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AN ISOMETRIC PLANE METHOD FOR LINEAR PROGRAMMING*1)

Nie Yi-yong
(*Shenyang Institute of Computing Technology, Academia Sinica, Shenyang, China*)
Xu Shu-rong
(*Department of Computer Science, Zhongshan University, Guangzhou, China*)

Abstract

In this paper the following canonical form of a general LP problem,

$$\max Z = C^T X,$$
$$\text{subject to } AX \geq b$$

is considered for $X \in R^n$. The constraints form an arbitrary convex polyhedron Ω^m in R^n. In Ω^m a strictly interior point is successively moved to a higher isometric plane from a lower one along the gradient defined in the paper. Finally, the highest boundary point which makes the objective function value maximum is found or the infinite value of the objective function is concluded. For an $m * n$ matrix A the arithmetic operations of a movement are $O(mn)$ in our algorithm. The algorithm enables one to solve linear equations with ill conditions as well as a general LP problem. Some interesting examples illustrate the efficiency of the algorithm.

§1. Introduction

The following cononical form of a general LP problem

$$\max Z = C^T X, \quad \text{subject to } AX \geq b \tag{1.1}$$

is considered in this paper, where C^T denotes the transpose of $C, C = (c_1, c_2, \cdots, c_n)^T$, $X = (x_1, x_2, \cdots, x_n)^T$, $A = (a_{ij}), i = 1, \cdots, m, j = 1, 2, \cdots, n, b = (b_1, \cdots, b_m)^T$. Note that problem (1.1) is called general LP problem because $X \geq 0$ is not required specially in the constraints, that is, the constraints of problem (1.1) can form an arbitrary convex polyhedron in an n-dimensional space R^n. The convex polyhedron is an empty set if the constraints of (1.1) are inconsistent. In this paper, (1.1) is said to have no solution if and only if the constraints are inconsistent.

Although L.G.Khachiyan published the first polynomial-time algorithm for linear programming, the ellipsoid algorithm[3], in 1979, the methods which are practically efficient are the simplex method and the method presented by N.Karmarkar in 1984[1,2,5]. In 1947, G.B.Dantzig designed his famous simplex method, but the idea can be traced back to J.B.J.Fourier in the 1820s[1,5]. In showing how to find the best L_∞ approximation to a

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solution of linear equations, he used the notion of a point moving along the edges of a polyhedron from one vertex to another[1].

If in the simplex method the objective point is moved along a broken line on the boundary of the convex polyhedron, then Karmarkar's algorithm is a method in which the objective point is moved along a curve in the convex polyhedron. The isometric plane method presented in this paper makes the objective point move from one isometric plane to another along a broken line in the convex polyhedron, and the objective function maximizes. In our view, the simplex method is a special case of the isometric plane method.

Compared with existing methods, the isometric plane method solves problem (1.1) while not increasing dimensionality n of the space and number m of the constraints. It needs no slack variables or duality principle.

Throughout the paper we always suppose that the dimension of the vector space R^n, $n \geq 2$, the number of constraints $m \geq 1$, and

$$C_i^T = (a_{i1}, a_{i2}, \cdots, a_{in}) \neq 0 \ , \ i = 1, \cdots, m \ ,$$

and naturally $C \neq 0$.

§2. Description of the Method

The m constraints of problem (1.1) form a convex polyhedron Ω^m in R^n. Set

$$C_i^T = (a_{i1}, a_{i2}, \cdots, a_{in}) \ , \quad i = 1, \cdots, m \ . \tag{2.1}$$

The polyhedron Ω^m may be denoted as

$$\Omega^m = \{X \in R^n \, | C_i^T X \geq b_i \ , \quad i = 1, \cdots, m\} \ . \tag{2.2}$$

We first suppose that Ω^m is nonempty and has interior points.

