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# THE SYMMETRICAL DISSIPATIVE DIFFERENCE SCHEMES FOR QUASI-LINEAR HYPERBOLIC CONSERVATION LAWS*

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**Abstract**

In this paper we construct a new type os symmetrical dissipative difference scheme. Except discontinuity these schemes have uniformly second-order accuracy. For calculation using these, the simple-wave is very exact, the shock has high resolution, the programing is simple and the CPU time is economical.

Since the paper [1] introduced that in some conditions Lax-Wendroff scheme would convergent to nonphysical solution, many researchers have discussed this problem. According to preserve the monotonicity of the solution preserving monotonical schemes and TVD schemes have been introduced by Harten, et. According to property of hyperbolic wave propagation the schemes of split-coefficient matrix(SCM) and split-flux have been formed. We emphasize the dissipative property of scheme for conservation laws and introduced a type of symmetrical dissipative difference scheme, these schemes are dissipative on arbitrary conditions.

## §1. The Symmetrical Dissipative Schemes for Hyperbolic Conservation Laws

The quasi-linear conservation law is represented by the following equation

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=0 \tag{1}$$

where $u$, $f(u)$ are column vectors with $m$ dimensions. $A=f_u$ is coefficient matrix of equation (1), it has $m$ real eigenvalues

$$\lambda_1\le\lambda_2\le\cdots\le\lambda_m$$

and a complete set of left (right) eigenvectors for all real $\lambda_i$.

Advantages of symmetrical scheme is that formula and programming become simpler, that the computation cost is law. We shall see that schemes constructed from the point of view of dissipative property has less severe limitations and often has weaker restriction of stability condition than that from the point of presevering monotonicity of the solution.

In papers [4] and [3], we have constructed the first-order, second-order least dissipative hybrid schemes of presevering monotonicity of the solution

$$u_j^{n+1}=L_2u_j^n+\frac{1}{2}\left[q^n_{j+\frac{1}{2}}\theta^n_{j+\frac{1}{2}}\left(u^n_{j+1}-u^n_j\right)-q^n_{j-\frac{1}{2}}\theta^n_{j-\frac{1}{2}}\left(u^n_j-u^n_{j-1}\right)\right] \tag{2}$$

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where $L_2 u_j^n$ stands for MacCormack two-step scheme or Lax-Wendroff scheme, we have defined

$$\theta_j = \begin{cases} \dfrac{\left||\Delta\sigma_j| - |\Delta\sigma_{j-1}|\right|}{|\Delta\sigma_j| + |\Delta\sigma_{j-1}|} & \text{for } |\Delta\sigma_j| + |\Delta\sigma_{j-1}| > \varepsilon_0, \\ 0 & \text{for } |\Delta\sigma_j| + |\Delta\sigma_{j-1}| \le \varepsilon_0 \end{cases} \tag{3}$$

and

$$\theta_{j+\frac{1}{2}} = \max\{\theta_j, \theta_{j+1}, \Delta t\}$$

for the single conservation law $\sigma = u$, for Euler equations, $\sigma = \rho$ or $c$. Assume that

$$\nu_j = \max_{1<k<m} |\lambda_{kj}| \tag{4}$$

we define

$$q = \nu \frac{\Delta t}{\Delta x}\left(1 - \nu \frac{\Delta t}{\Delta x}\right) \tag{5}$$

and

$$q_{j+\frac{1}{2}} = \frac{1}{2}(q_j + q_{j+1}) \tag{6}$$

or

$$q_{j+\frac{1}{2}} = \nu_{j+\frac{1}{2}} \frac{\Delta t}{\Delta x}\left(1 - \nu_{j+\frac{1}{2}} \frac{\Delta t}{\Delta x}\right) \tag{7}$$

where $\nu_{j+\frac{1}{2}}$ is one of the averages of $\nu_j$ and $\nu_{j+1}$.

Next we shall construct some new symmetrical dissipative schemes according to the view of dissipative property.

**Scheme 1.** MacCormack two-step scheme with $\theta/8$ dissipative modification

$$\begin{aligned} u_j^{\overline{n+1}} &= L_2 u_j^n, \\ u_j^{\overline{n+1}} &= u_j^{\overline{n+1}} + \frac{1}{8}\left[\tilde\theta_{j+1/2}^{n+1}(u_{j+1}^{\overline{n+1}} - u_j^{\overline{n+1}} - \tilde\theta_{j-1/2}^{n+1}(u_j^{\overline{n+1}} - u_{j-1}^{\overline{n+1}})\right] \end{aligned} \tag{8}$$

where

$$\tilde\theta_{j+1/2}^{n+1} = \max\{\theta_{j+1}^{\overline{n+1}}, \theta_j^{\overline{n+1}}, 8\eta\Delta t\}.$$

**Scheme 2.** MacCormack two-step scheme with $\frac{1}{2}\theta q$ dissipative modification

$$\begin{aligned} u_j^{\overline{n+1}} &= L_2 u_j^n, \\ u_j^{n+1} &= u_j^{\overline{n+1}} + \frac{1}{2}\left[q_{j+1/2}^{\overline{n+1}}\theta_{j+1/2}^{\overline{n+1}}(u_{j+1}^{\overline{n+1}} - u_j^{\overline{n+1}}) - q_{j-1/2}^{\overline{n+1}}\theta_{j-1/2}^{\overline{n+1}}(u_j^{\overline{n+1}} - u_{j-1}^{\overline{n+1}})\right]. \end{aligned} \tag{9}$$

