Journal of Computational Mathematics, Vol.9, No.4, 1991, 291–300.

A DECOMPOSITION METHOD FOR SOME BIHARMONIC PROBLEMS*

F. Scarpini
(*Dipartimento di Matematica, Universitá "La Sapienza", P.le A.Moro,* 00185 *Roma, Italia*)

Abstract

In this paper we consider two biharmonic problems [13] which will be conventionally indicated as "simply supported" and "clamped plate" problem.

We construct a decomposition method [16], [19] related to the partition of the plate in two, or more, subdomains. We carry on the numerical treatment of the method first decoupling these fourth order problems into two second order problems, then discretizing these problems by mixed linear finite element and obtaining an algebraic system. Moreover we present an iterative block algorithm for solving the foregoing system, which can be efficiently developed on parallel computers.

At the end we extend the method to the respective biharmonic variational inequalities [10].

Notations

Let:

— Ω be a bounded open set of R^2, whose boundary $\Gamma = \partial\Omega$ is "sufficiently" regular [8];

— $\Omega = \Omega_1 \cup \gamma \cup \Omega_2$, where $\Omega_i (i = 1, 2)$ are sufficiently regular open sets staying in the opposite sides of a regular curve γ;

— $\omega \subset \Omega$ be any regular open set containing γ. We will call ω "lacing" set of Ω_1 and Ω_2 owing to raisons that will be clear in future developments;

— $\Gamma_i = \partial\Omega_i$;

— γ_m be the trace operator $\partial_n^m \cdot$ defined on the boundary of an open set, n being the external normal vector [8].

We consider the following functional spaces:

(1) $H := L^2(\Omega)$ endowed with the scalar product $(u, v)_{0,\Omega} = \int_\Omega uvdx$ and the associated norm;

(2) $H_1 := \{v : v \in H/v = 0$ a.e. in $\Omega_{i+1}\} (i = 1, 2)$ where the indexes are counted modulo 2 when necessary;

(3) $V := H^1(\Omega)$ with the scalar product $(u, v)_V = \int_\Omega (\text{grad } u \cdot \text{grad } v + uv)$;

(4) $V_i := \{v : v \in V/v = 0$ a.e. in $\Omega_{i+1}\}, i = 1, 2$;

(5) $\overset{\circ}{V} := H_0^1(\Omega)$ with the scalar product $(u, v)_{1,\Omega} = \int_\Omega \text{grad } u \cdot \text{grad } vdx$;

(6) $\overset{\circ}{V}_i := \{v : v \in V/v = 0$ a.e. in $\Omega_{i+1}\}, i = 1, 2$;

* Received February 28, 1991.

(7) $\overset{\circ}{V}_3 := \{v : v \in \overset{\circ}{V} / v = 0 \text{ a.e. in } \Omega_i \backslash \omega,\ i = 1,2\}$;

(8) $W := H^2(\Omega) \cap \overset{\circ}{V}$ or $W = H_0^2(\Omega)$ endowed with the scalar product $(u,v)_W = \int_\Omega \Delta u \cdot \Delta v dx$, where $\Delta \cdot = \partial_{11} \cdot + \partial_{22} \cdot$ is the harmonic operator;

(9) $W_i := \{v : v \in W / v = 0 \text{ in } \Omega_{i+1}\}, i = 1,2$;

(10) $W_3 := \{v : v \in W / v = 0 \text{ in } \Omega_i \backslash \omega (i = 1,2)\}$.

Remark I. We remark the following obvious relations [20]:

(11) $V = V_1 + \overset{\circ}{V}_3 + V_2$;

(12) $\overset{\circ}{V} = \overset{\circ}{V}_1 + \overset{\circ}{V}_3 + \overset{\circ}{V}_2$;

(13) $W = W_1 + W_3 + W_2$.

First Part

1. A Minimum Problem in $W = H^2(\Omega) \cap \overset{\circ}{V}$. The Euler Equation. The Simply Supported Plate Problem.

We consider the following potential energy functional for a simply supported plate [13]:

$$F(v) = \frac{1}{2}|v|_W^2 - (f,v)_{0,\Omega}, \quad v \in W,\ f \in H. \tag{1.1.1}$$

It is well known that F is weakly lower semicontinuous, Frechet differentiable, strictly convex, coercive and that it results [8]:

$$< \text{grad } F(u), v >_W = (u,v)_W - (f,v)_{0,\Omega}, \quad \forall v \in W \tag{1.1.2}$$

where the symbol $< \circ, \circ >_W$ denotes the duality pairing between W and the dual space W'.

