Journal of Computational Mathematics, Vol.9, No.4, 1991, 369–377.

A GENERAL NUMERICAL METHOD FOR SOLVING RIEMANN PROBLEMS*[1)]

Wu Xiong-hua
(*Tongji University, Shanghai, China*)
Zhu You-lan Chen Bing-mu
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

In this paper a general numerical method for solving Riemann problems is discussed. It can be used to solve the Riemann problems of various hyperbolic systems of differential equations with two independent variables. The problem of reflection of discontinuities from external boundaries can also be solved by this general numerical method.

§1. Introduction

It is well-known that the Riemann problem plays a very important role in the theory of hyperbolic differential equations. Solving Riemann problems is the foundation of many numerical methods in this field. In the singularity-separating method[1],[2], in order to deal with interactions between discontinuities and reflection of discontinuities from boundaries accurately, it is also necessary to solve Riemann problems or some similar problems. In this paper a general numerical method is discussed. It can be used to solve the Riemann problem of various hyperbolic systems of differential equations with two independent variables. It is natural that users must give some necessary information when they use it. For example, the number of equations, the number of the distinct characteristic values, the multiplicity of every characteristic value, the physical quantities on the left and the right sides, the relations between physical quantities on the two sides of every kind of discontinuity, the entropy conditions, the relations among the physical quantities in every kind of central wave, the values of some parameters in equations and so on. If the above information is given, the solution of the Riemann problem, the types and the velocities of the discontinuity lines and weak discontinuity lines and the physical quantities between them, can be determined by our numerical method.

§2. Determining Equations for Riemann Problems

To solve a Riemann problem means to find out the solution of a system of quasilinear hyperbolic differential equations

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \tag{1}$$

* Received October 5, 1988.
1) This work is supported in part by The AvH Foundation, SFB123 in Heidelberg University and Chinese National Natural Science Foundation.

with the following kind of initial value

$$U(x,t_0)=\begin{cases} U_l, & x<0, \\ U_r, & x>0. \end{cases} \tag{2}$$

Here $U(x,t)$ is an n-dimensional vector, $A(U)$ is an $n\times n$ matrix with n real characteristic values and U_l and U_r are n-dimensional constant vectors.

Suppose among the n characteristic values there are l distinct ones, which are $\lambda^{(1)}$, $\lambda^{(2)}$, $\cdots$, $\lambda^{(l)}$ in the decreasing order, i.e.,

$$\lambda^{(1)}>\lambda^{(2)}>\cdots>\lambda^{(l)}.$$

In this case the structure of solution for a Riemann problem is as follows (see Fig.1)[3]: it consists of $l-1$ new constant-value subregions and l discontinuity lines—l boundary lines of the $l-1$ subregions. Here a discontinuity line means a real discontinuity or a central wave for the case that the state equation is convex, which we assume in this paper.