**Scheme 3.** Diffusive-antidiffusive second-order scheme with the least dissipative.

$$\begin{aligned} u_j^{\overline{n+1}} &= L_2 u_j^n + \frac{1}{2}\left[q_{j+1/2}^n(u_{j+1}^n - u_j^n) - q_{j-1/2}^n(u_j^n - u_{j-1}^n)\right], \\ u_j^{n+1} &= u_j^{\overline{n+1}} - \frac{1}{2}\left[q_{j+1/2}^{\overline{n+1}}(u_{j+1}^{\overline{n+1}} - u_j^{\overline{n+1}}) - q_{j-1/2}^{\overline{n+1}}(u_j^{\overline{n+1}} - u_{j-1}^{\overline{n+1}})\right]. \end{aligned} \tag{10}$$

**Scheme 4.** Modified diffusive-antidiffusive scheme with the least diffispative

$$\begin{aligned} u_j^{\overline{n+1}} &= L_2 u_j^n + \frac{1}{2}\left[q_{j+1/2}^n(u_{j+1}^n - u_j^n) - q_{j-1/2}^n(u_j^n - u_{j-1}^n)\right], \\ u_j^{n+1} &= u_j^{\overline{n+1}} - \frac{1}{2}\left[q_{j+1/2}^{*\overline{n+1}}(u_{j+1}^{\overline{n+1}} - u_j^{\overline{n+1}}) - q_{j-1/2}^{*\overline{n+1}}(u_j^{\overline{n+1}} - u_{j-1}^{\overline{n+1}})\right] \end{aligned} \tag{11}$$

where

$$q^{*\overline{n+1}}_{j+1/2} = -2\eta\Delta t + \begin{cases} q^{\overline{n+1}}_{j+1/2}, & \theta^{\overline{n+1}}_{j+1/2} > \theta_0, \\ \delta \cdot q^{\overline{n+1}}_{j+1/2}, & \theta^{\overline{n+1}}_{j+1/2} \le \theta_0 \end{cases} \tag{12}$$

$$0 < \eta \le 1, \quad 0 < \delta < 1, \quad \theta_0 > 0.2.$$

Suppose that $G_M$ represents increment matrix of MacCormack scheme, we have that increment matrices of Schemes 1 and 2 respectively are

$$G_1 = \left[1 - \frac{\theta}{4}(1-\cos\xi)\right]G_M, \tag{13}$$

$$G_2 = \left[1 - q\theta(1-\cos\xi)\right]G_M \tag{14}$$

hence if increment matrix $G_M$ of MacCormack Scheme satisfies

$$\|G_M\| \le 1$$

then

$$\|G_M\| \le 1 - \frac{\theta}{4}(1-\cos\xi) < 1 \quad \text{for } \forall\xi \ne 0, \tag{15}$$

$$\|G_2\| \le 1 - q\theta(1-\cos\xi) < 1 \quad \text{for } \forall\xi \ne 0 \tag{16}$$

thus, if stability condition of MacCormack scheme

$$r_{\max} = \max_{k,j}\left\{|\lambda_{kj}|\frac{\Delta t}{\Delta x}\right\} = C_N \le 1 \tag{17}$$

then schemes 1 and 2 are dissipative scheme.

Schemes 3 and 4 are diffusive-antidiffusive form when $r \le 1$, and antidiffusive-diffusive form when $r > 1$. By Fourier linear analysis we can obtain that the stability and dissipative condition is

$$r_{\max} = C_N \le 1.5 \tag{18}$$

consequently when we use with method of shock-fitting maxium time-step of both of these schemes can reach $C_N \le 1.5$, but in calculation of shock-capturing, predicated step must be stability, so $C_N \le 1$.

**Remark 1.** Scheme 3 is unformly second-order accuracy, but when $\varepsilon_0$ and $\theta_0$ are large scheme 1, scheme 2 and scheme 4 is first-order accuracy through discontinuity region, on the rest, these schemes are added second-order dissipative term of third infinite quantity that the coefficient is $\frac{1}{2}\eta\Delta t\Delta x^2$, consequently are uniformly second-order accuracy.

**Remark 2.** In constructing previously schemes, we don't claim that the schemes are presevering monotonicity, but we claim the schemes are dissipative i.e. statify

$$\|G\| \le 1 - \delta|\xi|^{2r} \quad |\xi| \ne 0, \quad r = 1 \text{ or } 2 \tag{19}$$

for $|\xi| \ne 0$, dissipative scheme is compression operator it decays propagation of error wave produced in calculating process, $|\xi|$ is bigger, decaying is quicker, hence short error wave can quickly be decreased and becomes zero. The calculating result everywhere no ossillation.

## §2. Classical Example and Dissension of Results

We have calculated following complicated problems by previous five schemes, incident shock on conical shock interaction flow fields, shock-on-shock refracted flow fields in shock-tube, incident shock on nose shock of sphere-cone interaction unsteady flow, compution of unviscous flow around sphere-cone body at angles of attack and around reentry-airship body. All calculations use shock-caputuring method and obtain reasonable results, for comparison of previous schemes, we calculate sod shock-tube example and give the results of time-step number $n = 60, 120$ space-step length $x = 0.01$.