The boundary $\partial\Omega^m$ of Ω^m consists of all or part of hyperplanes

$$P_i = \left\{X \in R^n \middle| \sum_{j=1}^{n} a_{ij} x_j = b_i \ , \ 1 \leq i \leq m\right\}. \tag{2.3}$$

The normal vector of P_i is C_i denoted by (2.1). The positive direction of vector C_i passing through X^i directs to the inside of Ω^m. The intersection of P_i and $\partial\Omega^m$ is called a surface of Ω^m, denoted as

$$\bar{P}_i = P_i \cap \partial\Omega^m \ .$$

Regarding C in the objective function of (1.1) as a vector, we define the c-line passing through Y :

$$L_y^c = \{X \in R^n \, | X - Y = tC, \ t \in R^1\} \ , \tag{2.4}$$

and denote $t > 0$ as the positive direction of L_y^c. The objective function value in (1.1) is increased with a point moving forward in the positive direction of L_y^c from Y. Therefore, the positive direction of L_y^c may be regarded as the c-height direction (or simply height direction), and the hyperplane

$$Q_y^c = \{X \in R^n \,|C^T(X-Y)=0\} \tag{2.5}$$

may be defined as the c-isometric plane (or simply isometric plane). Clearly, any two points on the isometric plane have the same objective function value (i.e. the same height).

Let $l_1, \cdots, l_m$ be any permutation of $1, \cdots, m$. The boundary points of the convex polyhedron Ω^m denoted by (2.2) can be divided into three classes: the surface point, the edge point and the vertex. The edge point is the point Y which satisfies

$$C_i^T Y = b_i,\ i = l_1, l_2, \cdots, l_r,\ 1 \le r \le n, \sum_{j=1}^{r} \alpha_j C_{l_j} \ne 0,\ \forall \alpha_1, \cdots, \alpha_r \in R^1, \sum_{j=1}^{r} \alpha_j^2 \ne 0,$$
$$C_i Y > b_i,\ i = l_{r+1}, \cdots, l_m. \tag{2.6}$$

In order to emphasize the degree of freedom, the edge point defined by (2.6) is usually called an $(n-r)$-dimensional edge point. When $r=1$, an $(n-1)$-dimensional edge point is a point on the surface of Ω^m, and it is called a surface point. And when $r=n$, a 0-dimensional edge point is a vertex of Ω^m. If the dimensions are not specified, an edge point will mean only the point satisfying (2.6) and $2 \le r \le n-1$ later on. Let Y be an $(n-r)$-dimensional edge point defined by (2.6). We now call the intersection of r hyperplanes in the form of (2.3),

$$E_y^r = \{x \in R^n | C_{l_j}^T (X-Y) = 0,\ j = 1, \cdots, r\}\ , \tag{2.7}$$

an $(n-r)$-dimensional edge of Ω^m. The vector

$$d_r = C - \sum_{j=1}^{r} \alpha_j C_{l_j} \tag{2.8}$$

is called the gradient of the edge (2.7), where the coefficients $\alpha_j (j = 1, \cdots, r)$ are chosen such that

$$C_{l_i}^T d_r = C_{l_i}^T C - \sum_{j=1}^{r} \alpha_j C_{l_i}^T C_{l_j} = 0,\ i = 1, \cdots, r\ . \tag{2.9}$$

In this section we suppose that the initial interior point X^0 is given. Assume the sequence of interior points $X^0, X^1, \cdots, X^k (0 \le k \le n-1)$ monotonously increasing height have been found, and the k linearly independent surfaces $P_{l_1}, P_{l_2}, \cdots, P_{l_k}$ recorded. Our algorithm finds a higher interior point X^{k+1} and records the corresponding surface $P_{l_{k+1}}$ (or turn to an appropriate exit) to complete an iteration or an iterative circle by the three steps as follows.

Step I. Construct the d_k-line

$$L_{x^k}^{d_k} = \{X \in R^n \,|X - X^k = t d_k,\ t \in R^1\} \tag{2.10}$$

passing through X^k, where the vector d_k equals C for $k=0$ or denotes the gradient of the edge intersected by $P_{l_1}, P_{l_2}, \cdots, P_{l_k}$ for $k>0$. As in (2.4), $t>0$ denotes the positive direction in (2.10). $L_{x^k}^{d_k}$ does not intersect any one of $P_{l_1}, P_{l_2}, \cdots, P_{l_k}$ for $k \ge 1$.

