpha(x,t),\alpha=1,\cdots,M,(x,t)\in\bar{\Omega}.$$

Putting $U_\alpha=Z_\alpha+u_\alpha$ into the difference equations (9), (10) we obtain the following equations which the error functions satisfy:

$$Z_{\alpha,\overset{\circ}{t}}^{n}-\mu(Z_\alpha^{n+1}-2Z_\alpha^n+Z_\alpha^{n-1})_{\bar{x},x}-\sum_{\beta=1}^{M}(a_{\alpha\beta}^n Z_{\beta,\bar{x}}^n)_x=\varphi_\alpha^n,$$
$$(x,t)\in\Omega,\quad \alpha=1,\cdots,M, \tag{11}$$

$$Z_{\alpha,j}^0=0,j=1,\cdots,J-1,\ Z_{\alpha,0}^n=Z_{\alpha,J}^n=0,\quad n=0,1,\cdots,N,\quad \alpha=1,\cdots,M, \tag{12}$$

where the term on the right-hand side of (11) is the truncation error:

$$\varphi_\alpha^n=-u_{\alpha,\overset{\circ}{t}}^{n}+\mu(u_\alpha^{n+1}-2u_\alpha^n+u_\alpha^{n-1})_{\bar{x},x}+\sum_{\beta=1}^{M}(a_{\alpha\beta}^n u_{\beta,\bar{x}}^n)_x,\quad \alpha=1,\cdots,M. \tag{13}$$

Taking the scalar product of $2\tau Z_{\alpha,\overset{\circ}{t}}^{n}$ in the α-th equation in (11), and then summing up the resulting relations for $\alpha=1,\cdots,M$, we obtain

$$2\tau\sum_{\alpha=1}^{M}(Z_{\alpha,\overset{\circ}{t}}^{n},Z_{\alpha,\overset{\circ}{t}}^{n})-2\tau\sum_{\alpha=1}^{M}(Z_{\alpha,\overset{\circ}{t}}^{n},\mu(Z_\alpha^{n+1}-2Z_\alpha^n+Z_\alpha^{n-1})_{\bar{x},x})$$
$$-2\tau\sum_{\alpha=1}^{M}(Z_{\alpha,\overset{\circ}{t}}^{n},\sum_{\beta=1}^{M}(a_{\alpha\beta}^n Z_{\beta,\bar{x}}^n)_x)=2\tau\sum_{\alpha=1}^{M}(Z_{\alpha,\overset{\circ}{t}}^{n},\varphi_\alpha^n). \tag{14}$$

By Green's difference formula (4) and the following relations

$$2\tau Z^n_{\alpha,\overset{\circ}{t}} = \tau(Z^n_{\alpha,t} + Z^n_{\alpha,\bar{t}}) = Z^{n+1}_\alpha - Z^{n-1}_\alpha,$$
$$Z^n_\beta = \frac{1}{2}(Z^{n+1}_\beta + Z^{n-1}_\beta) - \frac{1}{2}\tau^2 Z^n_{\beta,\bar{t}t} = \frac{1}{2}(Z^{n+1}_\beta + Z^{n-1}_\beta) - \frac{1}{2}\tau(Z^n_{\beta,t} - Z^n_{\beta,\bar{t}}),$$

equation (14) becomes

$$2\tau\sum_{\alpha=1}^{M}(Z^n_{\alpha,\overset{\circ}{t}}, Z^n_{\alpha,\overset{\circ}{t}}) + \tau^2\sum_{\alpha=1}^{M}\Big((Z^n_{\alpha,t} + Z^n_{\alpha,\bar{t}})_x, \mu(Z^n_{\alpha,t} - Z^n_{\alpha,\bar{t}})_x\Big]$$
$$+\frac{1}{2}\sum_{\alpha=1}^{M}\Big((Z^{n+1}_\alpha - Z^{n-1}_\alpha)_x, \sum_{\beta=1}^{M} a^n_{\alpha\beta}(Z^{n+1}_\beta + Z^{n-1}_\beta)_x\Big]$$
$$-\frac{\tau^2}{2}\sum_{\alpha=1}^{M}\Big((Z^n_{\alpha,t} + Z^n_{\alpha,\bar{t}})_x, \sum_{\beta=1}^{M} a^n_{\alpha\beta}(Z^n_{\beta,t} - Z^n_{\beta,\bar{t}})_x\Big]$$
$$= 2\tau\sum_{\alpha=1}^{M}(Z^n_{\alpha,\overset{\circ}{t}}, \varphi^n_\alpha).$$