Fig. 1 show result of least dissipative hybrid scheme when $C_N = 0.9, n = 60$. Fig. 2 expresses result of MacCormack two-step scheme with $\theta q/2$ dissipation modification. Fig. 3, Fig. 4 and Fig. 5 denote respectively results of MacCormack two-step scheme with $\theta/8$ dissipation modification when $C_N = 0.9, 0.95$, and $1.0$, $n = 60$, as $n = 120$ the results are show by Fig. 6, Fig. 7 and Fig. 8. Fig. 9, Fig. 10 and Fig. 11 represent respectively the results of modified diffusive-antidiffusive scheme with the least dissipation in which $C_N$ is 0.9, 0.95 and 1; as $n = 120$ the results are shown by Fig. 12, Fig. 13 and Fig. 14. Fig. 15 stands for result of diffusive-antidiffusive scheme with the least dissipation, in which $C_N$ is $0.9, n$ is 60.

From Figs. 1–15, we obtain that the numerical solution calculated by these schemes has high accuracy on the simple wave, and resolution on the shock. Modified diffusive-antidiffusive with the least dissipation and MacCormack two-step scheme with $\theta/8$ dissipation modification have many advantages. Courant number of both of these schemes can reach 1, the simple wave calculated by the former almostly agrees with theoretical solution, the transition region of contact discontinuity is narrow and only adds one net-point from $n = 60$ to $n = 120$, the shock has high resolution. The simple wave computed by the latter also is in agreement with theoretical solution, the shock has high resolution, and the algorithm is very simple, but the transition region of contact discontinuity is wider than that of the former. According to previous analysis we think that both schemes are very well.

Diffusive-antidiffusive scheme with the least dissipation is uniformly second-order accuracy, it preserves good point of traditional symmetrical scheme, it doesn't require judgement operation, and is particularly well adapted for parallel arithmetic, the simple wave is very agreement with theoretical solution, the transition region of contact discontinuity is very narrow, only shortcoming is that $u$ has down-oscillation, but we still think that it is also a good method.

The calculating work in this paper has been completed by my graduate Luo Wen-cang.