As a consequence of the above quoted properties we conclude that the extreme problem:

$$\inf_{v \in W} F(v) \tag{1.1.3}$$

has a unique minimum point u:

$$F(u) = \min_{v \in W} F(v) \tag{1.1.4}$$

which corresponds to the unique solution of the following W.elliptic variational problem:

$$u \in W : (u,v)_W = (f,v)_{0,\Omega}, \quad \forall v \in W. \tag{1.1.5}$$

As a consequence of (1.1.5) the strong formulation of the simply supported plate problem is, in formal way, the following:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ \gamma_0 u|_\Gamma = \gamma_0 \Delta u|_\Gamma = 0 \end{cases} \tag{1.1.6}$$

where Δ^2 is the biharmonic operator: $\Delta^2 \cdot = \partial_{1111} \cdot + 2\partial_{1122} \cdot + \partial_{2222} \cdot$ The condition $\gamma_0 \Delta u|_\Gamma = 0$ is "natural" that is in weak sense.

It is well known [17] that, if Ω is regular enough and $f \in H$, the solution u of (1.1.5) belongs to $H^4(\Omega) \cap W$. If Ω is a convex polygon $u \in H^3(\Omega) \cap W$ [7].

We consider now a function $w \in H^4(\Omega)$ and mention the so called Green's formula [8]:

$$\int_\Omega \Delta w \cdot \Delta v dx = \int_\Gamma (\gamma_0 \Delta w \cdot \gamma_1 v - \gamma_1 \Delta w \cdot \gamma_0 v) d\Gamma + \int_\Omega \Delta^2 w \cdot v dx, \quad \forall v \in H^2(\Omega). \quad (1.1.7)$$

With the aim of studying a decomposition method we consider the solutions $u_i (i = 1, 2)$, in the strong sense, of the following problems:

$$\begin{cases} \Delta^2 u_i = f_i \text{ a.e.} \quad \text{in } \Omega_i, \\ \gamma_0 u_i|_{\Gamma_i \setminus \gamma} = \gamma_0 \Delta u_i|_{\Gamma_i \setminus \gamma} = 0, \\ \gamma_0 u_i|_\gamma = \rho_i, \quad \gamma_0 \Delta u_i|_\gamma = \sigma_i \end{cases} \quad (1.1.8)$$

where $f_i = f|_{\Omega_i}; \rho_i, \sigma_i$ are regular functions.

By applying (1.1.7) we obtain the following relation:

$$\int_\Omega \Delta u_1 \cdot \Delta v dx + \int_\Omega \Delta u_2 \circ \Delta v dx = \int_\gamma (\gamma_0 \Delta(u_1 - u_2) \cdot \gamma_1 v - \gamma_1 \Delta(u_1 - u_2) \gamma_0 v) d\gamma$$
$$+ \int_\Omega f v dx, \quad \forall\, v \in H^2(\Omega). \quad (1.1.9)$$

(1.1.9) suggests the transmission conditions which guarantee that $u_1 + u_2$ coincides with the solution u of (1.1.6) belonging to $H^4(\Omega)$.

We obtain in fact the following necessary and sufficient conditions

$$\gamma_0(u_1 - u_2)|_\gamma = \gamma_1(u_1 - u_2)|_\gamma = \gamma_0 \Delta(u_1 - u_2)|_\gamma = \gamma_1 \Delta(u_1 - u_2)|_\gamma = 0 \quad (1.1.10)$$

which are equivalent to the trace conditions:

$$\gamma_0(u_1 - u_2)|_\gamma = \gamma_1(u_1 - u_2)|_\gamma = \gamma_2(u_1 - u_2)|_\gamma = \gamma_3(u_1 - u_2)|_\gamma = 0 \quad (1.1.11)$$

as it can be verified by some easy but tedious calculations.

It must be said that in (1.1.9)–(1.1.11) the normal vectors are conveniently oriented to avoid misleading changes of sign.