Through certain verification and simplification, we get the following equation:

$$\begin{aligned}
&2\tau\sum_{\alpha=1}^{M}\|Z^n_{\alpha,\overset{\circ}{t}}\|^2 + \frac{1}{2}\sum_{\alpha=1}^{M}((Z^{n+1}_\alpha - Z^{n-1}_\alpha)_x, \sum_{\beta=1}^{M} a^n_{\alpha\beta}(Z^{n+1}_\beta + Z^{n-1}_\beta)_x]\\
&+\tau^2\sum_{\alpha=1}^{M}((Z^n_{\alpha,t} + Z^n_{\alpha,\bar{t}})_x, (\mu Z^n_{\alpha,tx} - \frac{1}{2}\sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^n_{\beta,tx})]\\
&-\tau^2\sum_{\alpha=1}^{M}((Z^n_{\alpha,t} + Z^n_{\alpha,\bar{t}})_x, (\mu Z^n_{\alpha,\bar{t}x} - \frac{1}{2}\sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^n_{\beta,\bar{t}x})]\\
&= 2\tau\sum_{\alpha=1}^{M}(Z^n_{\alpha,\overset{\circ}{t}}, \varphi^n_\alpha).
\end{aligned} \tag{15}$$

In order to obtain energy estimate, we will derive each term of (15) in detail. The second term on the left-hand side of (15) can be written as

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n+1}_{\alpha,x} - Z^{n-1}_{\alpha,x}, \sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^{n+1}_{\beta,x} + \sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^{n-1}_{\beta x}]\\
&= \frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n+1}_{\alpha,x}, \sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^{n+1}_{\beta,x}] + \frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n+1}_{\alpha,x}, \sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^{n-1}_{\beta,x}]\\
&-\frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,x}, \sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^{n+1}_{\beta,x}] - \frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,x}, \sum_{\beta=1}^{M} a^n_{\alpha\beta}Z^{n-1}_{\beta,x}].
\end{aligned} \tag{16}$$

According to the coefficient condition K1°, we get

$$\frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n+1}_{\alpha,\bar{x}},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n-1}_{\beta,\bar{x}}] = \frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,\bar{x}},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n+1}_{\beta,\bar{x}}]. \tag{17}$$

Substituting (17) into (16), we have

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n+1}_{\alpha,\bar{x}} - Z^{n-1}_{\alpha,\bar{x}},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n+1}_{\beta,\bar{x}} + \sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n-1}_{\beta,\bar{x}}] \\
&\quad = \frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n+1}_{\alpha,\bar{x}},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n+1}_{\beta,\bar{x}}] - \frac{1}{2}\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,\bar{x}},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n-1}_{\beta,\bar{x}}] \\
&\quad = \frac{1}{4}\sum_{\alpha=1}^{M}((Z^{n+1}_{\alpha} + Z^{n}_{\alpha})_{\bar{x}},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}(Z^{n+1}_{\beta} + Z^{n}_{\beta})_{\bar{x}}] \\
&\quad -\frac{1}{4}\sum_{\alpha=1}^{M}((Z^{n}_{\alpha} + Z^{n-1}_{\alpha})_{\bar{x}},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}(Z^{n}_{\beta} + Z^{n-1}_{\beta})_{\bar{x}}] \\
&\quad +\frac{\tau^2}{4}\sum_{\alpha=1}^{M}(Z^{n}_{\alpha,tx},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n}_{\beta,tx}] - \frac{\tau^2}{4}\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,tx},\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n-1}_{\beta,tx}].
\end{aligned} \tag{18}$$