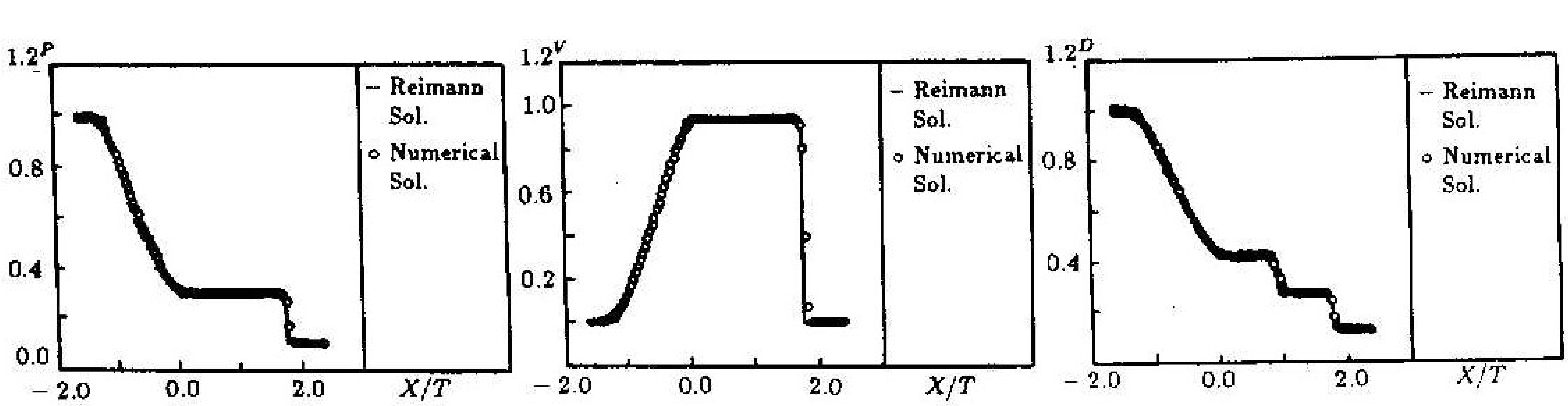


Fig.1. Hybrid Scheme with the least dissipation CFL =0.9

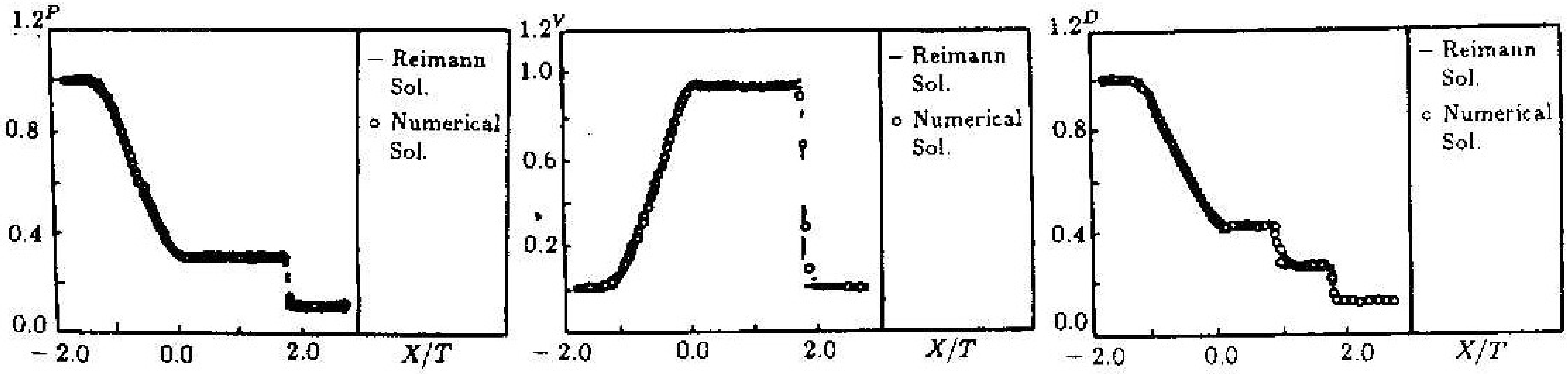


Fig.2. MacCormack scheme with $\theta q/2$ dissipation modification CFL =0.8

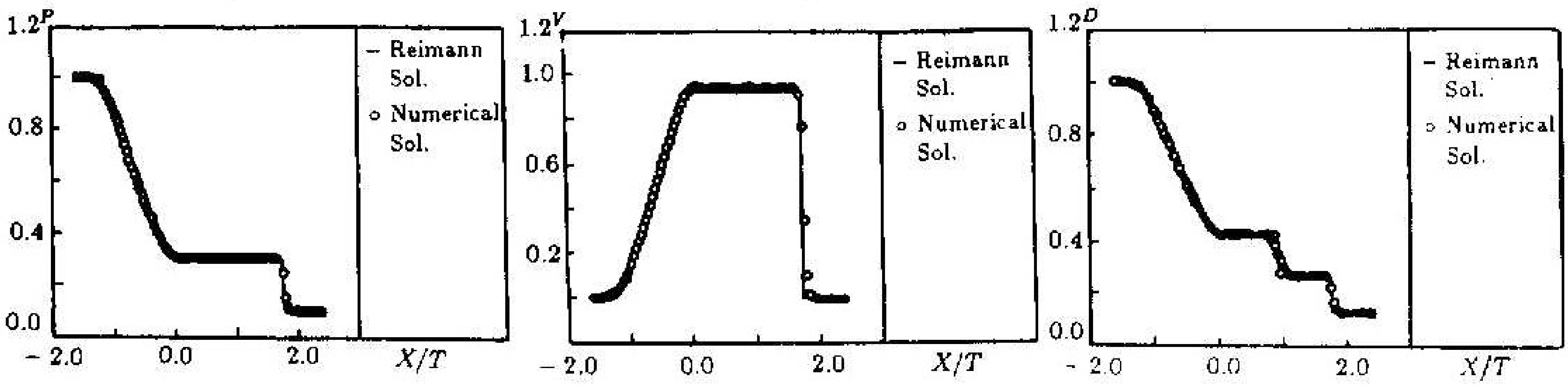


Fig.3. MacCormack scheme with $\theta/8$ dissipation modification CFL =0.9