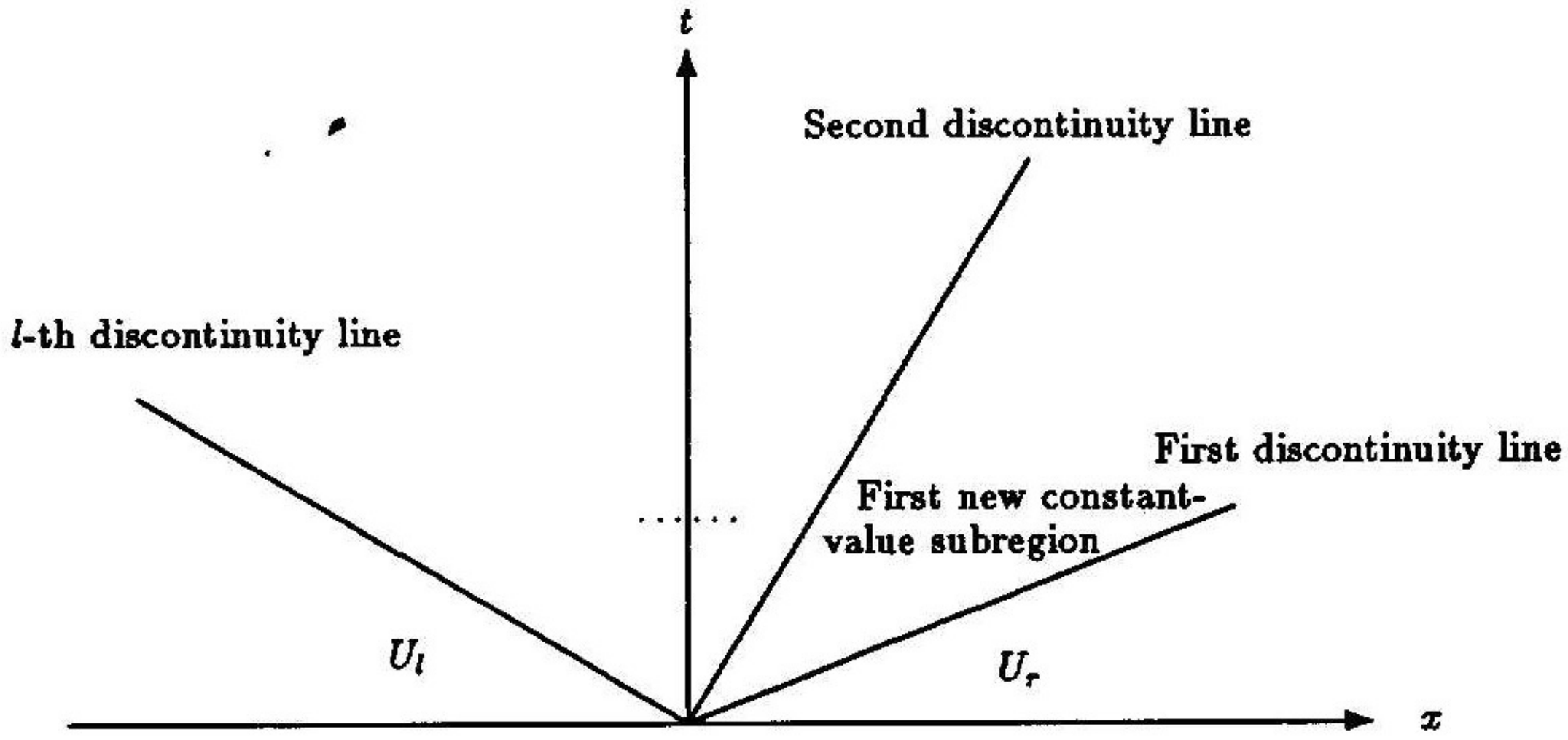

Fig. 1

If $\lambda^{(i)}$ is a $k^{(i)}$-fold characteristic value, there are $n+1-k^{(i)}$ discontinuity relations on the i-th kind of discontinuity line, where discontinuity relations means jump conditions on discontinuities or central wave relations. In practice, the jump conditions are nonlinear equations and the central wave relations, generally speaking, are ordinary differential equations. These discontinuity relations together form a system of equations which will be used to determine the solution of Riemann problems.

Because

$$\sum_{i=1}^{l}\left(n+1-k^{(i)}\right)=l(n+1)-\sum_{i=1}^{l}k^{(i)}=l(n+1)-n=(l-1)n+l,$$

the total number of equations in the system is $(l-1)n+l$. The unknown quantities are l velocities of discontinuity lines and $(l-1)n$ physical quantities in the $l-1$ subregions.

Therefore the number of unknowns is equal to the number of equations, i.e., the system is determined. However, it is an unusual nonlinear system: before the solution is obtained, we do not know which discontinuity relations— the jump conditions or the central wave relations—should be used for a certain boundary line. This makes the formulation of the problem more complicated. But it is clear that the i-th kind of discontinuity has the properties: (a) the velocity of the i-th characteristic line on its left-hand side must be greater than or equal to its velocity and (b) the velocity of the i-th characteristic line on its right-hand side must be less than or equal to its velocity. Consequently, if (a) or (b) is not satisfied, then the central wave relations should be adopted; otherwise the jump conditions should be used.