Using again condition K1°, we can write the third and fourth terms of (15) as

$$\begin{aligned}
&\tau^2\sum_{\alpha=1}^{M}(Z^{n}_{\alpha,tx} + Z^{n}_{\alpha,\bar{t}x},\mu Z^{n}_{\alpha,tx} - \frac{1}{2}\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n}_{\beta,tx}] \\
&\quad -\tau^2\sum_{\alpha=1}^{M}(Z^{n}_{\alpha,tx} + Z^{n}_{\alpha,\bar{t}x},\mu Z^{n}_{\alpha,\bar{t}x} - \frac{1}{2}\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n}_{\beta,\bar{t}x}] \\
&\quad = \tau^2\sum_{\alpha=1}^{M}(Z^{n}_{\alpha,tx},\mu Z^{n}_{\alpha,tx} - \frac{1}{2}\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n}_{\beta,tx}] \\
&\quad -\tau^2\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,tx},\mu Z^{n-1}_{\alpha,tx} - \frac{1}{2}\sum_{\beta=1}^{M}a^{n}_{\alpha\beta}Z^{n-1}_{\beta,tx}].
\end{aligned} \tag{19}$$

Putting (18), (19) into (15) and rearranging, we obtain

$$\begin{aligned}
&2\tau\sum_{\alpha=1}^{M}\|Z^{n}_{\alpha,\bar t}\|^2+\frac{1}{4}\sum_{\alpha=1}^{M}((Z^{n+1}_\alpha+Z^n_\alpha)_x,\sum_{\beta=1}^{M}a^n_{\alpha\beta}(Z^{n+1}_\beta+Z^n_\beta)_x]\\
&+\tau^2\sum_{\alpha=1}^{M}(Z^n_{\alpha,tx},\mu Z^n_{\alpha,tx}-\frac{1}{4}\sum_{\beta=1}^{M}a^n_{\alpha\beta}Z^n_{\beta,tx}]\\
&=\frac{1}{4}\sum_{\alpha=1}^{M}((Z^n_\alpha+Z^{n-1}_\alpha)_x,\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta}(Z^n_\beta+Z^{n-1}_\beta)_x]\\
&+\frac{\tau}{4}\sum_{\alpha=1}^{M}((Z^n_\alpha+Z^{n-1}_\alpha)_x,\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta,t}(Z^n_\beta+Z^{n-1}_\beta)_x]\\
&+\tau^2\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,tx},\mu Z^{n-1}_{\alpha,tx}-\frac{1}{4}\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta}Z^{n-1}_{\beta,tx}]\\
&+\tau^3\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,tx},-\frac{1}{4}\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta,t}Z^{n-1}_{\beta,tx}]+2\tau\sum_{\alpha=1}^{M}(Z^{n}_{\alpha,\bar t},\varphi^n_\alpha).
\end{aligned}\tag{20}$$

Let

$$E^n=\frac{1}{4}\sum_{\alpha=1}^{M}((Z^n_\alpha+Z^{n-1}_\alpha)_x,\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta}(Z^n_\beta+Z^{n-1}_\beta)_x]+\tau^2\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,tx},\mu Z^{n-1}_{\alpha,tx}-\frac{1}{4}\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta}Z^{n-1}_{\beta,tx}].\tag{21}$$

Then formula (20) can be rewritten as

$$2\tau\sum_{\alpha=1}^{M}\|Z^{n}_{\alpha,\bar t}\|^2+E^{n+1}=E^n+2\tau\sum_{\alpha=1}^{M}(Z^{n}_{\alpha,\bar t},\varphi^n_\alpha)+\tau\Psi^n,\tag{22}$$

where

$$\begin{aligned}
\Psi^n&=\frac{1}{4}\sum_{\alpha=1}^{M}((Z^n_\alpha+Z^{n-1}_\alpha)_x,\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta,t}(Z^n_\beta+Z^{n-1}_\beta)_x]\\
&+\tau^2\sum_{\alpha=1}^{M}(Z^{n-1}_{\alpha,tx},-\frac{1}{4}\sum_{\beta=1}^{M}a^{n-1}_{\alpha\beta,t}Z^{n-1}_{\beta,tx}].
\end{aligned}\tag{23}$$