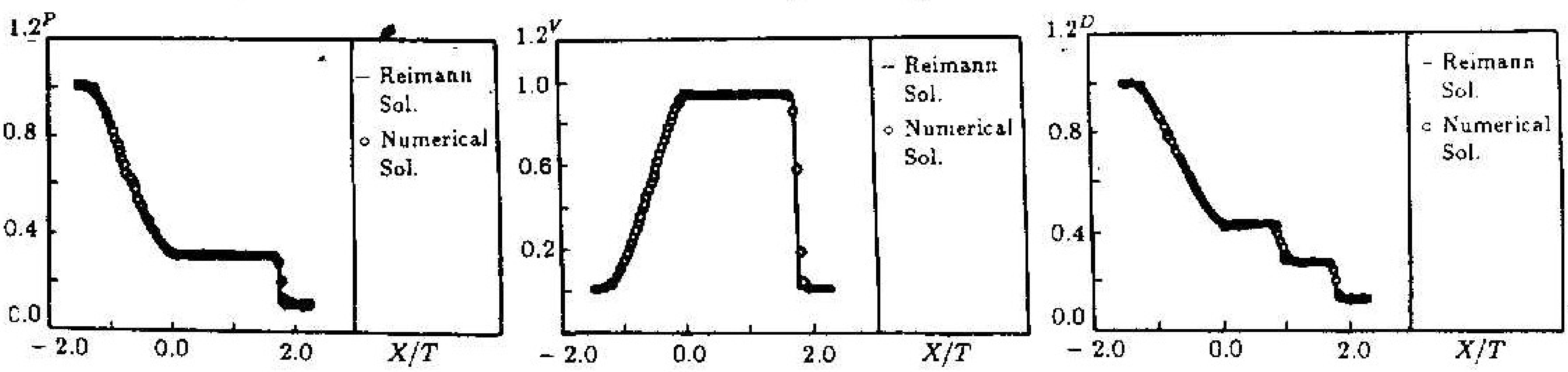


Fig.4. MacCormack scheme with $\theta/8$ dissipation modification CFL =0.95

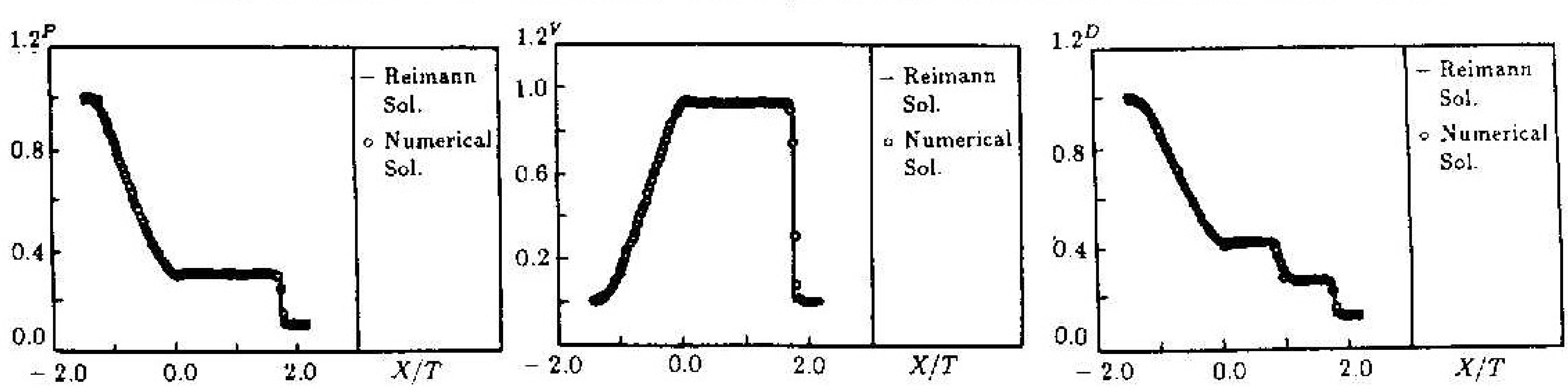


Fig.5. MacCormack scheme with $\theta/8$ dissipation modification CFL =1.0

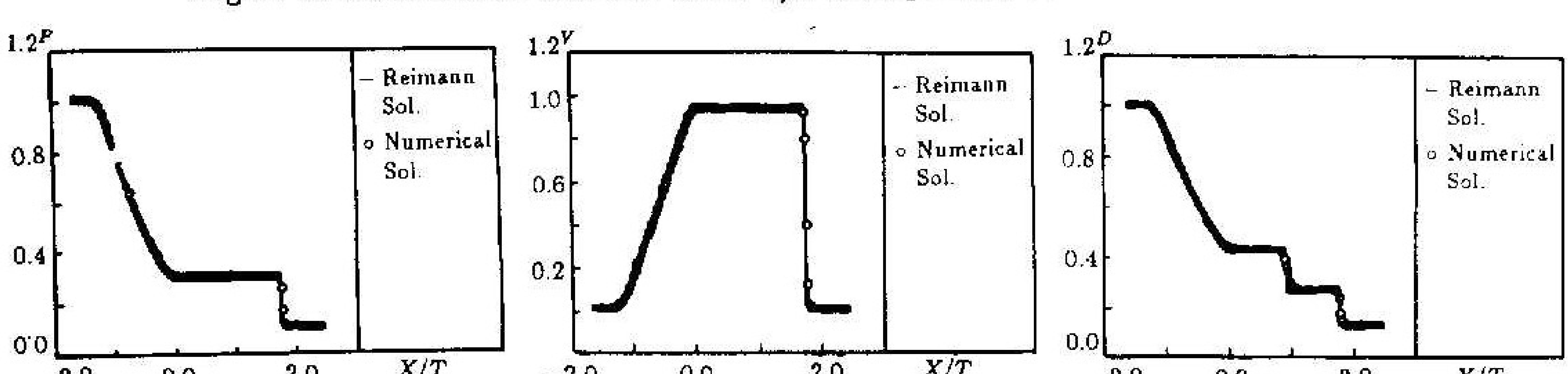


Fig.6. MacCormack scheme with $\theta/8$ dissipation modification CFL =0.9