i) $L_{x^k}^{d_k}$ does not intersect $\partial\Omega^m$ in the positive direction. Clearly, in this case Ω^m is non-closed and the positive direction of $L_{x^k}^{d_k}$ makes the objective function value increase infinitely. Problem (1.1) has been solved. Go to the exit INF.

ii) In the positive direction, $L_{x^k}^{d_k}$ intersects $\partial\Omega^m$ at an $(n-s)$-dimensional edge point $Y^s (1 \le s \le n)$, which is an intersection point of s linearly independent surfaces $P_{g_1}, \cdots, P_{g_s}$ (s is usually 1.)

$$Y^s = X^k + t_s d_k, \; t_s = \min_{1 \le j \le m} \{((b_j - C_j^T X^k)/C_j^T d_k) > 0\} .$$

If

$$C = \sum_{j=1}^{s} \lambda_j C_{g_j}, \quad \lambda_j \in R^1, \; 1 \le s \le n , \tag{2.11}$$

then, by (2.7) and (2.5), $E_{y^s}^s$ is located in the isometric plane $Q_{y^s}^c$. Theorem 3 in §3 gives us the sufficient and necessary condition to determine if Y^s is the highest $(n-s)$-dimensional edge point. If Y^s is the highest vertex, then the solution is obtained. If Y^s is the highest edge point, then any point Z of $E_{y^s}^s \cap \partial\Omega^m$ is a solution of (1.1) and $C^T Z = C^T Y^s$. Go to the multisolution exit MUL. Assume Y^s is not the highest $(n-s)$-dimensional edge point or (2.11) does not hold. Let

$$Y^0 = X^k + \theta_1 \theta (Y^s - X^k) = X^k + \theta_1 \theta t_s d_k,$$
$$0 < \theta_1 \le 1, \; \theta = \begin{cases} 1, & s = 1 \text{ or } \theta_1 \le 0.5, \\ 1/(2\theta_1), & s > 1 \text{ and } \theta_1 > 0.5, \end{cases} \tag{2.12}$$

and record the surface $\bar{P}_{l_{k+1}} = \bar{P}_{g_i}$ $(1 \le i \le s)$. Go to Step II.

$\bar{P}_{l_{k+1}}$ is linearly independent of $\bar{P}_{l_1}, \cdots, \bar{P}_{l_k}$, and, by $0 < \theta_1 \le 1$, $C^T Y^0 > C^T X^k$ and $Y^0 \in \Omega^m$.

Step II. Construct the projective ray of $C_{l_{k+1}}$:

$$J_{y^0}^{k+1} = \{X \in R^n \,|X - Y^0 = t(C_{l_{k+1}} - (C^T C_{l_{k+1}}/C^T C)C), \; t > 0\} \tag{2.13}$$

passing through X^k in the isometric plane $Q_{y^0}^c$, where

$$I_{k+1} = C_{l_{k+1}} - (C^T C_{l_{k+1}}/C^T C)C$$

is the projective vector of $C_{l_{k+1}}$. All or part of $J_{y^0}^{k+1}$ is located in Ω^m.

i) A portion of $J_{y^0}^{k+1}$ is located in Ω^m, namely $J_{y^0}^{k+1}$ intersects $\partial\Omega^m$ at the point

$$Z^0 = Y^0 + t_0 I_{k+1}, \; t_0 = \min_{1 \le j \le m} \{((b_j - C_j^T Y^0)/C_j^T I_{k+1}) > 0\} .$$

Take

$$X^{k+1} = \theta_2 Z^0 + (1-\theta_2) Y^0 = Y^0 + \theta_2 t_0 I_{k+1} \; , 0 \le \theta_2 < 1 \; , \tag{2.14}$$

and go to Step III.

ii) All of $J_{y^0}^{k+1}$ is located in Ω^m, namely $J_{y^0}^{k+1}$ does not intersect $\partial\Omega^m$ at any finite point. Let $t_0 = 1$ in (2.14) and take X^{k+1} and go to Step III.