§3. Numerical Methods

The general form of jump conditions corresponding to $\lambda^{(i)}$ is

$$f_j^{(i)}\left(U_l^{(i)}, U_r^{(i)}, V^{(i)}\right) = 0, \quad j = 1, 2, \cdots, n+1-k^{(i)}, \tag{3}$$

where we suppose $\lambda^{(i)}$ is a $k^{(i)}$-fold characteristic value, $V^{(i)}$ denotes the velocity of the discontinuity and $U_l^{(i)}$, $U_r^{(i)}$ stand for U on its left– and right– hand sides respectively. In many cases, the components of $U_l^{(i)}$ can be divided into two parts and every component in one part can be expressed as a function of $U_r^{(i)}$, $V^{(i)}$ and the components in the other. Therefore, (3) can usually be written in the following special form:

$$\begin{cases} F^{(i)}\left(U_l^{(i)}, U_r^{(i)}, V^{(i)}\right) = 0, \\ U_{l,m_i}^{(i)} = G^{(i)}\left(U_r^{(i)}, V^{(i)}, U_{l,n-m_i}^{(i)}\right), \end{cases} \tag{4}$$

where $F^{(i)}$, $G^{(i)}$, $U_{l,m_i}^{(i)}$, $U_{l,n-m_i}^{(i)}$ are n_i, $n+1-k^{(i)}-n_i$, $n+1-k^{(i)}-n_i$, $k^{(i)}+n_i-1$ dimensional vectors respectively, $U_{l,m_i}^{(i)}$ consists of some components of $U_l^{(i)}$, and $U_{l,n-m_i}^{(i)}$ consists of all the other components of $U_l^{(i)}$.

The general form of central wave relations corresponding to $\lambda^{(i)}$ is a system of ordinary differential equations of the following form

$$\frac{dU}{d\lambda^{(i)}} = W^{(i)}\left(U, \lambda^{(i)}\right)$$

with initial conditions

$$U\left(\lambda_r^{(i)}\right) = U_r^{(i)},$$

where $U_r^{(i)}$ is the U on the right boundary of the central wave and $\lambda_r^{(i)} \equiv \lambda^{(i)}\left(U_r^{(i)}\right)$. From these relations, we can have the relations

$$U_l^{(i)} = H^{(i)}\left(U_r^{(i)}, V^{(i)}\right) \tag{5}$$

implicitly or even explicitly, where $U_l^{(i)}$ is the U on the left boundary of the central wave and $V^{(i)} \equiv \lambda^{(i)}\left(U_l^{(i)}\right)$ (in order to make (5) in the form (4), we use $V^{(i)}$, instead of $\lambda_l^{(i)}$,

to represent $\lambda^{(i)}\left(U_l^{(i)}\right)$). Therefore, the discontinuity relations could be in the form (4) in any case and when $\lambda_r^{(i)} > V^{(i)}$, instead of (4), (5) should be used.

Therefore, solving a Riemann problem is reduced to solving a system of the form

$$\begin{cases} \begin{cases} \overline{F}^{(i)}\left(U_l^{(i)}, U_r^{(i)}, V^{(i)}\right) = 0, & (6) \\ U_{l,m_i}^{(i)} = \overline{G}^{(i)}\left(U_r^{(i)}, V^{(i)}, U_{l,n-m_i}^{(i)}\right), & (7) \end{cases} \\ \quad i = 1, 2, \cdots, l, \\ U_l^{(l)} = U_l, \quad (8) \end{cases}$$

where $U_r^{(1)} \equiv U_r$ is given and $U_r^{(i)} = U_l^{(i-1)}$, $i = 2, 3, \cdots, l$. This system can be solved in the following way:

(1) We guess $V^{(i)}$ and all the components of $U_{l,n-m_i}^{(i)}$, $i = 1, 2, \cdots, l$.

(2) All the components of $U_{l,m_i}^{(i)}$ are computed by using (7), $i = 1, 2, \cdots, l$, successively.