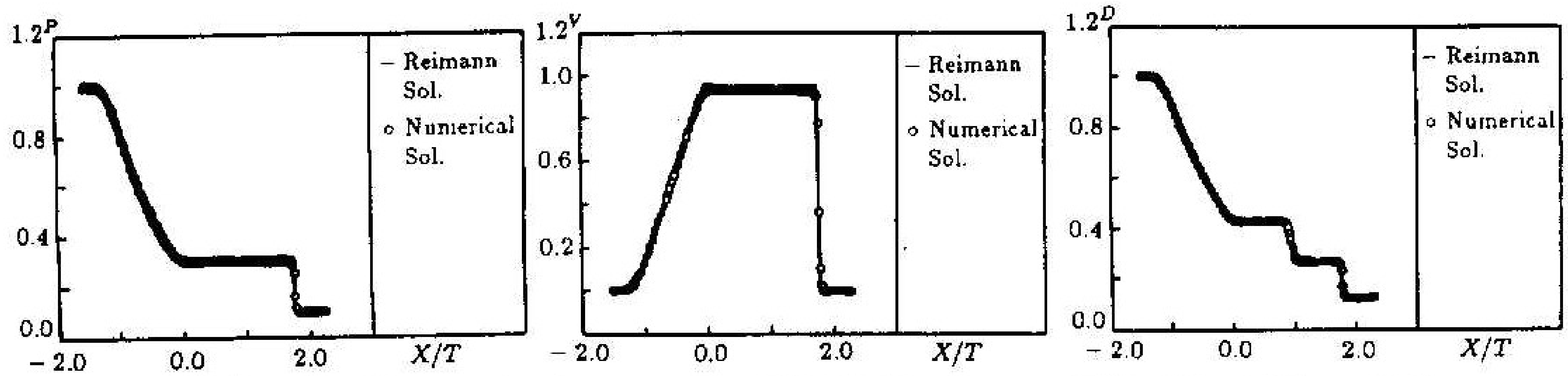


Fig.7. MacCormack scheme with $\theta/8$ dissipation modification CFL =0.95

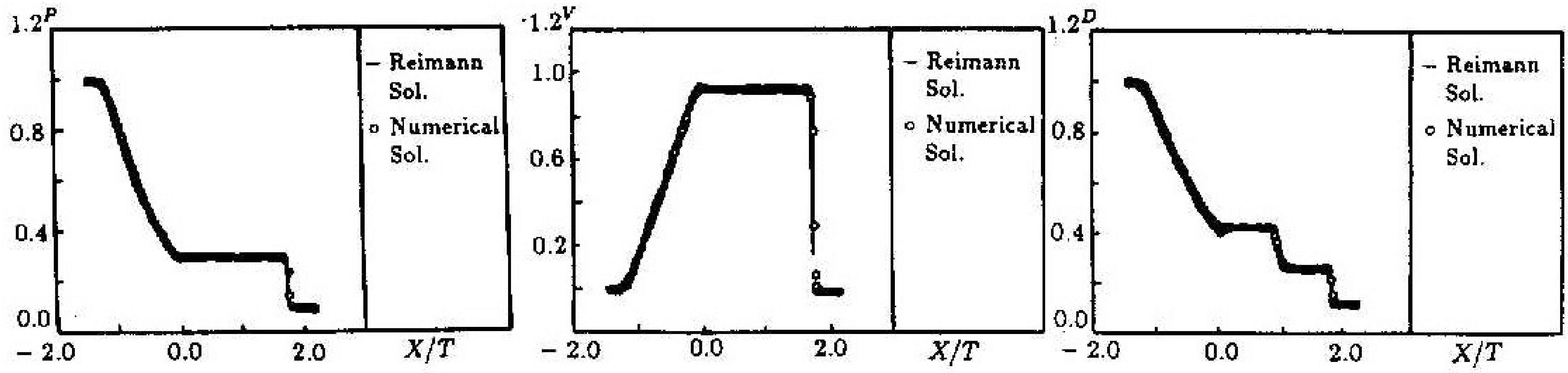


Fig.8. MacCormack scheme with $\theta/8$ dissipation modification CFL =1.0

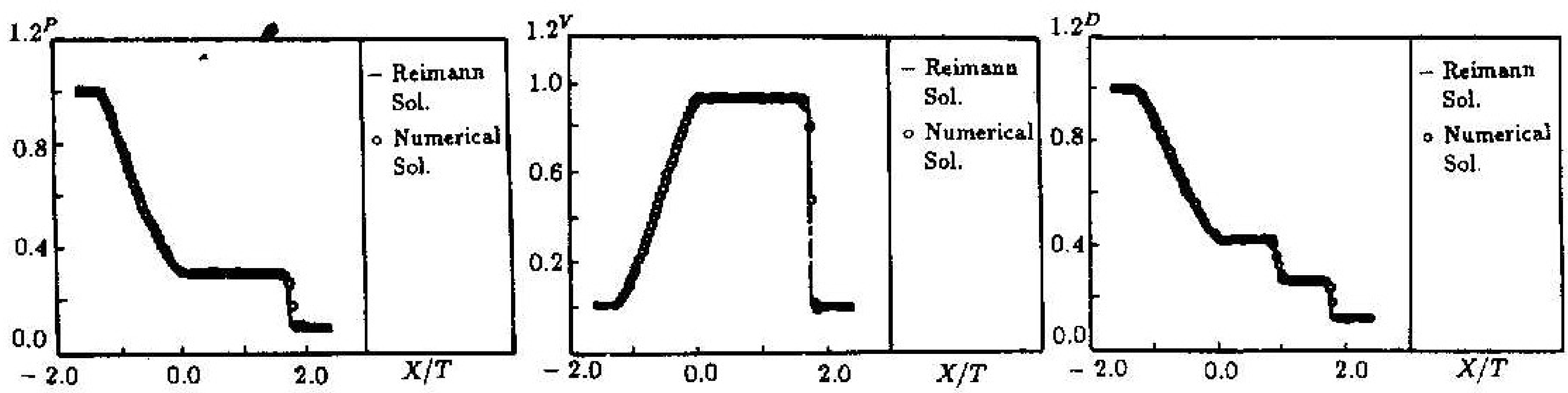


Fig.9. Modified diffusive — antidiffusive scheme CFL =0.9

