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THE SPECTRAL METHOD FOR THE GENERALIZED KURAMOTO-SIVASHINSKY EQUATION*

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Abstract

A spectral method is proposed, the existence and uniqueness of the global and smooth solution are proved for the periodic initial value problem of the generalized K-S equation. The error estimates are established and the convergence is proved for the approximate solution of the spectral method.

§1. Introduction

The Kuramoto-Sivashinsky equation

$$\Phi_t + \Phi_x^2 + \Phi_{xx} + \Phi_{xxxx} = 0 \tag{1.1}$$

was independently advocated by Kuramoto[1] in connection with reaction-diffusion systems, and then by Sivashinsky[2] in modeling flame propagation; it also arises in the context of viscous film flow[3], bifurcating solutions of the Navier-Stokes equation[4], etc.

Differentiating (1.1) with respect to x and setting $u = \Phi_x$, we get

$$u_t + (u^2)_x + u_{xx} + u_{xxxx} = 0. \tag{1.2}$$

In the present paper, we consider the generalized K-S equation of the form

$$u_t + f(u)_x + \alpha u_{xx} = \beta u_{xxxx} = g(u) \tag{1.3}$$

and its periodic initial value problem

$$u(x,0) = u_0(x), \quad x \in R^1, \quad u(x,t) = u(x+2\pi, t), \quad x \in R^1, \quad t \geq 0 \tag{1.4}$$

where $\alpha, \beta > 0$ are constants.

We propose a spectral method for the problem (1.3)–(1.4), prove the existence of the global smooth solution for the problem (1.3)–(1.4), and establish the error estimates and convergence for the approximate solution.

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§2. The Spectral Methods and a Priori Estimates

Here we adopt the usual notation and convention. Let $\Omega = [0, 2\pi]$, $L_p(\Omega)$ denotes the Lebesgue space with the norm $\|u\|_{L_p} = \left(\int_0^{2\pi} |u|^p dx\right)^{1/p}$. If we define the inner product

$$(u, v) = \int_0^{2\pi} u(x)v(x)dx, \quad \|u\|_{L_2}^2 = (u, u),$$

then $L_2(\Omega)$ is a Hilbert space; especially, $L_\infty(\Omega)$ denotes the Lebesgue space with norm $\|u\|_{L_\infty} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. Let $H^m(\Omega)$ denote the Sobolev space with the norm

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|_{L_2}^2\Big)^{1/2} \quad \text{or simply} \quad \|u\|_m.$$

Let $L^\infty(0, T; H^m)$ denote the space of the functions $u(x, t)$ each of which belongs to H^m as a function of x for every fixed $t, 0 \le t \le T$, and $\sup_{0\le t\le T} \|u(\cdot, t)\|_m < \infty$.

Let $H_p^m(\Omega) = \{u(x) | u \in H^m(\Omega),\ u^j(x) = u^j(x + 2\pi),\ 0 \le j \le m - 1\}$ be a periodic functional space, where $u^j = \dfrac{d^j u}{dx^j}$, $S_N = \text{Span}\ \{w_j(x), 1 \le j \le N\}$ is a subspace spanned on the basis $\{w_j(x)\}$, $j = 1, \cdots, N$, where $w_j(x) = \exp\{ijx\}$, $i = \sqrt{-1}$.

We construct an approximate solution of problem (1.3)–(1.4) as follows:

$$U_N(x, t) = \sum_{j=-N}^{N} \gamma_{jN}(t)w_j(x), \quad x \in \Omega$$

where the coefficient functions $\gamma_{jN}(t)$ should satisfy the equations

$$(U_{Nt} + f(U_N)_x + \alpha U_{Nxx} + \beta U_{Nxxxx}, w_j) = (g(U_N), w_j) \tag{2.1}$$

with the initial condition

$$U_N(x, 0) = U_{0N}(x), \quad x \in \Omega \tag{2.2}$$

where

$$U_{0N}(x) \overset{H^2(\Omega)}{\longrightarrow} u_0(x) \quad \text{as} \quad N \to \infty.$$

Problem (2.1)–(2.2) can be considered as an initial value problem of nonlinear ordinary differential equations of first order with unknown functions $\gamma_{jN}(t)$. Under the conditions of the lemmas and the a priori estimates in the present section, we know that there exists a global solution in the interval $[0, T]$ for the initial value problem (2.1)–(2.2).