2. The "Decouplage" Operation.

The problem (1.1.6) is equivalently expressed [6], [7] by:

$$-\Delta \phi = f \text{ a.e. in } \Omega, \quad -\Delta u = \phi \text{ in } \Omega; \ \gamma_0 \phi|_\Gamma = \gamma_0 u|_\Gamma = 0 \quad (1.2.1)$$

we consequently have the following weak formulation [7], [10]: Find $(\phi, u) \in \mathring{V} \times \mathring{V}$ such that

$$\begin{cases} (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, \quad \forall z \in \mathring{V}, \\ (u, z)_{1,\Omega} = (\phi, z)_{0,\Omega}, \quad \forall z \in \mathring{V}. \end{cases} \quad (1.2.2)$$

By this formulation we can construct the decomposition method.

3. The Decomposition Method.

Taking in mind the relation (12) of Remark 1, we have that (1.2.2) is equivalent to the following problem: Find $(\phi, u) \in \overset{\circ}{V} \times \overset{\circ}{V}$ such that

$$\begin{cases} (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}_i, \quad i = 1,2, \\ (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}_3, \\ (u, z)_{1,\Omega} = (\phi, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}_i, \quad i = 1,2, \\ (u, z)_{1,\Omega} = (\phi, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}_3 \end{cases} \tag{1.3.1}$$

in fact as a consequence of the Green's formula applied to the second and fourth equation in (1.3.1), we obtain a unique determination of:

$$\gamma_0 u|_\gamma, \quad \gamma_1 u|_\gamma, \quad \gamma_0 \phi|_\gamma = -\gamma_0 \Delta u|_\gamma, \quad \gamma_1 \phi|_\gamma = -\gamma_1 \Delta u|_\gamma \tag{1.3.2}$$

that is (1.3.1) imply (1.1.10) when we set

$$u_i = u|_{\Omega_i}, \quad \phi_i = \phi|_{\Omega_i}, \quad i = 1, 2.$$

4. The Mumerical Treatment of (1.3.1).

Just to simplify let us assume Ω to be a convex polygonal open set decomposed into two polygons $\Omega_i (i = 1, 2)$ by means a polygonal curve γ.

We operate a "regular triangulation" T_h of Ω [7] which includes γ among her reticulation and discretize (1.3.1) by finite linear elements [7]. We obtain in this way a linear algebraic system: Find $(\Phi, U) \in R^{N^0} \times R^{N^0}$ such that

$$\begin{cases} A_{I_i^0 \hat{I}_i} \Phi_{\hat{I}_i} = b_{I_i^0}, \quad i = 1, 2, \\ A_{G,J} \Phi_J = b_G, \\ A_{I_i^0, \hat{I}_i} U_{\hat{I}_i} = D_{I_i^0, \hat{I}_i} \Phi_{\hat{I}_i}, \\ A_{G,J} U_J = D_{G,J} \Phi_j \end{cases} \tag{1.4.1}$$

where: N^0 is the number of nodes in Ω,

$I^0 :=$ the set of indexes of nodes in Ω,

$I_i^0 :=$ the set of indexes of nodes in Ω_i, $i = 1, 2$,

$G :=$ the set of indexes of nodes on γ,

$\hat{I}_i := I_i^0 \cup G$,

$J :=$ the set of indexes of nodes in ω_h. ω_h being the stripeunion of triangles $T \in T_h$ such that $T \cap \gamma \neq \phi$,

$\phi_\Gamma^h :=$ the finite linear element related to node x_Γ,

$A_{I^0, I^0} = \{a_{r,s}\}_{I^0 \times I^0} = \{(\phi_s^h, \phi_r^h)_{1,\Omega}\}_{I^0 \times I^0}$ is a Gram's matrix (stiffness matrix),

$D_{I^0 \times I^0} = \{d_{r,s}\}_{I^0 \times I^0} = \{(\phi_s^h, \phi_r^h)_{0,\Omega}\}_{I^0 \times I^0}$ (mass matrix),

$b_{I^0} = \{b_s\}_{I^0} = \{(f, \phi_s^h)\}_{I^0}$ (load term).

The system (1.3.1) can be easily solved by an iterative method since the matrix A_{I^0,I^0} is positive definite. We start solving the first equation, assuming an initial guess Φ_G^0. Then we regularize Φ_G^0 by solving the second equation with respect to Φ_G and we continue using these new values as initial entries for another cycle. In the end we pass to solve the third and fourth equation, by means of the same computational technique.