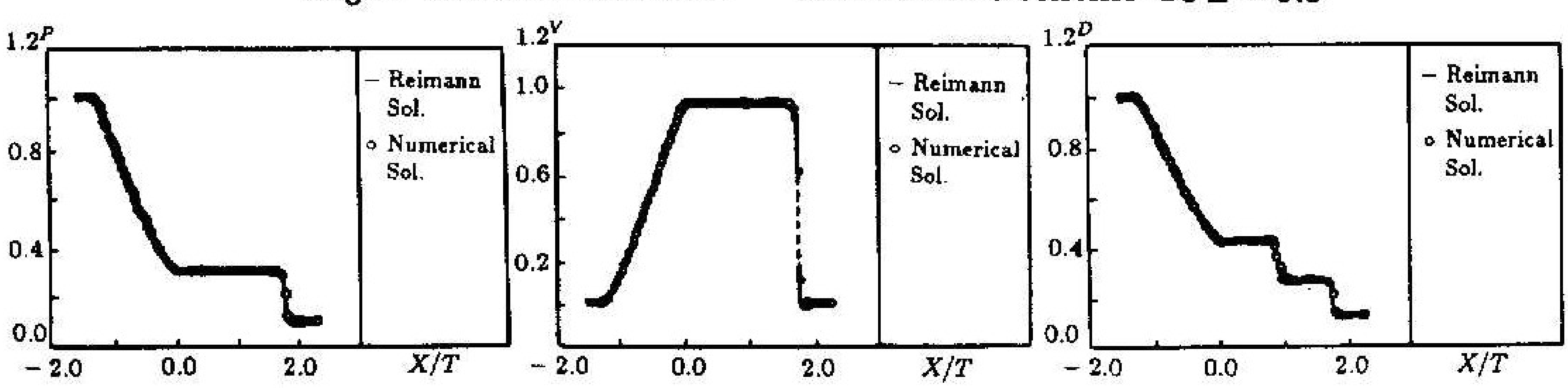


Fig.10. Modified diffusive — antidiffusive scheme CFL =0.95

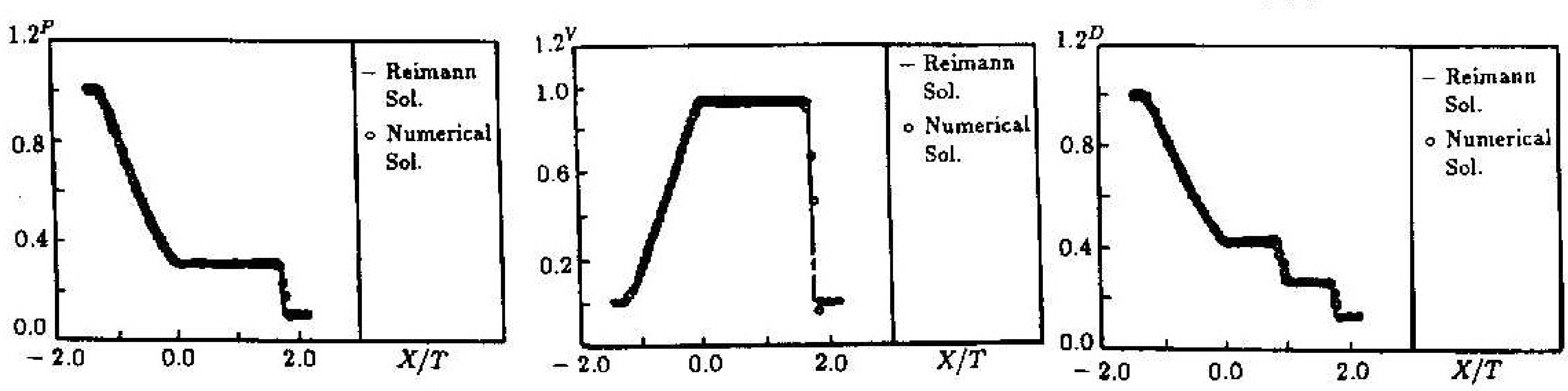


Fig.11. Modified diffusive — antidiffusive scheme CFL =1.0

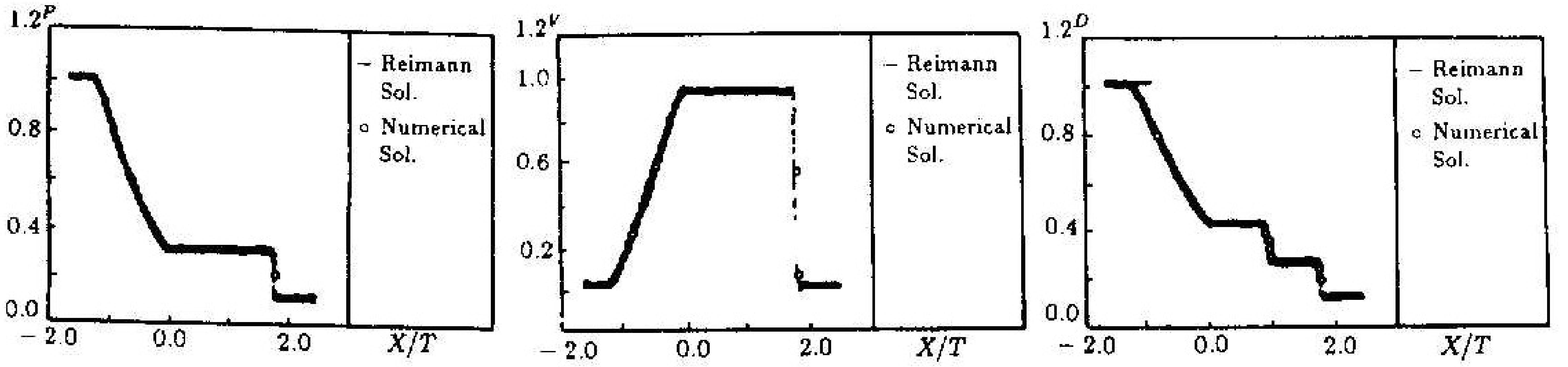


Fig.12. Modified diffusive — antidiffusive scheme CFL =0.9