Note that X^{k+1} is a strictly interior point of Ω^m unless $\theta_1 = 1$ in (2.12) and $\theta_2 = 0$ in (2.14). Evidently, we have

$$C^T X^{k+1} = C^T Y^0 > C^T X^k .$$

Step III. Investigate all the recorded surfaces $P_{l_1}, \cdots, P_{l_k}, P_{l_{k+1}}$.

i) If $d_{k+1} \neq 0$, then $k+1 < n$; an iteration is completed. $k+1 \Rightarrow k$, $X^{k+1} \Rightarrow X^k$. Go to step I.

ii) If $d_{k+1} = 0$, then $k+1 \leq n$; an iterative cycle is completed. The hyperplanes $P_{l_1}, \cdots, P_{l_{k+1}}$ determine a point V_n. (Let $\theta = \theta_1 = 1$ in (2.12) and $\theta_2 = 0$ in (2.14). We can obtain an intersection point V_n of $P_{l_1}, \cdots, P_{l_{k+1}}$ after the surfaces $\bar{P}_{l_1}, \cdots, \bar{P}_{l_{k+1}}$ are recorded.)

If V_n does not satisfy the condition of Theorem 3, then, by $d_{k+1} = 0$,

$$C = \sum_{j=1}^{q} \alpha_j C_{l_j} + \sum_{j=q+1}^{k+1} \alpha_j C_{l_j} ,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_q$ are all negative coefficients and $1 \leq q < k+1$. Record the surfaces $\bar{P}_{l_1}, \bar{P}_{l_2}, \cdots, \bar{P}_{l_q}$, $q \Rightarrow k$, $X^{k+1} \Rightarrow X^k$. Go to Step I. There is at least one different recorded surface between two successive cycles. If V_n satisfies the conditions of Theorem 3, then $C^T V_n > C^T X^{k+1}$. Create the vector $V = V_n - X^{k+1}$ which generates an acute angle with C. And construct the v-line $L^v_{x^{k+1}}$ passing through X^{k+1} similarly to (2.10). In the positive direction, $L^v_{x^{k+1}}$ intersects $\partial\Omega^m$ at Y^*. When $Y^* = V_n$, Y^* is the highest vertex or edge point of Ω^m. When $Y^* \neq V_n$, Y^* is usually an $(n-1)$-dimensional edge point, namely $L^v_{x^{k+1}}$ intersects the surface $\bar{P}_l$ at Y^*. Clearly, the surface $\bar{P}_l$ is different from any one of $\bar{P}_{l_1}, \cdots, \bar{P}_{l_{k+1}}$. Record $\bar{P}_l$, $l \Rightarrow k$, $X^{k+1} \Rightarrow X^k$. Go to Step I and enter iterations of the next cycle. There is also at least one different recorded surface between two successive cycles in the case.

The convergence of the isometric plane algorithm is clear. Both bounded and unbounded solutions of problem (1.1) can be obtained in a finite time of iterations. An iterative cycle consists of n iterations at most. (Note that a number of iterations required for each circle are not necessarily the same.)

We now investigate the parameters θ_1 and θ_2 in the algorithm. When $\theta_1 = 1$ and $\theta_2 = 0$, the objective point rises monotonously along gradients of different edges. This is essentially the simplex method [1]. When $0 < \theta_1 < 1$ and $\theta_2 = 0$, the objective point rises monotonously along a broken line in Ω^m. This is similar to Karmarkar's algorithm in a sense.

§3. Theoretical Analysis

The possibility of the algorithm described in §2 will be concluded by the following Theorems 1–4.

Theorem 1. *Let Y be the edge point which satisfies (2.6) and $r < n$, and X^k an interior point of Ω^m. Assume*

$$C \neq \sum_{j=1}^{r} \lambda_j C_{l_j} , \quad \lambda_j \in R^1 , \quad j = 1, \cdots, r .$$

Then the vector d_r defined by (2.8) and (2.9) does not equal zero, and the line $Y + td_r (t \in R^1)$ is located in the edge (2.7). And

$$C^T d_r > 0 ,$$

and d_r is the vector which generates the minimal acute angle with C in the edge E^r_y. The d_r-line (2.10) passing through X^k does not intersect any boundary face linearly dependent on $P_{l_1}, \cdots, P_{l_r}$, where $P_{l_1}, \cdots, P_{l_r}$ are the hyperplanes which form the edge (2.7).