(3) Relations (6) and (8) are checked. If all those relations are satisfied, then we have a solution; if not, we adjust $V^{(i)}$ and all the components of $U_{l,n-m_i}^{(i)}$ and do Steps (2) and (3) again.

Therefore solving (6)–(8) can be reduced to solving a nonlinear system with $l+\sum_{i=1}^{l}\left(k^{(i)}+n_i-1\right)$ unknowns, which are $V^{(i)}$ and all the components of $U_{l,n-m_i}^{(i)}$. Any numerical method for systems of nonlinear equations can be adopted. A discretized Newton method[1] or a generalized linear method by Brown[4] is used in our computation.

Clearly, if, instead of (4), we have jump conditions of the form

$$\begin{cases} F^{(i)}\left(U_l^{(i)}, U_r^{(i)}, V^{(i)}\right) = 0, \\ U_{r,m_i}^{(i)} = G^{(i)}\left(U_l^{(i)}, V^{(i)}, U_{r,n-m_i}^{(i)}\right), \end{cases} \quad (9)$$

the problem can be solved similarly. If for $i = 1, 2, \cdots, l_1$, for example, $l_1 = [l/2]$, jump conditions are in form (4) and for $i = l_1+1, \cdots, l$, they are in form (9), a similar method can be constructed for solving the Riemann problem. In many cases, solving a Riemann problem in this way gives some benefits. For example, if $\lambda^{(l_1)}$ is a multi-fold characteristic value, solving a system in such a form usually needs less computer time than solving a system in (6)–(8) because the number of unknowns for the iteration procedure will be smaller. Other benefits will be described in Section 4.

It is well-known that for these iterative methods of Newton-type, only if the initial value is quite good will the convergence of the iterative procedure be guarantteed. Also, it is clear that if $U_l = U_r$, the solution will be a constant-value one: in every constant-value subregion, $U = U_l$ and the discontinuity lines are characteristic lines. If $U_l \simeq U_r$, the solution should be close to the above constant-value solution. Therefore, if $U_l \simeq U_r$, we can always find a good initial value and the iterative method will work. Consequently, instead of solving a Riemann problem with initial value U_l and U_r, we solve a sequence of Riemann problems with initial values U_l and $U_l + \dfrac{i(U_r - U_l)}{m}$, $i = 1, 2, \cdots, m$, in the following way. First we solve a Riemann problem with initial values U_l and $U_l + \dfrac{U_r - U_l}{m}$

taking a constant-value solution $U = U_l + \dfrac{U_r - U_l}{2m}$ as the initial iterative value. Then we solve the Riemann problem with initial value U_l and $U_l + \dfrac{i(U_r - U_l)}{m}$ taking the solution of the Riemann problem with initial value U_l and $U_l + \dfrac{(i-1)(U_r - U_l)}{m}$ as the initial iterative value, $i = 2, 3, \cdots, m$. When m is large enough (i.e., $(U_r - U_l)/m$ is small enough), these iterative procedures will converge. In practice, first we take a small m, and try the iterative procedure. If the iteration does not converge, we take a larger m and try again until the iteration is convergent. Of course, in the procedure of solving the problem, if we find the initial iterative value is very good, we can take the solution of the Riemann problem with initial value U_l and $U_l + \dfrac{(i-1)(U_r - U_l)}{m}$ as the initial iterative value for the Riemann problem with initial value U_l and $U_l + \dfrac{(i+k)(U_r - U_l)}{m}$, $k > 0$, instead of $\dfrac{i(U_r - U_l)}{m}$. In this way, we can obtain the solution with less computer time. If for every initial value U_l and $U_l + \xi(U_r - U_l)$, $\xi \in [0, 1]$, the Riemann problem has a solution, the method mentioned above will work.