Now we make the a priori estimates for the solution of problem (2.1)–(2.2).

Lemma 1. *If the following conditions are satisfied* :

(i) $f(u) \in C^1$, $\alpha > 0$, $\beta > 0$, (ii) $g(0) = 0, g'_u \le b$, (iii) $u_0(x) \in L_2(\Omega)$,

then for the solution $U_N(x, t)$ *of problem* (2.1)–(2.2) *there is the estimate*

$$\|U_N\|_{L^\infty(0,T;L_2(\Omega))} + \|U_{Nxx}\|_{L^2(0,T;L_2(\Omega))} \le E_0 \tag{2.3}$$

where the constant E_0 *is independent of* N.

Proof. Multiplying (2.1) by $\gamma_{jN}(t)$ and summing them up for j from 1 to N, we have

$$(U_{Nt} + f(U_N)_x + \alpha U_{Nxx} + \beta U_{Nxxxx}, U_N) = (g(U_N), U_N).$$

Since

$$(U_{Nt},U_N)=\frac{1}{2}\frac{d}{dt}\|U_N\|_{L_2}^2,\quad (f(U_N)_x,U_N)=-\int F(U_N)_x dx=0,$$

$$F(u)=\int_0^u f(z)dz,\quad (\alpha U_{Nxx},U_N)=-\alpha\|U_{Nx}\|_{L_2}^2,$$

$$(\beta U_{Nxxxx},U_N)=-\beta\|U_{Nxx}\|_{L_2}^2,$$

and by condition (ii), we have

$$(U_N,g(U_N))=(U_N,g(U_N)-g(0))\le b\|U_N\|_{L_2}^2.$$

According to the above estimates, we get

$$\frac{1}{2}\frac{d}{dt}\|U_N\|_{L_2}^2+\beta\|U_{Nxx}\|_{L_2}^2-\alpha\|U_{Nx}\|_{L_2}^2\le b\|U_N\|_{L_2}^2.$$

By using Sobolev's inequality

$$\|u_x\|_{L_2}^2\le\varepsilon\|u_{xx}\|_{L_2}^2+C\|u\|_{L_2}^2$$

it follows that

$$\alpha\|U_{Nx}\|_{L_2}^2\le\alpha\varepsilon\|U_{Nxx}\|_{L_2}^2+\alpha C\|U_N\|_{L_2}^2\le\frac{1}{2}\beta\|U_{Nxx}\|_{L_2}^2+\alpha C\|U_N\|_{L_2}^2$$

where ε is so chosen that $\alpha\varepsilon<\frac{1}{2}\beta$. Hence we have

$$\frac{d}{dt}\|U_N\|_{L_2}^2+\frac{\beta}{2}\|U_{Nxx}\|_{L_2}^2\le\alpha C\|U_N\|_{L_2}^2+b\|U_N\|_{L_2}^2.$$

Therefore, by using Gronwall's inequality, it follows that

$$\|U_N\|_{L_2}^2\le e^{(\alpha c+b)T}\|U_{0N}(x)\|_{L_2}^2\le E_0',$$

$$\frac{\beta}{Z}\int_0^T\|U_{Nxx}\|_{L_2}^2dt\le(\alpha C+b)\int_0^T\|U_N\|_{L_2}^2dt+\|U_{0N}\|_{L_2}^2-\|U_N\|_{L_2}^2$$

$$\le(\alpha C+b)E_0'T+\|U_{0N}\|_{L_2}^2-E_0'\le E_0''.$$

Take $E_0=\max(E_0',2E_0''/\beta)$, and the lemma has been proved.

Lemma 2 (Sobolev's estimates)[5]. *Let $D^m u\in L_r(\Omega)$, $u\in L_q(\Omega)$, $\Omega\subseteq R^n$. Then, there is a constant C such that*

$$\|D^j u\|_{L_p(\Omega)}\le C\|D^m u\|_{L_r(\Omega)}^{\alpha}\|u\|_{L_q(\Omega)}^{1-\alpha}$$

where

$$\frac{1}{p}=\frac{j}{n}+a\Big(\frac{1}{r}-\frac{m}{n}\Big)+(1-a)\frac{1}{q},\quad 0\le j\le m,\quad 0\le a\le 1.$$

Lemma 3. *Suppose that the conditions of Lemma 1 are satisfied and assume that*

(i) $f(u)\in C^2$, $|f(u)|\le A(u)^p+B$, $A,B>0$, $1\le p\le 7$,

(ii) $u_0(x)\in H^1(\Omega)$.