Theorem 2. *Let Y be a surface point of Ω^m, which is located in the boundary face P_g. Assume*

$$C_g \neq tC, \quad \forall t \in R^1 .$$

Then the projective ray J_y^g defined by (2.13) is located in the isometric plane Q_y^c, and the intersection of J_y^g and Ω^m is a line segment or ray, which belongs to the inside of Ω^m except the end points.

Theorem 3. *Let Y be the $(n-r)$-dimensional edge point satisfying (2.6). Assume*

$$C = \sum_{j=1}^{r} \lambda_j C_{l_j}, \ \lambda_j \in R^1, \ j = 1, \cdots, r, \ 1 \leq r \leq n ,$$

and there is at least a negative real number among $\lambda_1, \lambda_2, \cdots, \lambda_r$. Then Y is the highest edge point ($r < n$) or vertex ($r = n$) in the c-direction if and only if there is no positive real number among $\lambda_1, \lambda_2, \cdots, \lambda_r$.

Theorem 4. *Let d_r be the gradient defined by (2.8) and (2.9) of the edge (2.7), $1 \leq r \leq n$, and X^k an interior point of Ω^m. Remove a hyperplane P_{l_i} from the hyperplanes $P_{l_1}, \cdots, P_{l_r}$ which form the edge (2.7). The remainder hyperplanes form the $(n-r+1)$-dimensional edge*

$$E_y^{r-1} = \{X \in R^n \mid C_{l_j}^T (X - Y) = 0, \ j = 1, \cdots, i-1, i+1, \cdots, r, \ 1 \leq r \leq n, \ 1 \leq i \leq r\} . \tag{3.1}$$

The gradient of (3.1) is denoted by d_{r-1}. Let $L_{x^k}^{d_{r-1}}$ be the d_{r-1}-line in the form of (2.10). We have that, in the positive direction, $L_{x^k}^{d_{r-1}}$ intersects P_{l_i} if and only if $\alpha_i < 0$ in (2.8).

In order to solve successively (2.9) and to determine the gradient d_k, we need $O(nk)$ arithmetic operations using the positive definite symmetrical decomposition (LDL^T decomposition). And in order to find the intersection point of $\partial\Omega^m$ and $L_{x^k}^{d_k}$ in the positive direction, we need $O(mn)$ operations. Usually $m > n \geq k$; therefore an iteration needs $O(mn)$ operations, and an iterative cycle needs at most $O(mn^2)$ operations.

§4. Determination of Initial Interior Point

In this section we deal with the problem of finding the initial interior point X^0, and simultaneously solve the problem of consistence of (1.1).

First, if Ω^m is degenerate, i.e. Ω^m is not empty but has no interior point, then the equality holds in the constraints of (1.1), and we can decrease the dimensionality of the space by using the equality or simply replace the equality with two perturbation inequalities[4]. Therefore we can suppose that Ω^m is non-degenerate. We now solve the inequalities

$$AX \geq b , \tag{4.1}$$

where A and b are as in (1.1).

Introducing a variable, we can convert (4.1) into the canonical form of a general LP problem. In fact, let X^τ be an arbitrary point in R^n; then the problem

$$\begin{aligned} &\max Z = -\xi, \\ &\text{subject to } AX + (b - AX^\tau + \lambda e_m)\ \xi \geq b, \ \xi \geq 0 \end{aligned} \tag{4.2}$$

is clearly a canonical LP problem in R^{n+1}, where the $(n+1)$-dimensional unknown vector $\begin{pmatrix} X \\ \xi \end{pmatrix} = (x_1, x_2, \cdots, x_n, \xi)^T$, and

$$e_m = (1, 1, \cdots, 1)^T \ , \tag{4.3}$$

$$\lambda = \max\Big(0, \max_{1 \le i \le m} \Big(\sum_{j=1}^{n} a_{ij} x_j^{\tau} - b_i\Big)\Big) + \theta_3 \ , \ \theta_3 > 0 \ . \tag{4.4}$$