This method is an iterative block method whose blocks are linked with the selected subdomains. The lacing set ω_h and the matrix $A_{G,J}$ enables the discrete approximation of all the above quoted transmission conditions.

5. Another Numerical Treatment.

For a further simplification we remark that we can diagonalize the mass matrix D_{I^0,I^0} if we compute the terms $d_{r,s}$ by numerical integration formulas exact for polynomials of degree less or equal to one instead of degree less or equal to two. In fact in this way we have [11], [12]:

$$d_{r,s} = h^2 \delta_{r,s}, \quad \delta_{r,s} \text{ Kronecker symbol.} \tag{1.5.1}$$

With this simplification the system [10]:

$$\begin{cases} A_{I^0,I^0}\Phi_{I^0} = b_{I^0}, \\ A_{I^0,I^0}U_{I^0} = D_{I^0,I^0}\Phi_{I^0} \end{cases} \tag{1.5.2}$$

which is the discretization of (1.2.2), is equivalent to [10]:

$$T_{I^0,I^0}U_{I^0} = b_{I^0} \tag{1.5.3}$$

where $T_{I^0,I^0} = 1/h^2(A_{I^0,I^0})^2$ is the "bending" matrix, that approximates Δ^2 and the boundary conditions, $T_{I^0,I^0}u_{I^0}$ is, after all, a bending difference formula with central term $t_{r,r}U_r$.

For using the operator T_{I^0,I^0} we write the following direct decomposition method: Find $u \in W$ such that

$$\begin{cases} (u,z)_W = (f,z)_{0,\Omega}, & \forall z \in W_i, \quad i = 1,2, \\ (u,z)_W = (f,z)_{0,\Omega}, & \forall z \in W_3. \end{cases} \tag{1.5.4}$$

Owing to (13) in the Remark 1, (1.5.4) is equivalent to the problem (1.1.5) and gives the following discrete method: Find U_{I^0} such that

$$\begin{cases} T_{J_i^0,I_i}U_{I_i} = b_{J_i^0}, \quad i = 1,2, \\ T_{\Lambda,N}U_N = b_\Lambda \end{cases} \tag{1.5.5}$$

where: $\Lambda :=$ the set of indexes of nodes appearing in the foregoing difference formula whose central terms are on G;

$J_i^0 := I_i^0 \backslash \Lambda, \quad i = 1,2,$

$N :=$ the set of indexes containing Λ, that has an extension suitable to express the complete difference formula with contral term on Λ.

Remark II. The decomposition method (1.5.4) can be seen as a computational method for the minimum point of the functional (1.1.1). In fact starting with an initial guess u^0 defined on Ω, we construct the following iterative procedure [20]:

n.th step. We compute the minimum point $u^{n-1/2}$, one by one on Ω_1 and Ω_2 with data $\gamma_0(u^{n-1/2} - u^{n-1})|_\gamma = 0$, we then compute the minimum point on ω and, with a welding operation, obtain the function u^n on Ω.

In this way obtain a sequence $\{u^n\}_n$ that approximates the solution u of (1.5.4) and of (1.1.4). The numerical treatment in section 5 carries out this algorithm in the finite dimensional space.

Remark III. The convergence of the mixed finite element solution of (1.5.3) to the continuous solution of (1.1.5) was theoretically verified and numerically tested in many concrete experiments [10]–[12], [15].

Second Part

1. A Minimum Problem in $W = H_0^2(\Omega)$. The Euler Equation. The Clamped Plate Problem.

Let us consider now the functional (1.1.1) with the new meaning of W, we reach the same conclusion (1.1.4) and the weak formulation (1.1.5). The strong problem of clamped plate is the following [13]:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ \gamma_0 u|_\Gamma = \gamma_1 u|_\Gamma = 0. \end{cases} \tag{2.1.1}$$

It holds the same regularity theorem quoted in the First Part and the same transmission conditions on γ (1.1.10) related to the partition $\Omega = \Omega_1 \cup \gamma \cup \Omega_2$.

2. The Mixed Formulation of (2.1.1).

The weak formulation of (2.1.1) differs from (1.2.2) for the condition $\gamma_1 u|_\Gamma = 0$ which replaces $\gamma_0 \Delta u|_\Gamma = 0$.