After we obtain the solution, we check if the entropy condition is satisfied. For the hyperbolic problem with a convex equation of state, if $\lambda^{(i)}$ is not a multifold characteristic value, the entropy condition for the i-th kind of discontinuity seems to be

$$\begin{cases} \lambda_r^{(i)} \le V^{(i)} \le \lambda_l^{(i)}, & (10.1) \\ V^{(i)} < \lambda_r^{(i-1)}, \quad \text{if } i \ne 1\,, & (10.2) \\ \lambda_l^{(i+1)} < V^{(i)}, \quad \text{if } i \ne n\,. & (10.3) \end{cases}$$

In our method, $\lambda_r^{(i)} \le V^{(i)}$ or $V^{(i)} \le \lambda_l^{(i)}$ is always satisfied, so the entropy condition must be satisfied at least partly. Moreover, from Section 4 we can see that for some cases, (10.2) and (10.3) can be deduced from (10.1) and the left inequality in (10.1) can be deduced from the right inequality in (10.1) or the right inequality in (10.1) can be deduced from the left inequality. In addition, the initial iterative value seems to be quite close to the entropy-satisfying solution. Therefore it is hopeful that our method often gives an entropy-satisfying solution.

§4. Examples of Application

As an example, the Riemann problem of a one–dimensional unsteady gasdynamic system has been calculated. The following informations is given:

$$n = 3, \quad l = 3, \quad \gamma = 1.4,$$
$$U = \begin{pmatrix} u \\ \rho \\ p \end{pmatrix}, \quad U_l = \begin{pmatrix} 0.0 \\ 1.0 \\ 1.0 \end{pmatrix}, \quad U_r = \begin{pmatrix} 0.0 \\ 0.125 \\ 0.1 \end{pmatrix}.$$

Here and in what follows, u stands for velocity, ρ for density, p for pressure, γ for the ratio of specific heats, c for sound velocity, V for velocity of the discontinuity line, the quantities with the subscript 0 for the physical quantities in front of the discontinuity line, and the

quantities with the subscript 1 for the physical quantities behind the discontinuity line. The relations which correspond to $\lambda^{(1)} = u + c$ is as follows:

1) The relations on shock wave are

$$\rho_0(u_0 - V) = \rho_1(u_1 - V), \quad \rho_0(u_0 - V)^2 + p_0 = \rho_1(u_1 - V)^2 + p_1,$$
$$\frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0} + 0.5(u_0 - V)^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + 0.5(u_1 - V)^2.$$

This system has a trivial solution and the other solution is

$$\begin{cases} \rho_1 = \rho_0((\gamma+1)/(\gamma-1))M_0^2/((2/\gamma - 1) + M_0^2), \\ u_1 = u_0 - (u_0 - V)\left(1 - \dfrac{\rho_0}{\rho_1}\right), \qquad \text{where} \quad M_0^2 = \dfrac{(u_0 - V)^2}{\gamma p_0/\rho_0}. \\ p_1 = p_0 + \rho_0(u_0 - V)^2\left(1 - \dfrac{\rho_0}{\rho_1}\right), \end{cases}$$

2) The relations of rarefaction wave are

$$u_1 = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2}u_0 + \lambda_1^{(1)} - c_0\right), \quad p_1 = p_0\left(\frac{\lambda_1^{(1)} - u_1}{c_0}\right)^{\frac{2\gamma}{\gamma-1}},$$
$$\rho_1 = \rho_0\left(\frac{\lambda_1^{(1)} - u_1}{c_0}\right)^{\frac{2}{\gamma-1}}, \quad \text{where} \quad c = \sqrt{\frac{\gamma p}{\rho}}.$$

The discontinuity relations which correspond to $\lambda^{(2)} = u$ are as follows:

$$u_1 = u_0, \quad p_1 = p_0, \quad V = u_0.$$

The relations which correspond to $\lambda^{(3)} = u - c$ are as follows:

1) The relations on shock wave are the same as in the case for $\lambda^{(1)}$.