If (4.1) has a solution X^0, then $\begin{pmatrix} X^0 \\ 0 \end{pmatrix}$ is clearly a solution of (4.2). Conversely, if (4.2) has a solution $\begin{pmatrix} X^0 \\ 0 \end{pmatrix}$, then X^0 is the solution of (4.1). If (4.2) has a solution, but the solution is $\begin{pmatrix} X^0 \\ \xi_0 \end{pmatrix}$, $\xi_0 > 0$, then (4.1) is inconsistent. In fact, (4.3) and (4.4) lead to

$$b - AX^{\tau} + \lambda e_m > 0 \ . \tag{4.5}$$

Noting (4.5) and $\xi_0 > 0$ in (4.2), we know that (4.1) is an inconsistent system.

It is easy to see that the constraints of (4.2) form the convex polyhedron Ω^{m+1} in R^{n+1}, and that, by (4.3) and (4.4),

$$\begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} X^{\tau} \\ 1 \end{pmatrix}$$

is a strictly interior point of Ω^{m+1}. Therefore, using the isometric plane method, we can find the solution of (4.2).

With the solution of (4.2) we can usually obtain an initial interior point of (1.1) if (4.1) is consistent and non-degenerate.

§5. Numerical Examples

We have coded our algorithm. The program running on ND100 computer uses LDL^T decomposition to solve successively (2.9). In order to make the most of the algorithm, the program allows us to choose properly X^{τ} in (4.2) and the following six parameters,

ST1 — θ_1 in (2.12)	ER — error bound,
ST2 — θ_2 in (2.14)	EU — infinitely great bound,
ST3 — θ_3 in (4.4)	ED — infinitely small bound.

Example 1. Solve the linear equations formed by 40th-order Hilbert matrix

$$AX = b, \ a_{ij} = 1/(i+j), \ b_i = \sum_{j=1}^{n} a_{ij}, \ n = 40 \ . \tag{5.1}$$

The exact solution of (5.1) is

$$x_i = 1, \quad i = 1, 2, \cdots, 40 \ .$$

We want to find the perturbation solution of the right-hand members with the perturbation quantity 10^{-8}; hence (5.1) is replaced by[4]

$$b - 10^{-8}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} < AX < b + 10^{-8}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}. \tag{5.2}$$

Using the method described in §4 and choosing:

$$\theta_1 = 0.8,\ \theta_2 = 0.5,\ \theta_3 = 0.1,\ \mathrm{ER} = 10^{-12},\ \mathrm{EU} = 10^{64},\ \mathrm{ED} = 10^{-16},$$
$$X^\tau = (0.9,\ 0.9, \cdots, 0.9)^T$$

we obtain a solution of (5.2) as follows:

1.00000548,	0.99994156,	1.00013361,	1.00000466,	0.99990923,
0.99990284,	0.99994867,	1.00000707,	1.00005449,	1.00008161,
1.00008798,	1.00007752,	1.00005570,	1.00002794,	0.99999886,
0.99997199,	0.99994974,	0.99993358,	0.99992415,	0.99992145,
0.99992502,	0.99993406,	0.99994755,	0.99996434,	0.99998323,
1.00000301,	1.00002249,	1.00004057,	1.00005620,	1.00006843,
1.00007642,	1.00007940,	1.00007670,	1.00007677,	1.00005211,
1.00002933,	0.99999909,	0.99996115,	0.99991531,	0.99986143.

The number of iteration cycles is 6, and the total number of iterations is 28. If we choose

$$\theta_1 = 0.8,\ \theta_2 = 0.5,\ \theta_3 = 0.1,\ \mathrm{ER} = 10^{-6},\ \mathrm{EU} = 10^{64},\ \mathrm{ED} = 10^{-12},$$
$$X^\tau = (0,\ 0, \cdots, 0)^T\ ,$$

then, after 40 iterations of 8 cycles, we have that $X =$

0.999517,	1.005445,	0.985203,	1.005554,	1.010444,
1.005098,	0.998457,	0.994318,	0.993145,	0.994134,
0.996270,	0.998738,	1.001001,	1.002770,	1.003935,
1.004501,	1.004540,	1.004155,	1.003459,	1.002561,
1.001558,	1.000532,	0.999549,	0.998662,	0.997906,
0.997309,	0.996886,	0.996644,	0.996584,	0.996703,
0.996992,	0.997440,	0.998034,	0.998761,	0.999605,
1.000552,	1.001585,	1.002691,	1.003855,	1.005064.