Then for the solution of the initial value problem (2.1)–(2.2), there is the estimate

$$\|U_{Nx}\|_{L^\infty(0,T;L_2(\Omega))}+\|U_{Nxxx}\|_{L^2(0,T;L_2(\Omega))}\le E_1 \tag{2.4}$$

where the constant E_1 is independent of N.

Proof. Because $-w_j''(x) = \lambda_j w_j(x)$ and (2.1), it is known that

$$(U_{Nt} + f(U_N)_x + \alpha U_{Nxx} + \beta U_{Nxxxx}, \quad -w_j''(x)) = (g(U_N), -w_j''(x)).$$

Multiplying the above equality by $\gamma_j(t)$, then summing them up for j from 1 to N and setting $V_N = U_{Nx}$, we get

$$(V_{Nt} + f(U_N)_{xx} + \alpha V_{Nxx} + \beta V_{Nxxxx}, V_N) = (g'(U_N)V_N, V_N). \tag{2.5}$$

Since

$$(V_{Nt}, V_N) = \frac{1}{2}\frac{d}{dt}\|V_N\|_{L_2}^2, \quad (\alpha V_{Nxx}, V_N) = -\alpha\|V_{Nx}\|_{L_2}^2,$$

$$(\beta V_{Nxxxx}, V_N) = \beta\|V_{Nxx}\|_{L_2}^2, \quad (g'(U_N)V_N, V_N) \le b\|V_N\|_{L_2}^2,$$

$$(f(U_N)_{xx}, V_N) = (f(U_N), V_{Nxx}) \le \|f(U_N)\|_{L_2}\|V_{Nxx}\|_{L_2},$$

by condition (i) of this lemma, it is known that

$$\|f(U_N)\|_{L_2} \le A\|U_N\|_{L_{2p}}^p + B.$$

By Lemma 2, we have

$$\|U_N\|_{L_{2p}} \le C\|U_{Nxxx}\|_{L_2}^a \|U_N\|_{L_2}^{1-a}$$

where $\frac{1}{2p} = a(\frac{1}{2} - 3) + (1-a)\frac{1}{2} = \frac{1}{2} - 3a$. Taking $ap = \frac{p-1}{6} < 1$, that is $p < 7$, we have

$$\|f(U_N)\|_{L_2} \le C\|U_{Nxxx}\|_{L_2}^{1-\delta} + C_1 \le \frac{\beta}{6}\|U_{Nxxx}\|_{L_2} + C_1', \quad \delta > 0.$$

Hence

$$|(f(U_N)_{xx}, V_N)| \le \frac{\beta}{6}\|V_{Nxx}\|_{L_2}^2 + C_1\|V_{Nxx}\|_{L_2} \le \frac{\beta}{3}\|V_{Nxx}\|_{L_2}^2 + C_2.$$

Furthermore,

$$\alpha\|V_{Nx}\|_{L_2}^2 \le \frac{\beta}{3}\|V_{Nxx}\|_{L_2}^2 + C_3\|V_N\|_{L_2}^2.$$

Thus from (2.5), it follows that

$$\frac{1}{2}\frac{d}{dt}\|V_N\|_{L_2}^2 + \frac{\beta}{3}\|V_{Nxx}\|_{L_2}^2 \le (C_3 + b)\|V_N\|_{L_2}^2 + C_2.$$

Finally, by Gronwall's inequality, the estimate (2.4) is obtained.

From Lemma 3 and Sobolev's estimate, we have

Corollary.

$$\sup_{0\le t\le T} \|U_N\|_{L_\infty(\Omega)} \le E_2 \tag{2.6}$$

where the constant E_2 is independent of N.

Lemma 4. *Suppose that the conditions of Lemma 3 are satisfied, and assume that*

(i) $f(u) \in C^{m+2}, \quad g(u) \in C^{m+1}$,

(ii) $u_0(x) \in H^{m+1}(\Omega), \quad m \ge 1$.