According to the coefficient conditions K2°, K3° and the assumption $\mu>\frac{1}{4}\sigma_2$, the following inequalities hold:

$$|(Z_\alpha^n + Z_\alpha^{n-1}, -\sum_{\beta=1}^{M} a_{\alpha\beta,t}^{n-1}(Z_\beta^n + Z_\beta^{n-1})_{x\bar{x}})| \le C_2(Z_\alpha^n + Z_\alpha^{n-1}, -\sum_{\beta=1}^{M} a_{\alpha\beta}^{n-1}(Z_\beta^n + Z_\beta^{n-1})_{x\bar{x}}),$$

$$|(Z_{\alpha,t}^{n-1}, -\mu Z_{\alpha t x\bar{x}}^{n-1} + \frac{1}{4}\sum_{\beta=1}^{M} a_{\alpha\beta t}^{n-1} Z_{\beta t x\bar{x}}^{n-1})| \le C_2(Z_{\alpha,t}^{n-1}, -\mu Z_{\alpha t x\bar{x}}^{n-1} + \frac{1}{4}\sum_{\beta=1}^{M} a_{\alpha\beta}^{n-1} Z_{\beta t x\bar{x}}^{n-1}). \tag{24}$$

Thus, expression (23) becomes

$$|\Psi^n| \le C_2\Big\{\frac{1}{4}\sum_{\alpha=1}^{M}((Z_\alpha^n + Z_\alpha^{n-1})_x, \sum_{\beta=1}^{M} a_{\alpha\beta}^{n-1}(Z_\beta^n + Z_\beta^{n-1})_x]$$

$$+\tau^2 \sum_{\alpha=1}^{M}(Z_{\alpha,tx}^{n-1}, \mu Z_{\alpha,tx}^{n-1} - \frac{1}{4}\sum_{\beta=1}^{M} a_{\alpha\beta}^{n-1} Z_{\beta,tx}^{n-1}]\Big\} = C_2 E^n.$$

Therefore, equation (22) can be written as

$$E^{n+1} + \tau \sum_{\alpha=1}^{M} \|Z_{\alpha,\hat{t}}^{n}\|^2 \le E^n + \tau \sum_{\alpha=1}^{M} \|\varphi_\alpha^n\|^2 + \tau C_2 E^n,$$

namely

$$E^{n+1} \le (1 + C_2\tau)E^n + \tau \sum_{\alpha=1}^{M} \|\varphi_\alpha^n\|^2. \tag{25}$$

Summing over $n = 1, \cdots, N-1$, we have

$$E^N \le E^1 + \tau C_2 \sum_{n=1}^{N-1} E^n + \sum_{n=1}^{N-1}\sum_{\alpha=1}^{M} \|\varphi_\alpha^n\|^2 \tau. \tag{26}$$

By means of inequality (26), we can derive the following lemmas and theorems about the stability and convergence of the difference scheme.

§3

Lemma 1. *Suppose that the coefficients satisfy* K1°–K3°. *Then the difference scheme* (11)–(13) *is stable with respect to the right-hand side terms* $\varphi_\alpha^n, \tilde{\phi}_\alpha^1, \tilde{\phi}_\alpha^2$, *where* $\tilde{\phi}_\alpha^1$ *and* $\tilde{\phi}_\alpha^2$ *denote the truncation errors with the difference schemes* (7) *and* (8) *at* $t = \tau^*$ *and* $t = 2\tau^*$ *respectively.*

Proof. From the expression (21) for E^n, it is shown that

$$E^1 = \frac{1}{4}\sum_{\alpha=1}^{M}((Z_\alpha^1 + Z_\alpha^0)_x, \sum_{\beta=1}^{M} a_{\alpha\beta}^0(Z_\beta^1 + z_\beta^0)_x] + \tau^2 \sum_{\alpha=1}^{M}(Z_{\alpha,tx}^0, \mu Z_{\alpha tx}^0 - \frac{1}{4}\sum_{\beta=1}^{M} a_{\alpha\beta}^0 Z_{\beta,tx}^0].$$

By initial condition $Z^0_{\alpha,j}=0$, we get

$$\begin{aligned}E^1 &= \frac{1}{4}\sum_{\alpha=1}^{M}(Z^1_{\alpha,x},\sum_{\beta=1}^{M}a^0_{\alpha\beta}Z^1_{\beta,x}] + \sum_{\alpha=1}^{M}(Z^1_{\alpha,x},\mu Z^1_{\alpha,x} - \frac{1}{4}\sum_{\beta=1}^{M}a^0_{\alpha\beta}Z^1_{\beta,x}] \\ &= \mu\sum_{\alpha=1}^{M}(Z^1_{\alpha,x},Z^1_{\alpha,x}] = \mu\sum_{\alpha=1}^{M}\|Z^1_{\alpha,x}\|^2.\end{aligned} \tag{27}$$

In the case of the trilayer difference scheme, besides the given initial value(called the value of zeroth step), the first step's value, which is calculated by the two-layer difference scheme, has to be given. In order to guarantee accuracy of the numerical solution, generally we use scheme of the same order of convergence. Since the trilayer difference scheme is of second order convergence, we consider using the alternating difference scheme (7)–(8) with computing setp $\tau^*=0.5\tau$, and use the values calculated at the time $t=2\tau^*=\tau$ as the value of the first setp for the trilayer difference scheme.