Example 2. Consider Klee-Minty's counter-example of a LP problem

$$\max \quad Z = \sum_{j=1}^{n} 10^{n-j} x_j,$$

$$\text{subject to} \quad \left(2\sum_{j=1}^{i-1} 10^{i-j} x_j\right) + x_i \le 100^{i-1}\ ,\ i = 1, 2, \cdots, n\ , \tag{5.3}$$

$$x_j > 0,\ j = 1, 2, \cdots, n\ .$$

In order to get the highest vertex, the simplex method has to go through $2^n - 1$ iterative cycles[1].

We have tested solving (5.3) for $n = 6, 7, 8$. Since the constraints of (5.3) are clearly consistent, we take

$$X^{\tau} = (0.1, 0.1, \cdots, 0.1)^T ,$$

which is a strictly interior point of the constraint polyhedron of (5.3). The test results are as follows. For $n = 6$, we choose

$$\theta_1 = 0.999999,\ \theta_2 = 0.000001,\ \theta_3 = 1,\ \mathrm{ER} = 10^{-8},\ \mathrm{EU} = 10^{72},\ \mathrm{ED} = 10^{-12} .$$

After 15 iterations of 6 cycles, we have

$$X = (0, 0, 0, 0, 0, 0.99999999999875 * 10^{10})^T ,$$
$$\max \quad Z = 0.99999999999875 * 10^{10} .$$

For $n = 7$, we choose

$$\theta_1 = 0.9,\ \theta_2 = 0.5,\ \theta_3 = 1,\ \mathrm{ER} = 0.496873 * 10^{-6},\ \mathrm{EU} = 10^{72},\ \mathrm{ED} = 10^{-16} .$$

After 18 iterations of 6 cycles, we record the highest vertex

$$V_7 = (0, 0, 0, 0, -0.73880 * 10^{-4},\ -0.12190 * 10^{-2},\ 1.00025 * 10^{12})^T$$

but have no max Z because of ER. Therefore the iterations restart from some interior point. Altogether, using 31 iterations of 11 cycles, we obtain

$$X = (0, 0, 0, 0, -0.76419 * 10^{-4},\ -0.52122 * 10^{-4},\ 0.99999 * 10^{12})^T ,$$
$$\max \quad Z = 0.99999 * 10^{12} .$$

For $n = 8$, we choose

$$\theta_1 = 0.8,\ \theta_2 = 0.5,\ \theta_3 = 1,\ \mathrm{ER} = 10^{-8},\ \mathrm{EU} = 10^{72},\ \mathrm{ED} = 10^{-20} .$$

After 25 iterations of 8 cycles, we have

$$X = (0, 0, 0, 0, -20 * 10^{-1},\ -1.47, -261.28,\ 0.97 * 10^{14})^T ,$$
$$\max \quad Z = 0.97 * 10^{14} .$$

Example 3. Consider the cutting-stock problem[1]

$$\min \quad Z = \sum_{j=1}^{37} x_j ,$$

$$\text{subject to} \quad \sum_{j=1}^{37} a_{ij} x_j = b_i \ , i = 1, 2, 3, 4;\ x_j \geq 0;\ j = 1, 2, \cdots, 37 , \tag{5.4}$$

where $b_1 = 97$, $b_2 = 610$, $b_3 = 395$, $b_4 = 211$, and $a_{ij} (i = 1, 2, 3, 4,\ j = 1, 2, \cdots, 37)$ are listed in the following table.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
a_{1j}	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
a_{2j}	0	1	1	0	0	0	0	0	0	2	2	2	1	1	1	1	1	1	1
a_{3j}	0	0	0	1	1	0	0	0	0	0	0	0	2	1	1	1	0	0	0
a_{4j}	0	1	0	1	0	3	2	1	0	2	1	0	0	2	1	0	4	3	2

j	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
a_{1j}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_{2j}	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_{3j}	0	0	3	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0
a_{4j}	1	0	0	2	1	0	4	3	2	1	0	7	6	5	4	3	2	1

