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THE CONVEXITY OF FAMILIES OF ADJOINT PATCHES FOR A BEZIER TRIANGULAR SURFACE

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Abstract

A necessary and sufficient condition for the convexity of adjoint patches for a Bezier triangular surface is presented. Furthermore, it is proved that this condition is equivalent to the fact that the adjoint patches form a decreasing sequence as the corresponding degree decreases. The condition can be easily computationally verified.

§1. Introduction

Consider a given triangle $T \subset R^2$. A Bernstein-Bezier surface over T is usually expressed as

$$B^n := B^n(p) := \sum_{i+j+k=n} f_{i,j,k} J^n_{i,j,k}(p) \tag{1}$$

where

$$J^n_{i,j,k}(p) := \frac{n!}{i!\, j!\, k!} u^i v^j w^k,$$

$p := (u, v, w) \in T$ is a point given by its barycentric coordinates, and $F := \{f_{i,j,k} \in R \mid i + j + k = n,\ i, j, k \geq 0\}$ is a set of prescribed real numbers. The de Casteljau algorithm[1] provides a stable and efficient tool for the evaluation of $B^n(p)$. It is well known that it has also a simple geometric interpretation, i.e. it can be viewed as a sequence of plain interpolations. To be precise, let us follow [2] and define partial shift operators:

$$E_1 g_{i,j,k} := g_{i+1,j,k}, \quad E_2 g_{i,j,k} := g_{i,j+1,k}, \quad E_3 g_{i,j,k} := g_{i,j,k+1}. \tag{2}$$

Let the nodes $P_{i,j,k}$ that correspond to $f_{i,j,k}$ be given by

$$P_{i,j,k} := (i/n,\ j/n,\ k/n), \quad i + j + k = n.$$

For a given $P \in T$, the de Casteljau algorithm computes the values

$$f^m_{i,j,k} := f^m_{i,j,k}(p) := (uE_1 + vE_2 + wE_3)^m f_{i,j,k} = \sum_{\substack{\alpha+\beta+\gamma=m \\ i+j+k=n-m}} f_{i+\alpha,j+\beta,k+\gamma} J^m_{\alpha,\beta,\gamma}(p), \tag{3}$$

that correspond to the nodes

$$P^m_{i,j,k} := P^m_{i,j,k}(p) := (uE_1 + vE_2 + wE_3)^m P_{i,j,k} = \Big(\frac{i+mu}{n}, \frac{j+mv}{n}, \frac{k+mw}{n}\Big). \tag{4}$$

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In particular, $f^n_{0,0,0}$ is the value of B^n at the point $P^n_{0,0,0} = P$. Let $F^m := \{f^m_{i,j,k}\}$ and consider $P^m_{i,j,k}$. These nodes belong to a smaller triangle with vertices

$$P^m_{n-m,0,0}, \quad P^m_{0,n-m,0}, \quad P^m_{0,0,n-m}.$$

We denote it by T^m. Quite obviously, F^m and $\{P^m_{i,j,k}\}$ depend on the point $P \in T$ at which we are evaluating B^n. Nevertheless, we can adjoin to F^m an $(n-m)$th degree Bezier surface over T_m. In barycentric coordinates with respect to T^m it reads

$$B^{n-m}_p := \sum_{i,j+k=n-m} f^m_{i,j,k} J^{n-m}_{i,j,k}. \tag{5}$$

It is called $(n-m)$th adjoint patch of B^n (for the given point p). In [4] it is shown that the original surface B^n is an envelope of the family $\{B^m_p\}$. This explains why the study of adjoint patches could be useful. In the next section we shall discuss the convexity of families of adjoint patches and provide a simple necessary and sufficient condition.

§2. Convexity of Adjoint Patches

In [4] the following conclusion was proved: If the inequalities

$$\begin{aligned} D_1 f_{i,j,k} &:= (E_1 - E_2)(E_1 - E_3) f_{i,j,k} \geq 0, \\ D_2 f_{i,j,k} &:= (E_2 - E_1)(E_2 - E_3) f_{i,j,k} \geq 0, \\ D_3 f_{i,j,k} &:= (E_3 - E_1)(E_3 - E_2) f_{i,j,k} \geq 0 \end{aligned} \tag{6}$$

hold for $i + j + k = n - 2$, then the adjoint patch B^{n-m}_p is convex over T^m, for all $m = 1, 2, \cdots, n$. The condition (6) is only sufficient, not necessary. We proceed with a necessary and sufficient condition that can be easily verified.