Let $\tilde{Z}_\alpha$ denote the errors of the alternating calculation difference scheme. Then we have

$$Z^1_\alpha = \tilde{Z}^2_\alpha, \quad \alpha=1,\cdots M. \tag{28}$$

In view of the estimate of $\tilde{Z}_\alpha$[2],

$$\sum_{\alpha=1}^{M}\|\tilde{Z}^2_{\alpha,x}\|^2 \le \frac{1}{\sigma_1}\tilde{E}^2, \tag{29}$$

where

$$\tilde{E}^i = \sum_{\alpha=1}^{M}(\tilde{Z}^i_{\alpha,x},\sum_{\beta=1}^{M}a^i_{\alpha\beta}\tilde{Z}^i_{\beta,x}], \quad i=0,1,2,\cdots. \tag{30}$$

Simultaneously, from the estimate of $\tilde{E}^2$[2], we get

$$\tilde{E}^2 \le (1+C_3\tau^*)\tilde{E}^0 + (1+\frac{C_4\tau^*}{T})\sum_{\alpha=1}^{M}(\|\tilde{\phi}^1_\alpha\|^2+\|\tilde{\phi}^2_\alpha\|^2)\tau^*. \tag{31}$$

So from (27)–(31), we obtain

$$\begin{aligned}E^1 &\le \mu\sum_{\alpha=1}^{M}\|Z^1_{\alpha,x}\|^2 \le \frac{\mu}{\sigma_1}\tilde{E}^2 \\ &\le \frac{\mu}{\sigma_1}\{(1+C_3\tau)\tilde{E}^0 + (1+\frac{C_4\tau}{T})\sum_{\alpha=1}^{M}(\|\tilde{\phi}^1_\alpha\|^2+\|\tilde{\phi}^2_\alpha\|^2)\tau\}.\end{aligned} \tag{32}$$

Substituting (32) into (26), we have

$$E^N \le \frac{\mu}{\sigma_1}\tilde{E}^0 + C_5\tau\sum_{n=0}^{N-1}E^n + T\sum_{\alpha=1}^{M}\Big(\|\tilde{\phi}^1_\alpha\|^2+\|\tilde{\phi}^2_\alpha\|^2+\max_{1\le n\le N-1}\|\varphi^n_\alpha\|^2\Big). \tag{33}$$

Thus, according to the initial condition (12) and expression (30), and by the Gronwall inequality, it follows that

$$E^N \le e^{C_6 T} \cdot T \sum_{\alpha=1}^{M} \left(\|\tilde{\phi}_\alpha^1\|^2 + \|\tilde{\phi}_\alpha^2\|^2 + \max_{1\le n\le N-1} \|\varphi_\alpha^n\|^2 \right). \tag{34}$$

On the other hand, in view of the definition (21) of E^n, coefficient condition K2° and assumption $\mu > \frac{1}{4}\sigma_2$, we can have estimate

$$\begin{aligned} E^n &\ge \frac{\sigma_1}{4} \sum_{\alpha=1}^{M} \|Z_{\alpha,x}^n + Z_{\alpha,x}^{n-1}\|^2 + \tau^2(\mu - \frac{1}{4}\sigma_2) \sum_{\alpha=1}^{M} \|Z_{\alpha,tx}^{n-1}\|^2 \\ &\ge \frac{\sigma_1}{4} \sum_{\alpha=1}^{M} \|Z_{\alpha,x}^n + Z_{\alpha,x}^{n-1}\|^2 \end{aligned}$$

for $n = 1, 2, \cdots, N$. Hence

$$\sum_{\alpha=1}^{M} \|Z_{\alpha,x}^N + Z_{\alpha,x}^{N-1}\|^2 \le \frac{4}{\sigma_1} E^N. \tag{35}$$