Then, for the approximate solution of problem (1.3)–(1.4), there is the estimate

$$\sup_{0\le t\le T} \|U_N(\cdot,t)\|_{H^{m+1}}^2 + \|U_N(\cdot,t)\|_{L^2(0,T;H^{m+3})} \le E_m \tag{2.7}$$

where the constant E_m is dependent on $\|u_0(x)\|_{H^{m+1}(\Omega)}$.

Proof. By using the inductive method and the a priori estimate method analogous to Lemma 3, the estimate (2.7) is easily obtained.

From the uniform estimation for the norm of every term of the approximate solution $U_N(x,t)$ with N terms and compact argument, similarly to the proof in [6], we have directly the following theorem.

Theorem 1. *If the following conditions are satisfied:*

(i) $f(u) \in C^{s+1}$ and $|f(u)| \le A|u|^p + B,\ 1 \le p < 7,\ \beta > 0,$

$g(u) \in C^s,\ \ g'(u) \le b,\ \ b > 0,\ \ s \ge 1,$

(ii) $u_0(x) \in H^s(\Omega)$,

then there exists a global solution for the initial value problem (1.3)–(1.4):

$$u(x,t) \in L^\infty(0,T;H^s(\Omega)).$$

By means of the method of energy, we may easily prove the following theorem.

Theorem 2. *If* $f(u) \in C^2,\ \ g(u) \in C^1$, *then the global solution of problem* (1.3)–(1.4) *is unique.*

§3. The Error Estimation of an Approximate Solution by the Spectral Method

For a periodic function $u(x,t)$, setting

$$u_N(x,t) = \sum_{j=1}^{N} \mu_{jN}(t) w_j(x),$$

which represents the sum of N terms of the Fourier series for $u(x,t)$, we have the following lemma.

Lemma 5[7]. *For any real number* $0 \le \mu \le \sigma$ *and function* $u \in H_p^\sigma(\Omega)$, *there is a constant C such that*

$$\|u - u_N\|_\mu \le CN^{\mu-\sigma}\|u\|_\sigma.$$

Taking the inner product of (1.3) and $w_j(x)$, we have

$$(u_t + f(u)_x + \alpha u_{xx} + \beta u_{xxxx}, w_j) = (g(u), w_j). \tag{3.1}$$

Then again by (2.1)

$$(U_{Nt} + f(U_N)_x + \alpha U_{Nxx} + \beta U_{Nxxxx}, w_j) = (g(U_N), w_j). \tag{3.2}$$

Set $\varsigma_N = U - u_N, \Psi_N = U_N - u_N$. Then,

$$u - U_N = (u - u_N) - (U_N - u_N) = \varsigma_N - \Psi_N.$$

Subtracting (3.2) from (3.1), we get

$$\begin{aligned}(\Psi_{Nt} + \alpha\Psi_{Nxx} + \beta\Psi_{Nxxxx}, w_j) = (&\varsigma_{Nt} + \alpha\varsigma_{Nxx} + \beta\varsigma_{Nxxxx} \\ &+ [f(u)_x - f(U_N)_x] - [g(u) - g(U_N)], w_j).\end{aligned} \tag{3.3}$$

Take

$$\Psi_N = \sum_{j=1}^{N} \gamma_j(t) w_j(x).$$

Multiplying (3.3) by $\gamma_j(t)$ and summing them up for j from 1 to N, we get

$$(\Psi_{Nt} + \alpha\Psi_{Nxx} + \beta\Psi_{Nxxxx}, \Psi_N) = (\varsigma_{Nt} + \alpha\varsigma_{Nxx} + \beta\varsigma_{Nxxxx} + [f(u)_x - f(U_N)_x] - [g(u) - g(U_N)], \Psi_N). \tag{3.4}$$

Since

$$f(u)_x - f(U_N)_x = [f(u) - f(U_N)]_x = \Big[\int_0^1 f'_u(zu + (1-z)U_N)dz(\varsigma_N - \Psi_N)\Big]_x$$

$$= \int_0^1 f''_{uu}(zu_x + (1-z)U_{Nx})dz(\varsigma_N - \Psi_N)$$

$$+ \int_0^1 f'_u(zu + (1-z)U_N)dz(\varsigma_{Nx} - \Psi_{Nx})$$

$$g(u) - g(u_N) = \int_0^1 g'_u(zu + (1-z)U_N)dz(\varsigma_N - \Psi_N)$$

from lemma 5, it follows that

$$\|\varsigma_{Nt}\|_{L_2} \le CN^{-r}\|u_t\|_r, \quad u_t \in H_p^r(\Omega), \quad \|\varsigma_{Nxx}\|_{L_2} \le CN^{2-r}\|u\|_r.$$