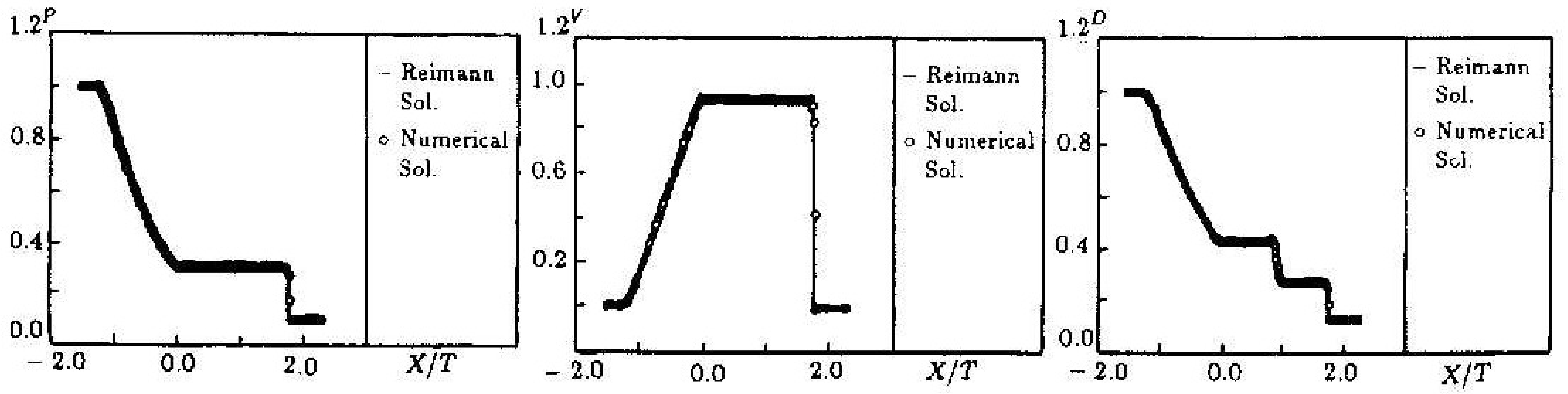


Fig.13. Modified diffusive — antidiffusive scheme CFL =0.95

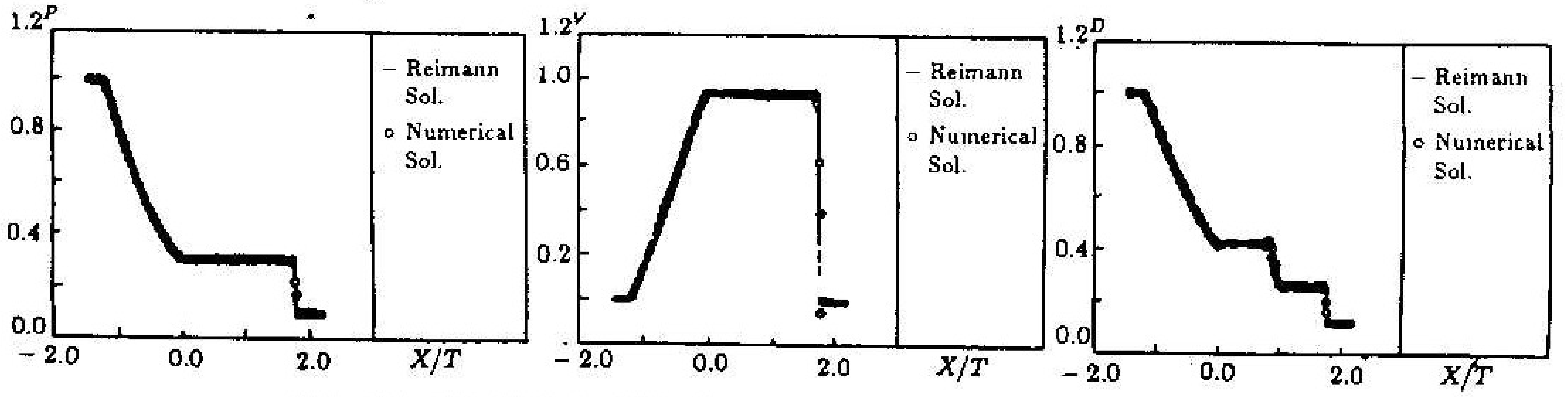


Fig.14. Modified diffusive — antidiffusive scheme CFL =1.0

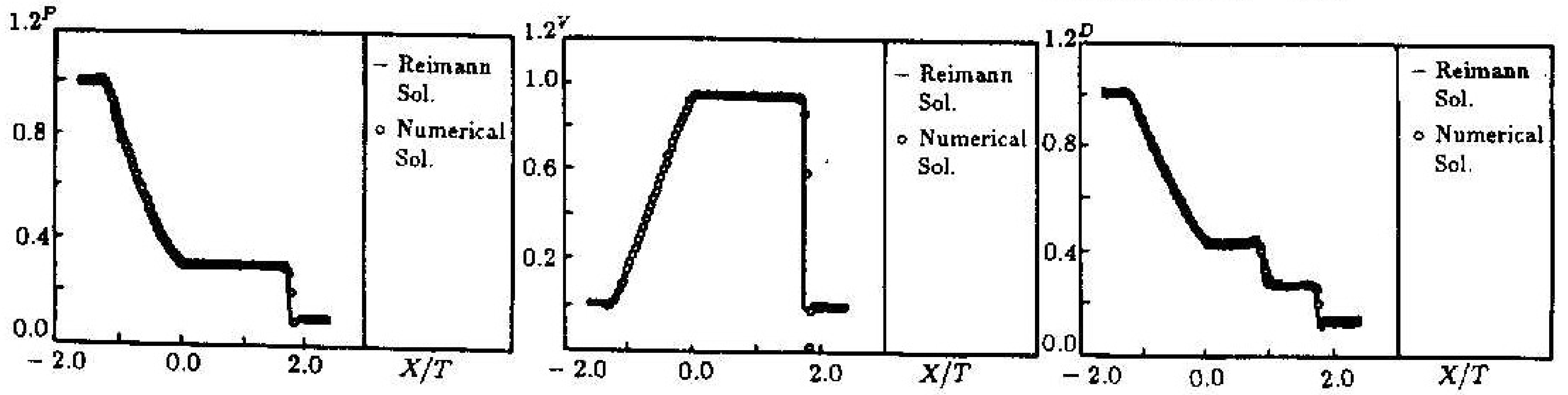


Fig.15. Modified diffusive — antidiffusive scheme CFL =0.9

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