As a cutting-stock problem the numbers x_j must be nonnegative integers, but we first consider (5.4) as a LP problem. Transform (5.4) into the canonical form

$$\max\,(-Z) = -\sum_{j=1}^{37} x_j,$$

$$\text{subject to } \sum_{j=1}^{37} a_{ij}x_j = b_i - \varepsilon_i,\ -\sum_{j=1}^{37} a_{ij}x_j > -b_i - \varepsilon_i,\ i = 1,2,3,4,$$

$$x_j \geq 0,\ j = 1,2,\cdots,37\,, \tag{5.5}$$

with $\varepsilon_1 = 10^{-7}, \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 10^{-6}$. Using the method described in §4 and choosing

$$\theta_1 = 0.9,\ \theta_2 = 0.5,\ \theta_3 = 10^{-2},\ \mathrm{ER} = 10^{-8},\ \mathrm{EU} = 10^{64},\ \mathrm{ED} = 10^{-12}\,,$$

$$X^{\tau} = (0,0,\cdots,0)^T,$$

we obtain an interior point of the constraint polyhedron after 46 iterations of 4 cycles. Then, after 34 iterations of one cycle we obtain the solution of (5.5) as follows. Problem (5.5) has multi-solution; one solution is

$$x_1 = 13.32499999,\ x_2 = 36.12499981,\ x_3 = 34.22500011, x_4 - x_9 = 0\,,$$

$$x_{10} = 58.92499963,\ x_{11} = 57.02499993,\ x_{12} = 55.12500023\,,$$

$$x_{13} = 197.49999950,\ x_{14} - x_{37} = 0\,,$$

and

$$\max(-Z) = -452.2499992\,.$$

Moreover, 34 iterations record the surfaces

$$\bar{P}_{43}, \bar{P}_5\,, \bar{P}_{42}, \bar{P}_1\,, \bar{P}_{41}, \bar{P}_7\,\,, \bar{P}_3\,, \bar{P}_{15}, \bar{P}_{16}, \bar{P}_{37}, \bar{P}_{45}, \bar{P}_{44}, \bar{P}_{35}, \bar{P}_{40}, \bar{P}_{27}, \bar{P}_{36}, \bar{P}_{34}\,,$$

$$\bar{P}_{14}, \bar{P}_{39}, \bar{P}_{26}, \bar{P}_{25}, \bar{P}_{31}, \bar{P}_{17}, \bar{P}_{32}, \bar{P}_{12}, \bar{P}_{38}, \bar{P}_{28}, \bar{P}_{29}, \bar{P}_{33}, \bar{P}_{13}, \bar{P}_{22}, \bar{P}_{23}, \bar{P}_{24}, \bar{P}_{30}\,.$$

Therefore, all optimal fractional-valued solutions belong to a 3-dimensional edge, for which we know that

$$x_4 - x_9 = 0,\ x_{14} - x_{37} = 0,\ x_{13} = 197.4999995\,,$$

$$\sum_j a_{ij}x_j = b_i - \varepsilon_i - a_{i,13}x_{13},\ i = 1,2,4;\ x_j \geq 0;\ j = 1,2,3,10,11,12\,. \tag{5.6}$$

We thus conclude that the LP problem (5.5) has no integer-valued solution, and that the minimal objective function value of the cutting-stock problem (5.4)

$$\min \sum_{j=1}^{37} x_j \geq 453 .$$

Let

$$x_4 - x_9 = 0,\ x_{14} - x_{29} = 0,\ x_{31} - x_{37} = 0,\ \ x_{13} = 197,\ x_{30} = 1 ,$$

$$2x_1 + x_2 + x_3 = 97 ,$$

$$x_2 + x_3 + 2x_{10} + 2x_{11} + 2x_{12} = 610 - 197 = 413 ,$$

$$x_2 + 2x_{10} + x_{11} = 211 .$$

Any integer-valued solution of (5.7) is an optimal integer-valued solution of (5.4), which makes

$$\sum_{j=1}^{37} x_j = 453 .$$

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