It is well known [7], [10] that one obtains the following problem: Find $(\phi, u) \in V \times \overset{\circ}{V}$ such that

$$\begin{cases} (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}, \\ (u, z)_{1,\Omega} = (\phi, z)_{0,\Omega}, & \forall z \in V. \end{cases} \tag{2.2.1}$$

It is important to remark that the second equation in (2.2.1) implies, in weak sense, the condition $\gamma_1 u|_\Gamma = 0$, in fact, by applying the Green's formula, we have:

$$(u, z)_{1,\Omega} = \int_\Omega \text{grad } u \cdot \text{grad } z dz = \int_\Gamma \gamma_1 u \cdot \gamma_0 z d\Gamma - \int_\Omega \Delta u \cdot z dx = (\phi, z)_{0,\Omega} \ \forall z \in V. \tag{2.2.2}$$

So if $z \in \overset{\circ}{V}$ we obtain:

$$-\int_\Omega \Delta u \cdot z dx = (\phi, z)_{o,\Omega} \tag{2.2.3}$$

that is:

$$-\Delta u = \phi \quad \text{in } \Omega; \tag{2.2.4}$$

if $z \in V$ it results:

$$\gamma_1 u|_\Gamma = 0. \tag{2.2.5}$$

3. The Decomposition Method.

By analogy to the process of the First Part, section 3, we construct the problem: Find $(\phi, u) \in V \times \overset{\circ}{V}$ such that

$$\begin{cases} (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}_i, \quad i = 1,2, \\ (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}_3, \\ (u, z)_{1,\Omega} = (\phi, z)_{0,\Omega}, & \forall z \in V_i, \quad i = 1,2, \\ (u, z)_{1,\Omega} = (\phi, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}_3 \end{cases} \tag{2.3.1}$$

where we use the usual notations. By taking in mind (11), (12) in the Remark I, we verify the equivalence of (2.3.1) to (2.2.1).

4. The Numerical Treatment.

Discretizing (2.3.1) we obtain: Find $(\Phi, U) \in R^N \times R^{N^0}$ such that

$$\begin{cases} A_{I_i^0, I_i} \Phi_{I_i} = b_{I_i^0}, \quad i = 1,2, \\ A_{G,J} \Phi_J = b_G, \\ A_{I_i^0, \hat{I}_i} U_{\hat{I}_i} = D_{I_i^0, I_i} \Phi_{I^i}, \\ A_{G,J} U_J = D_{G,J} \Phi_J \end{cases} \tag{2.4.1}$$

where: N is the number of nodes in $\bar{\Omega}$,

$I :=$ the set of indexes of nodes on $\bar{\Omega}$,

$I_i :=$ the set of indexes of nodes on $\bar{\Omega}_i$.

Diagonalizing the matrix $D_{I,I}$, as explained in the section 5 of the First Part, we obtain the system: Find U_{I^0} such that

$$\begin{cases} T_{J_i^0, \hat{I}_i} U_{\hat{I}_i} = b_{J_i^0}, \quad i = 1,2, \\ T_{\Lambda, N} U_N = b_\Lambda \end{cases} \tag{2.4.2}$$

where, in this case, an easy check [10] gives:

$$T_{I^0, I^0} = 1/h^2 A_{I^0, I} A_{I, I^0}.$$

The bending matrix T_{I^0, I^0} approximates Δ^2 on Ω and the boundary condition $\gamma_0 u|_\Gamma = \gamma_1 u|_\Gamma = 0$.

Third Part

1. A Minimum Problem on a Convex Set. The Biharmonic Variational Inequalities.

We consider the functional (1.1.1) on the closed convex set K [1], [21]:

$$K := \{v : v \in W / v \geq \alpha\} \text{ (as an example)} \tag{3.1.1}$$

where α is a fixed sufficiently regular function, such that $\alpha|_\Gamma \leq 0$. The unique minimum point u is the solution of the variational inequality [21]:

$$u \in K : (u, v-u)_W \geq (f, v-u)_{0,\Omega}, \quad \forall v \in K. \tag{3.1.2}$$

If Ω is sufficiently regular, $f \in H$, and $\alpha|_\Gamma < 0$ the solution $u \in H^3(\Omega) \cap K$[9].