Theorem 1. *The adjoint patch B^{n-m}_p is convex over $T^m, m = 1, 2, \cdots, n$, for any $P \in T$ if and only if the data F satisfy*

$$\begin{aligned} &(D_1 + D_2) f_{i,j,k} \geq 0, \quad (D_2 + D_3) f_{i,j,k} \geq 0, \quad (D_1 + D_3) f_{i,j,k} \geq 0, \\ &D_1 f_{i,j,k} D_2 f_{i,j,k} + D_1 f_{i,j,k} D_3 f_{i,j,k} + D_2 f_{i,j,k} D_3 f_{i,j,k} \geq 0 \end{aligned} \tag{7}$$

for all $i + j + k = n - 2$.

Proof. The conditions (7) imply the convexity of B^n over T[3]. But any B^{n-m}_p is also a Bezier surface corresponding to F^m. Therefore, it is sufficient to prove that (7) hold for any F^m. Assume that F^m satisfies (7) for some fixed $m \geq 0$. We obtain by (3) and (6)

$$D_1 f^{m+1}_{i,j,k} = D_1(uE_1 + vE_2 + wE_3) f^m_{i,j,k} = uD_1 f^m_{i+1,j,k} + vD_1 f^m_{i,j+1,k} + wD_1 f^m_{i,j,k+1} \tag{8}$$

and similar equalities for $D_2 f^{m+1}_{i,j,k}, D_3 f^{m+1}_{i,j,k}$. Thus by assumption on F^m,

$$(D_1 + D_2) f^{m+1}_{i,j,k} = [u(D_1 + D_2) f^m_{i+1,j,k} + v(D_1 + D_2) f^m_{i,j+1,k} + w(D_1 + D_2) f^m_{i,j,k+1}] \geq 0, \tag{9}$$

and

$$(D_1 + D_3) f^{m+1}_{i,j,k} \geq 0, \tag{10}$$

$$(D_2 + D_3) f^{m+1}_{i,j,k} \geq 0 \tag{11}$$

for all $i + j + k = n - (m + 1) - 2$.

Further,

$$D_1 f_{i,j,k}^{m+1} D_2 f_{i,j,k}^{m+1} + D_1 f_{i,j,k}^{m+1} D_3 f_{i,j,k}^{m+1} + D_2 f_{i,j,k}^{m+1} D_3 f_{i,j,k}^{m+1} = \det A_{i,j,k}^{m+1} \tag{12}$$

with

$$A_{i,j,k}^r := \begin{pmatrix} (D_1 + D_2) f_{i,j,k}^r & D_1 f_{i,j,k}^r \\ D_1 f_{i,j,k}^r & (D_1 + D_3) f_{i,j,k}^r \end{pmatrix}.$$

Note that again by (3) and (6),

$$A_{i,j,k}^{(m+1)} = u A_{i+1,j,k}^m + v A_{i,j+1,k}^m + w A_{i,j,k+1}^m. \tag{13}$$

All the matrices appearing on the right side of (13) are nonnegative definite, and $A_{i,j,k}^{m+1}$ has to be nonnegative definite also. This finally proves the induction step of the argument. (7) thus imply convexity.

On the other hand, the convexity of B_p^3 implies that F^{n-3} has to satisfy (7)[3]. Assume now that F^m satisfies (7) for any P and consider F^{m-1}. A particular choice of $p = (1,0,0), p = (0,1,0), p = (0,0,1)$ reveals that F^{m-1} has to satisfy (7) also. By induction F satisfies (7) too.

We proceed to point out that conditions (7) have an equivalent geometric formulation for adjoint patches of different degrees that belong to the same point P. First, we prove the following.

Lemma. *B^n is convex over T if and only if* $\sigma = (\xi, \eta, \varsigma) H \begin{pmatrix} \xi \\ \eta \\ \varsigma \end{pmatrix} \geq 0$, *where* $\xi + \eta + \varsigma = 0$ *and*

$$H := n(n-1) \sum_{i+j+k=n-2} \begin{pmatrix} E_1^2 f_{i,j,k} & E_1 E_2 f_{i,j,k} & E_1 E_3 f_{i,j,k} \\ E_2 E_1 f_{i,j,k} & E_2^2 f_{i,j,k} & E_2 E_3 f_{i,j,k} \\ E_3 E_1 f_{i,j,k} & E_3 E_2 f_{i,j,k} & E_3^2 f_{i,j,k} \end{pmatrix} J_{i,j,k}^{n-2}(P^*) \tag{14}$$

for any $P^* \in T$.