2) The relations of rarefaction wave are

$$u_1 = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2}u_0 + \lambda_1^{(3)} + c_0\right),$$
$$p_1 = p_0\left(\frac{u_1 - \lambda_1^{(3)}}{c_0}\right)^{\frac{2\gamma}{\gamma-1}}, \quad \rho_1 = \rho_0\left(\frac{u_1 - \lambda_1^{(3)}}{c_0}\right)^{\frac{2}{\gamma-1}}.$$

The entropy conditions of the first family of shock are

$$\begin{cases} \lambda_0^{(1)} = u_0 + \sqrt{\dfrac{\gamma p_0}{\rho_0}} \le V^{(1)} \le \lambda_1^{(1)} = u_1 + \sqrt{\dfrac{\gamma p_1}{\rho_1}}, \\ u_1 = \lambda_1^{(2)} \le V^{(1)}. \end{cases}$$

From the relations on shock wave we see that $V^{(1)} - u_0$ has the same sign of $V^{(1)} - u_1$. Therefore we can deduce the second inequality from the left part of the first inequality. That

is, we can think that the entropy condition for this case consists of only the first inequality. Clearly, the left part of the first inequality can be rewritten as $M_0 \equiv \dfrac{V-u_0}{\sqrt{\gamma p_0/\rho_0}} \geq 1$ and the right part as $M_1 \equiv \dfrac{V-u_1}{\sqrt{\gamma p_1/\rho_1}} \leq 1$. From (11) we know that $\dfrac{\rho_1}{\rho_0} \geq 1$ is equivalent to $M_0^2 \geq 1$. Moreover, the following relation similar to (11)

$$\rho_0 = \rho_1 \Big(\frac{\gamma+1}{\gamma-1}\Big) M_1^2 \Big/ \Big(\frac{2}{\gamma-1} + M_1^2\Big)$$

holds. Thus $\rho_1/\rho_0 \geq 1$ is equivalent to $M_1^2 \leq 1$, i.e., $M_0^2 \geq 1$ is equivalent to $M_1^2 \leq 1$. From the relations on shock wave, we know $M_1 \geq 0$ if and only if $M_0 \geq 0$. Therefore, if $M_0 > 0$ or $M_1 > 0$, $M_0 \geq 1$ is equivalent to $M_1 \leq 1$. Consequently, if the left part of the first inequality holds, so does the right part; if the right part holds and $M_1 \geq 0$, so does the left part. Noticing $M_1 \geq 0$ if $\lambda_1^{(2)} = u_1 \leq V^{(1)}$, we know that, when

$$\lambda_0^{(1)} \leq V^{(1)} \quad \text{or} \quad \lambda_1^{(2)} \leq V^{(1)} \leq \lambda_1^{(1)},$$

the first inequality is satisfied, i.e., the entropy conditions are satisfied.

For the second family of shock, the entropy conditions

$$\begin{cases} \lambda_1^{(3)} \leq V^{(3)} \leq \lambda_0^{(3)}, \\ V^{(3)} \leq \lambda_1^{(2)} \end{cases}$$

can be reduced to $V^{(3)} \leq \lambda_0^{(3)}$ or $\lambda_1^{(3)} \leq V^{(3)} \leq \lambda_1^{(2)}$. Therefore, if we take (4) for $i = 1$ and (9) for $i = 3$ in the method in Section 3, we can always obtain an entropy-satisfying solution.

For the above Riemann problem, our method gives the following results:

The types of the boundary lines are

DIS1, DDI2, CHA3, CHA3,

where $DIS1$ stands for a shock wave which corresponds to $\lambda^{(1)}$, $DDI2$ for a contact discontinuity which corresponds to $\lambda^{(2)}$, and $CHA3$ for a characteristic line which corresponds to $\lambda^{(3)}$. The velocities of the boundary lines are

$$1.7522,\ 0.9275,\ -0.07025,\ -1.1832.$$

In the first constant-value subregion from the right,

$$u = 0.9275, \quad \rho = 0.2656, \quad p = 0.3031.$$

In the second constant-value subregion,

$$u = 0.9275, \quad \rho = 0.4263, \quad p = 0.3031.$$

In the region of rarefaction wave, for $\lambda^{(3)} = \lambda_0^{(3)}$, $\lambda_0^{(3)} + \frac{1}{5}\left(\lambda_0^{(3)} - \lambda_1^{(3)}\right)$, $\cdots$, $\lambda_0^{(3)} + \frac{4}{5}\left(\lambda_0^{(3)} - \lambda_1^{(3)}\right)$, $\lambda_1^{(3)}$,