Hence by (3.4), we have

$$\frac{1}{2}\frac{d}{dt}\|\Psi_N\|_{L_2}^2 - \alpha\|\Psi_{Nx}\|_{L_2}^2 + \beta\|\Psi_{Nxx}\|_{L_2}^2 \le \|\varsigma_{Nt}\|_{L_2}\|\Psi_N\|_{L_2} + \alpha\|\varsigma_N\|_{L_2}\|\Psi_{Nxx}\|_{L_2}$$

$$+\beta\|\varsigma_{Nxx}\|_{L_2}\|\Psi_{Nxx}\|_{L_2} + \Big\|\int_0^1 (f'_u)_x dz\Big\|_{L_\infty}(\|\varsigma_N\|_{L_2}\|\Psi_N\|_{L_2} + \|\Psi_N\|_{L_2}^2)$$

$$+\Big\|\int_0^1 f'_u dz\Big\|_{L_\infty}\|\varsigma_{Nx}\|_{L_2}\|\Psi_N\|_{L_2} + \Big\|\int_0^1 (f'_u)_x dz\Big\|_{L_\infty}\frac{1}{2}\|\Psi_N\|_{L_2}^2$$

$$+\Big\|\int_0^1 g'_u dz\Big\|_{L_\infty}(\|\varsigma_N\|_{L_2}\|\Psi_N\|_{L_2} + \|\Psi_N\|_{L_2}^2)$$

Therefore,

$$\frac{d}{dt}\|\Psi_N\|_{L_2}^2 + \frac{\beta}{3}\|\Psi_{Nxx}\|_{L_2}^2 \le C[\|\Psi_N\|_2^2 + N^{2(2-r)}.$$

It implies

$$\|\Psi_N\|_{L_2} = O(N^{2-r}).$$

Hence

$$\|u - U_N\|_{L_2(\Omega)} + \|u - U_N\|_{L^2(0,T;H^2(\Omega))} = O(N^{2-r}).$$

From the above discussion, we obtain the following theorem.

Theorem 3. *If the following conditions are satisfied:*

(i) $u(x,t)$ is a solution of problem (1.3)-(1.4) and

$$u_t \in L^\infty(0,T;H_p^r(\Omega)), \quad u_0(x) \in H_p^r(\Omega), \quad r > 2,$$

(ii) $f(u) \in C^2, \quad g(u) \in C^1.$

then for an approximate solution of the spectral method, we have the following error estimate:

$$\|u - U_N\|_{L_2(\Omega)} + \|u - U_N\|_{L^2(0,T;H^2(\Omega))} = O(N^{2-r}).$$

§4. The Discrete Spectral Method

For the discrete spectral method we consider the difference quotient

$$U_{N\bar{t}} = \frac{U_N(x,t) - U_N(x,t-\Delta t)}{\Delta t}$$

which is approximate to U_{Nt}, where Δt is the step of time $t, t = n\Delta t$, $n \in \left[0, \left[\frac{T}{\Delta t}\right]\right]$. Then, problem (2.1)–(2.2) becomes

$$\begin{aligned}(U_{N\bar{t}}^{n+1} &+ f((U_N^{n+1} + U_N^n)/2)_x + \alpha U_{Nxx}^{n+1} + \beta U_{Nxxxx}^{n+1}, w_j) \\ &= \left(g\left(\frac{1}{2}(U_N^{n+1} + U_N^n)\right), w_j\right), \quad j = 1, 2, \cdots, N, \end{aligned} \tag{4.1}$$

$$U_N(x,0) = U_0(x), \quad x \in \Omega, \tag{4.2}$$

wher $U_{0N}(x) \overset{H^2(\Omega)}{\longrightarrow} u_0(x)$, as $N \to \infty$.

We have the following theorem.

Theorem 4. *If the conditions of Theorem 3 hold, then for the approximate solution $U_N(x,t)$ of problem (4.1)–(4.2), there is the estimate*

$$\|u - U_N\|_{L_2(\Omega)} = O(N^{2-r} + \Delta t^2)$$

where $u(x,t)$ is the smooth solution of problem (1.3)–(1.4).

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