Proof. For any $P_0, P_1 \in T$, we assume that

$$P_0 = (u_0, v_0, w_0), \quad P_1 = (u_1, v_1, w_1).$$

The point P on the line segment will have the following barycentric coordinates:

$$p(t) = ((1-t)u_0 + tu_1,\ (1-t)v_0 + tv_1,\ (1-t)w_0 + tw_1), \quad 0 \leq t \leq 1.$$

The curve intersected by the surface $Z = B^n(p)$ and the plane perpendicular to the domain triangle and containing the line segment $\overline{P_0 P_1}$ has the following equation

$$z = B^n(p(t)), \quad 0 \leq t \leq 1. \tag{15}$$

It is obvious that $z = B^n(p)$ is convex over T if and only if the curve (15) is convex for any $P_0, P_1 \in T$, or $\frac{d^2 z}{dt^2} \geq 0$ for $t \in [0,1]$, and $P_0, P_1 \in T$. A straight forward calculation shows that

$$\frac{d^2 z}{dt^2} = (\xi, \eta, \varsigma) H \begin{pmatrix} \xi \\ \eta \\ \varsigma \end{pmatrix} = \sigma. \tag{16}$$

The lemma is confirmed.

We now prove the following

Theorem 2. *The adjoint patches are increasing with respect to the degree, i.e.*

$$B^n \geq B_p^{n-i} \geq B_p^{n-j} \quad on\ T^j, \quad 1 \leq i \leq j \leq n \tag{17}$$

for any $P \in T$, if and only if the conditions (7) are satisfied.

Proof. The same argument as in [4] shows that for any $p = (u, v, w) \in T$, we have

$$B^n = B_p^{n-1} + \sigma/2n^2. \tag{18}$$

By the lemma, we know that $\sigma \geq 0$ if and only if B^n is convex. But this implies that

$$B_p^{n-i} \geq B_p^{n-j}, \quad 1 \leq i < j \leq n$$

if and only if B_p^{n-i}, $i = 1, 2, \cdots, n-1$, are convex, or by Theorem 1, conditions (7) hold.

At last, we add some remarks.

1. The conditions (7) are superior to that presented in [4]. For example, take $n = 3$, and $f_{3,0,0} = f_{1,2,0} = f_{0,1,2} = f_{0,0,3} = -f_{2,1,0} = 1$, $f_{2,0,1} = f_{1,0,2} = f_{1,1,1} = 0$, $f_{0,3,0} = 3$, $f_{0,2,1} = 2$. It is obvious that data F^3 satisfy conditions (7). Therefore Theorems 2 and 3 hold, but the criterion in [4] fails to be valid.

2. The convexity of B^n implies the inequality $B^n \geq B_p^{n-1}$ for any $P \in T$, but B_p^{n-1} may not be convex. For example, take $n = 4$ and the only nonzero data are

$$f_{4,0,0} = f_{0,4,0} = f_{0,0,4} = 1, \quad f_{0,2,2} = f_{2,0,2} = f_{2,2,0} = \frac{1}{3}. \tag{19}$$

The corresponding Bernstein-Bezier polynomial is

$$B^4 = (u^2 + v^2 + w^2)^2 \tag{20}$$

and is obviously convex. Take $p = (1, 0, 0)$. Then F^3 do not satisfy conditions (7) and B_p^3 cannot be convex.

3. The convexity of B^n is not sufficient to guarantee inequality (17). Take the same example in 2 and $p = (1, 0, 0)$. Then

$$B^4 = (u^2 + v^2 + w^2)^2, \quad B_p^3 = u(u^2 + v^2 + w^2), \quad B_p^2 = u^2 + (v^2 + w^2)/3. \tag{21}$$

Let $Q = (0, 1, 0) \in T^2$. Then Q has barycentric coordinates $(1/3, 2/3, 0), (1/2, 1/2, 0)$ with respect to T^1 and T. It follows that

$$B_p^4(Q) = 1/4 > B_p^3(Q) = 5/27, \quad B_p^3(Q) < B_p^2(Q) = 1/3$$

and (17) does not hold.

References

[1] W. Boehm, G. Farin and J. Kahmann, A survey of curve and surface methods in CAGD, *Computer Aided Geometric Design,* **1** (1984), 1–60.

[2] G.Z. Chang and P.J. Davis, The Convexity of Bernstein polynomials over triangles, *J. Approx. Theory,* **40** (1984), 11–28.

[3] G.Z. Chang and Y.Y. Feng, An improved condition for the convexity of Bernstein-Bezier surfaces over triangles, *Computer Aided Geometric Design,* **1** (1984), 279–283.

[4] G.Z. Chang and S.C. Su, Families of adjoint patches for a Bezier triangular surface, *Computer Aided Geometric Design,* **2** (1985), 37–42.