By means of a decoupling procedure, we obtain in the case $W = H^2(\Omega) \cap \overset{\circ}{V}$ ($W = H_0^2(\Omega)$), the equivalent weak formulation[10]: Find $(\phi, u) \in \overset{\circ}{V} \times \hat{K}$ ($(\phi, u) \in V \times \hat{K}$) such that

$$\begin{cases} (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in \overset{\circ}{V}, \\ (u, z-u)_{1,\Omega} \geq (\phi, z-u)_{0,\Omega}, & \forall z \in \hat{K} \end{cases} \tag{3.1.3}$$

where

$$\hat{K} := \{z : z \in \overset{\circ}{V} (z \in V) / z \geq \alpha \text{ a.e. in } \Omega\}.$$

Equivalent to (3.1.3), (3.1.2) are the following systems:

Find $(\phi, u) \in \overset{\circ}{V} \times \hat{K}$ ($(\phi, u) \in V \times \hat{K}$) such that:

$$\begin{cases} (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in V_i, \quad i = 1,2, \\ (\phi, z)_{1,\Omega} = (f, z)_{0,\Omega}, & \forall z \in V_3, \\ (u, z-u)_{1,\Omega} \geq (\phi, z-u)_{0,\Omega}, & \forall z \in \hat{K}_i, \quad i = 1,2, \\ (u, z-u)_{1,\Omega} \geq (\phi, z-u)_{0,\Omega}, & \forall z \in \hat{K}_3 \end{cases} \tag{3.1.4}$$

where

$$\hat{K}_i := \{z : z \in u + \overset{\circ}{V}_i (z \in u + V_i) / (z-\alpha)|_{\Omega_i} \geq 0 \text{ a.e. }\},$$

$$\hat{K}_3 := \{z : z \in u + \overset{\circ}{V}_3 / (z-\alpha)|_\omega \geq 0 \text{ a.e. }\};$$

Find $u \in K$ such that:

$$\begin{cases} (u, v-u)_W \geq (f, v-u)_{0,\Omega}, & \forall v \in K_i, \quad i = 1,2, \\ (u, v-u)_W \geq (f, v-u)_{0,\Omega}, & \forall v \in K_3 \end{cases} \tag{3.1.5}$$

where

$$K_i := \{v : v \in u + W_i / (v-\alpha)|_{\Omega_i} \geq 0\},$$

$$K_3 := \{v : v \in u + W_3 / (v-\alpha)|_\omega \geq 0\}.$$

2. Numerical Treatment.

By following the arguments in the First and Second Part, we construct the complementarity system:

$$\begin{cases} U_{I^0} \geq A_{I^0}, \\ M_{J_i^0} = T_{J_i^0, I_i} U_{I_i} - b_{J_i^0} \geq 0, \quad i = 1,2, \\ M_\Lambda = T_{\Lambda, N} U_N - b_N \geq 0, \\ M_{J_i^0}(U_{J_i^0} - A_{J_i^0}) = 0, \quad i = 1,2, \\ M_\Lambda (U_\Lambda - A_\Lambda) = 0 \end{cases} \tag{3.2.1}$$

where: A_{I^0} is the vector $\{\alpha(x_r)\}_{I^0}$; the matrix T_{I^0,I^0} is defined in the First and Second Part respectively for simply supported and clamped plate. We solve (3.2.1) by the same technique, adding a projection of $[A_{I^0,+\infty}]$.

Remark IV. In addition to (11), (12), (13) in Remark I, we can verify the following relations[20]:

(14) $V(O_1) = V_1 + \overset{\circ}{V}_3, \quad V(O_2) = \overset{\circ}{V}_3 + V_2$,

(15) $\overset{\circ}{V}(O_1) = \overset{\circ}{V}_1 + \overset{\circ}{V}_3, \quad \overset{\circ}{V}(O_2) = \overset{\circ}{V}_3 + \overset{\circ}{V}_2$,

(16) $W(O_1) = W_1 + W_3, \quad W(O_2) = W_3 + W_2$

where $O_1 = \Omega_1 \cup \omega, O_2 = \omega \cup \Omega_2$. We have the new decomposition $\Omega = O_1 \cup O_2$ of partially overlapping open sets O_1, O_2. This decomposition enables us to construct the alternating Schwarz method for biharmonic variational inequalities[20].