From the equivalence relation of norms, (5), we get

$$\sum_{\alpha=1}^{M} \|Z_\alpha^N + Z_\alpha^{N-1}\|_\infty^2 \le \frac{1}{\sigma_1} E^N. \tag{36}$$

Substituting (34) into (36), we have

$$\sum_{\alpha=1}^{M} \|Z_\alpha^N + Z_\alpha^{N-1}\|_\infty^2 \le \frac{T}{\sigma_1} e^{C_6 T} \sum_{\alpha=1}^{M} \left(\|\tilde{\phi}_\alpha^1\|_\infty^2 + \|\tilde{\phi}_\alpha^2\|_\infty^2 + \max_{1\le n\le N-1} \|\varphi_\alpha^n\|_\infty^2 \right). \tag{37}$$

It follows that the difference scheme (11)-(13) with respect to the right-hand side is stable in the maximum norm.

Lemma 2. *Suppose that the coefficients satisfy* K1° – K3°. *Moreover, assume the right-hand side terms* $\varphi_\alpha^n, \tilde{\phi}_\alpha^1, \tilde{\phi}_\alpha^2$ *of difference scheme* (11)-(13) *equal zero, but the initial condition is not zero. Then the scheme with respect to the initial condition is stable.*

Proof. By assumptions inequality (33) becomes

$$E^N \le \frac{\mu}{\sigma_1} \tilde{E}^0 + C_5 \tau \sum_{n=0}^{N-1} E^n \le \frac{\mu}{\sigma_1} e^{C_7 T} \tilde{E}^0. \tag{38}$$

Substituting (36) into it, we have

$$\sum_{\alpha=1}^{M} \|Z_\alpha^N + Z_\alpha^{N-1}\|_\infty^2 \le \frac{\mu}{\sigma_1^2} e^{C_7 T} \max_{\alpha,\beta} a_{\alpha\beta}^0 \sum_{\alpha,\beta=1}^{M} (|Z_{\alpha,x}^0|, |Z_{\beta,x}^0| \le \frac{\mu}{\sigma_1^2} e^{C_7 T} M K \sum_{\alpha=1}^{M} \|Z_{\alpha,x}^0\|^2. \tag{39}$$

It follows that the difference scheme (11)–(13) with respect to the initial condition is stable in the maximum norm.

Therefore, from Lemma 1 and Lemma 2 we can give the following theorems directly.

Theorem 1. *Suppose that the coefficients satisfy* K1° – K3°. *Then the difference scheme* (9)–(10) *is stable with respect to the initial condition.*

Theorem 2. *Suppose that the coefficients* $K_{\alpha\beta}(\alpha, \beta = 1, \cdots, M)$ *of the parabolic system* (1) *are sufficiently smooth and satisfy* K1° – K3°. *Moreover, there exist unique sufficiently smooth solutions* $u_\alpha(\alpha = 1, \cdots, M)$ *for equations* (1) *with initial and boundary conditions* (2). *Then the solutions* $U_\alpha(\alpha = 1, \cdots, M)$ *of the difference scheme* (9)–(10) *unconditionally converge to* $u_\alpha(\alpha = 1, \cdots, M)$ *as* $h \to 0, \tau \to 0$, *and the rate of convergence is* $O(\tau^2 + h^2)$, *namely*

$$\sum_{\alpha=1}^{M} \|U_\alpha^N - u_\alpha^N\|_\infty \le C(\tau^2 + h^2). \tag{40}$$

In fact, $U_\alpha - u_\alpha = Z_\alpha$, and Z_α satisfy the difference equations and additional conditions (11), (12), where φ_α^n are expressed by (13). According to the assumptions of Theorem 2, we get $\varphi_\alpha^n = O(\tau^2 + h^2)$. In addition, according to the error estimate for the alternating difference scheme[2], we have $\tilde{\phi}_\alpha^1 = O(\tau^2 + h^2), \tilde{\phi}_\alpha^2 = O(\tau^2 + h^2)$. Therefore, from (37) we obtain

$$\sum_{\alpha=1}^{M} \|U_\alpha^N - u_\alpha^N\|_\infty \le \sum_{\alpha=1}^{M} \|Z_\alpha^N + Z_\alpha^{N-1}\|_\infty^2 \le C(\tau^2 + h^2).$$

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