$$u = 0.0000,\ 0.1855,\ 0.3710,\ 0.5565,\ 0.7420,\ 0.9275;$$
$$\rho = 1.0000,\ 0.8528,\ 0.7234,\ 0.6102,\ 0.5117,\ 0.4263;$$
$$p = 1.0000,\ 0.8001,\ 0.6355,\ 0.5008,\ 0.3914,\ 0.3031$$

respectively.

Another numerical example is shallow water equations[5]. The hyperbolic quasilinear system of equations governing the flow of an ideal imcompressible fluid in a gravitational field is known as the shallow water equations. In the solution of these equations, discontinuities arise when bores or hydraulic jumps are present. An example of such a phenomenon is provided by the problem of the break down of a dam.

The homogeneous shallow water system is

$$\begin{cases} h_t + (hv)_x = 0\,, \\ (hv)_t + \left(hv^2 + \frac{1}{2}gh^2\right)_x = 0\,. \end{cases} \tag{23}$$

Here, h stands for the depth of water, v for mean velocity, and g for gravitational acceleration, which is 9.8066 m/sec^2.

We take the following initial values:

$$v_l = 0.2667,\ h_l = 10.8,\ v_r = 1.6,\ h_r = 1.8.$$

For the above Riemann problem, our method gives the following results:
The types of the boundary lines are

DIS1, CHA2, CHA2.

The velocities of the boundary lines are:

10.74, 0.4536, −10.02.

In the constant-value subregion,

$$v = 7.252, \quad h = 4.713.$$

In the region of rarefaction wave, for $\lambda^{(2)} = \lambda_0^{(2)}$, $\lambda_0^{(2)} + \frac{1}{5}\left(\lambda_1^{(2)} - \lambda_0^{(2)}\right)$, $\cdots$, $\lambda_1^{(2)}$,

$$v = 0.2667,\ 1.664,\ 3.061,\ 4.458,\ 5.856,\ 7.252,$$
$$h = 10.80,\ 9.384,\ 8.067,\ 6.849,\ 5.732,\ 4.713,$$

respectively.

§5. Reflection of discontinuities from external boundaries

Solving a problem of reflection of a discontinuity from an external boundary is similar to solving a Riemann problem. After reflection some new constant-value subregions are generated. We must determine the physical quantities in the constant-value subregions and the types and velocities of discontinuity lines between these subregions. The problem of reflection of discontinuities from external boundaries can also be solved by our general numerical method. It is natural that the boundary conditions should be given by the user.

§6. Concluding remarks

A sub-software for the Riemann problem based on the above method is a component part of numerical software for the initial-boundary-value problems of hyperbolic differential equations. The sub-software can deal with various interaction and reflection problems which appear frequently in the initial-boundary-value problems of hyperbolic differential equations. We believe it will also be very useful for other purposes.

References

[1] Zhu You-lan, Zhong Xi-chang, Chen Bing-mu and Zhang Zuo-min, Difference Methods for Initial-Boundary Value Problems and Flow Around Bodies, Science Press, Beijing, China, 1980; English Edition, Springer, Heidelberg, FRG and Science Press, Beijing, China, 1988.

[2] Wu Xionghua and Zhu Youlan, Numerical solution of multimedium flow with various discontinuities, *Journal of Computational Mathematics*, **1**: 4 (1983), 303–316.

[3] P.D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1973.

[4] K. Brown, A quadratically convergent Newton-like method based upon Gaussian elimination, *SIAM J. Numer. Anal.*, **6** (1969), 560–569.

[5] Magdi M.Shoucri, Numerical solution of the shallow water equations, *J.C.P.*, **63** (1986), 240–245.