Remark V. The numerical approach to the mentioned problems remains unchanged if the solution $u \in H^3(\Omega) \cap W$, in fact in this case $\gamma_1 \Delta u|_\gamma \in H^{-1/2}(\gamma)$ and a "generalized" Green's formula holds[8,17].

References

[1] H. Brezis and G. Stampacchia, Remarks on some fourth order variational inequalities, *Ann. Scuola Norm. Pisa,* 4 (4) (1977), 363–371.

[2] F. Brezzi and P.A. Raviart, A Mixed Finite Element Methods for 4th Order Elliptic Equations, Rapp. Interne $N^0$9 Ecole Polytechnique, Palaiseau, France, 1976, 1–312.

[3] L.A. Caffarelli and A. Friedman, The obstacle problem for the biharmonic operator, *Ann. Scuola Norm. Pisa,* (6) (1979), 151–186.

[4] L.A. Caffarelli, A. Friedman and A. Torelli, The free boundary for a fourth order variational inequality, *Ill. J. of Math.,* **25** (1981), 402–422.

[5] L.A. Caffarelli, A. Friedman and A. Torelli, The two obstacle problem for the biharmonic operator, *Pacific J. of Math.,* **103** (1982), 325–336.

[6] P.G. Ciarlet and P.A. Raviart, A Mixed Finite Element Method for the Biharmonic Equation, in Math. Aspects of Finite Element in Partial Differential Equations (C.de Boor Ed.), Academic Press, 1974, 125–145.

[7] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[8] R. Dautray and J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 1–6, Springer-Verla, 1985.

[9] J. Frehse, On the regularity of the solution of the biharmonic variational inequality, *Manuscripts Math.,* **9** (1973), 119–129.

[10] A. Fusciardi and F. Scarpini, A mixed finite element solution of some biharmonic unilateral problem, *Num. Funct. Anal. and Optimiz.,* **2** (5) (1980), 397–420.

[11] R. Glowinski, Approximations Extérnes, par Eléments Finis de Lamigrange d'ordre Un et Deux, du Probléme de Dirichlet pour l'Ope rateur Biharmonique, Méthodes Itératives de Résolution des Problémes Approchés, In Topics in Numerical Analysis (J.J.H. Miller Ed.), Academic Press, London, 1973, 123–171.

[12] R. Glowinski and G. Pironneau, Sur la résolution via une approximation par eléments finis mixtes du probléme de dirichlet pour L'opérateur biharmonique par une méthode "Quasi Directe" et diverses méthodes iteratives, *Rapp. de Recherque,* 197, 1976.
[13] R. Glowinski, J.L. Lions and R. Tremolieres, Analyse Numérique des Inéquations Variationnelles, Dunod, Paris, 1976.
[14] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer-Verlag, 1984.
[15] R. Glowinski, L.D. Marini and M. Vidrascu, Finite element approximation and iterative solution of a fourth-order elliptic variational inequality, *IMA J. of Num. Anal.,* **4** (1984), 127–167.
[16] R. Glowinski and M.F. Wheler, Domain Decomposition and Mixed Element Methods for Elliptic Problems, In First International Symposium on Domain Decomposition Methods for Partial Differential Equations (R. Glowinski, G.H. Golub, G.A. Meurant, G. Periaux, Eds) Siam, Philadelfia, 1988, 144–172.
[17] J.L. Lions and E. Magenes, Problémes aux Limites Non-Homogénes, Vol. I, Dunod, Paris, 1968.
[18] P.L. Lions, On the Schwarz Alternating Method I, In First Intern. Symp. on Domain Decomposition Methods for Partial Differential Equations (R. Glowinski, G.H. Golub, G.A. Meurant, G. Periaux Eds.), Siam, Philadelfia, 1988, 1–42.
[19] L.D. Marini and A. Quarteroni, An Iterative Procedure for Domain Decomposition Methods, A Finite Element Approach, In First Inter. Symp. on Domain, Decomposition Methods for Partial Differential Equations (R. Glowinski, G.H. Golub, G.A. Meurant, G. Periaux, Eds.), Siam, Philadelfia, 1988, 129–143.
[20] F. Scarpini, The alternating schwarz method applied to some biharmonic variational inequalities (to appear in *Calcolo*).
[21] G. Stampacchia, Variational Inequalities, In Theory and Applications of Monotone Operators (A. Ghizzetti Ed.), 1969, 101–191.